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ELECTROMAGNETIC FIELDS, WAVES AND NUMERICAL METHODS

Z. Haznadar and Ž. Štih



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ELECTROMAGNETIC FIELDS, WAVES AND NUMERICAL METHODS

Studies in Applied Electromagnetics and Mechanics

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Electromagnetic Fields, Waves and Numerical Methods

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INTRODUCTION

The aim of this book is to present classical electromagnetic theory, which includes electromagnetic fields and waves, in a modern way suitable for application in engineering problems and research. It has been written with the intention of serving as a resource to scientists and engineers, and as a textbook and source of knowledge in the study of electrical engineering. This book may be utilised as a textbook in advanced undergraduate and postgraduate courses on electromagnetic fields and waves in engineering universities, especially in electrical engineering faculties.

Electromagnetism is a fundamental topic with a variety of different applications. Its inherent complexity results in the need for usage of sophisticated mathematical procedures. Therefore, we assume that the reader has a knowledge of vector calculus, differential and integral equations and matrix algebra, together with a basic knowledge of electricity and magnetism. The mathematical prerequisites are summarised in the appendices.

Classical electromagnetic theory has been founded on a macroscopic approach that enables the neglecting of the granular structure of matter and charge. We may summarise all knowledge about macroscopic electricity and magnetism using only a few equations and definitions. Accordingly, our approach to problems in electrical engineering has mostly been deductive. Everything originates from Maxwell's equations. Therefore, within this book, we have not followed a historical sequence, but have chosen a logical sequence of subject presentation.

In the first chapter, we start by using Maxwell's equations as postulates upon which the entirety of classical electromagnetic theory has been based. Initially, the study of the basic laws of electromagnetism is restricted to those in free space. After that we adjust these laws to their application in materials.

The introduction of new supplementary functions known as electromagnetic potentials to represent the field is the subject of the second chapter. These mathematical functions, called vector and scalar potentials, are introduced in accordance with Maxwell's equations and their usage significantly simplifies the analysis of electromagnetic fields.

Electromagnetic energy is tackled in the third chapter by analysing the power an electromagnetic field delivers to charge moving in free space. Afterwards, we include the charge on which the macroscopic model of a material has been built. The resulting equation of energetic balance, known as Poynting's theorem, will be our guide in establishing the different forms of energy stored and transmitted by means of an electromagnetic field in both free space and a material.

In continuation, we shall develop the application of this theory to solutions of problems in electromagnetism. The simplest problems involve static fields, where field quantities are independent of time. So, in the fourth chapter, we go on to analyse problems in a static electric field caused by charge at rest. Special attention will be paid to the analytical solution procedures used for solving static electric fields, i.e. the direct solutions of

Laplace's equation by a separation of variables, the solution of problems in unbounded space and a determination of Green's functions by the method of images.

The next logical step in the study of electromagnetism is the analysis of an electromagnetic field caused by charge in uniform motion. The fundamental characteristic of such a field is that it can be split into two independent fields: a static current and static magnetic field. In the fifth chapter, we shall study a static current field in a conductive material and prove its analogy with a static electric field.

The fundamental characteristics of a static magnetic field are described at the beginning of the sixth chapter. In continuation, we look into how to solve problems in current-free regions by means of an antipotential, and how to solve problems relating to given currents in an unbounded space by means of the magnetic vector potential. We shall solve problems associated with given distributions of current in a bounded space by application of the method of images.

Fields in which we may neglect displacement currents are called quasi-static electromagnetic fields. In the seventh chapter, we shall describe a reduced system of Maxwell's equations and derive diffusion equations for field quantities in such a field. Afterwards, we shall introduce a phasor notation which greatly simplifies the solution of sinusoidal fields. The solution of quasi-static field problems is arranged into two classes: problems involving conductors of large thickness and unknown current distribution, and problems involving thin conductors and uniformly distributed currents.

In the eighth chapter, we shall study the propagation of homogeneous transverse plane waves through conductive materials and dielectrics. The phenomenon of dispersion of a wave is analysed in both dielectrics and metals. Afterwards, we go on to analyse the incidence of electromagnetic waves on a plane boundary between two materials. The propagation of guided plane waves is applied in the analysis of transmission lines, waveguides and resonant cavities. Finally, we study the Hertz dipole and a linear antenna in an unbounded space as sources of radiation.

The finite element method, which is a wide-spread numerical tool for solving field problems, is described in the ninth chapter. We have applied a step-by-step approach based on describing the solving of actual electromagnetic problems. We begin with a detailed description of all the essential steps needed to be taken in the solution of simple, two-dimensional static problems. In continuation, we outline non-linear problems and finally we describe the solving of dynamic quasi-static fields.

The method of moments is introduced at the beginning of the tenth chapter. Afterwards, we describe the application of this method to the solution of three-dimensional static electric field problems and the scattering of electromagnetic waves. Finally, we go on to describe thin-wire structures, introduce Pocklington's equation and finally show the application of the method of moments to the solution of radiation problems.

Throughout this book, we have presented a lot of elaborate examples that explain the introduced methods and concepts in detail. These examples were solved and graphical presentations were generated using the programming system *Mathematica*[®].

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To:

Biserka and Leila
Alemka and Vilim

1. THE ELECTROMAGNETIC FIELD

The basic laws of electromagnetic theory were founded at the beginning of the nineteenth century. In 1862, James Clerk Maxwell (1831 – 1879) expressed them in the form of a harmoniously coupled set of equations. Since then we have used *Maxwell's equations* for the precise mathematical modelling of macroscopic electromagnetic phenomena. These fundamental equations of classical electromagnetic theory describe the physical properties of an electromagnetic field and connect them with the charges and currents which represent sources.

The approach to the theory of electromagnetic fields applied in this book is a deductive one, contrary to the usual historical approach. It starts with a presentation of the fundamental postulates focussed upon in Maxwell's equations that completely covers the results of experimental research over many years, as well as research of electromagnetic phenomena. Subsequently, using these postulates as laws of electromagnetism, we shall derive and thus describe the mathematical and physical properties of electromagnetic fields and waves.

Originally, studies of electromagnetic theory tended to develop an "acting at a distance" concept, meaning that the forces acting on a particular charge or current element were taken to be directly caused by forces on that charge or current element resulting from all other sources. By contrast, Maxwell accepted Faraday's ideas. Michael Faraday (1791 – 1867) introduced the idea of an electric field that exists at all points of the media surrounding the source charge. He presented a field in analogy to stresses in elastic materials and called the media aether. By the introduction of "displacement currents", Maxwell completed and mathematically formulated a unique theory of electromagnetic fields. He wrote his classic equations in 1862.

In this chapter, we shall define electric and magnetic fields by means of the law for Lorentz force (Hendrik Antoon Lorentz, 1853 – 1928) separately from the fundamental postulates of electromagnetic fields. Initially, the study of the basic laws of electromagnetism will be restricted to those in free space. After that we shall adjust these laws to their application in materials. We shall analyse the macroscopic behaviour of fields in conductors, dielectric and magnetic materials, thus postulating that the influence of polarised and magnetised materials on the laws of electromagnetic fields can be modelled by the addition of polarized and magnetized charge and currents to free charge and currents already existing within the fundamental laws.

1.1. A PRESENTATION OF AN ELECTROMAGNETIC FIELD

For our purpose, we can define a *field* as a set of values associated to every point of a specific region that defines the acting of a particular quantity in the region. Accordingly, if a quantity is associated with every point of the region and if the changing in time and space of this quantity is defined by laws expressed as functions of space and time coordinates, that field is associated with the region.

In the study of classical electromagnetism in this book, we shall analyse scalar and vector fields in three-dimensional Euclidian space. We shall study static and time-dependent dynamic fields.

It is very important to note that an electromagnetic field is defined by a set of fundamental laws of physics that can be expressed in mathematical form by differential equations in three-dimensional space and time. Differential equations join the character of a field at a particular point to those of other fields and sources at the same point independently of causes placed everywhere else. Hence the fact that the character of an electromagnetic field depends directly upon the sort and properties of the material at a given point and in its enclosure. Consequently we can analyse the general features of various fields in some bounded region of space without any interest in the state in space outside of the chosen region. This is one of the fundamental characteristics of Faraday's and Maxwell's approach to electromagnetism.

1.1.1. Definitions of the fields E and B

We associate the forces observed in electromagnetic experiments with the existence of an electric charge and its motion. We can represent the distribution of charge and its motion as either a cause or a source, and an observed force as a consequence or an effect. The existence of charge and its motion produces electromagnetic field in a space. There are a few laws of forces that describe the physical properties of every electromagnetic system and associated electric field. In order to start the study of an electromagnetic field and its character directly, we shall first of all define certain field quantities. Hence we do not start with *Coulomb's law of force* (Charles Auguste de Coulomb, 1736 – 1806) as has been historically established. We shall define an electric and magnetic field using the *Lorentz force law* separately from other postulates. A number of experiments with electromagnetic forces show agreement with a few fundamental postulates:

1. *The existence of an electric charge*

According to the atomic-electron theory of electricity, there are in existence two types of electric charges: *positive* and *negative*. All charges in nature are multiples of the charge of the electron, whose present-day accepted value is $q_e = (1.6021892 \pm 0.0000007) \cdot 10^{-19}$ coulombs. The unit of electric charge is defined by means of Coulomb's law and we measure it in *coulombs* (C), which is equivalent to ampere-seconds (As). In this book, we

shall consequently use the international system of units (System International d'Unites-SI). This system of units is based on seven fundamental units: metre, kilogram, second, ampere, kelvin, candela and mol. The SI system in electromagnetism is also called the MKSA-system. It is based on four fundamental units: *metre* (m) for length, *kilograms* (kg) for mass, *second* (s) for time and *ampere* (A) for current.

2. Conservation of electric charge

In every isolated system, the algebraic sum of all charges is constant. This postulate introduces electric charge as a fundamental physical quantity in electromagnetic theory.

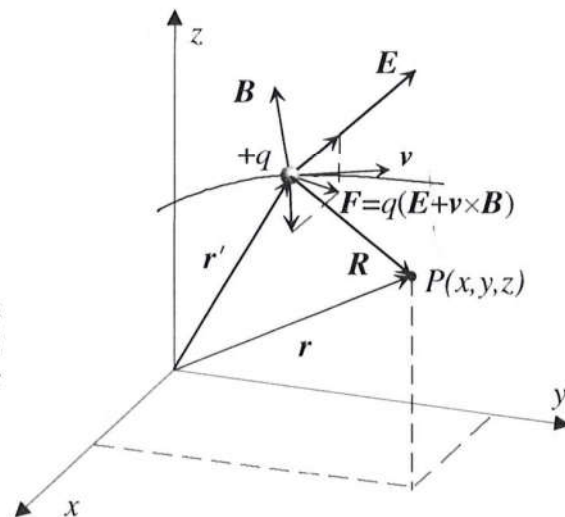
3. The Lorentz force law

We can express the electromagnetic force on a test charge placed at a given point of a space with the help of electromagnetic field quantities at the same point. Thus we express a total force $\mathbf{F}$ on a point charge q which is moving with velocity $\mathbf{v}$ with the help of two vector fields called *the electric field strength* $\mathbf{E}$ and *the magnetic induction* $\mathbf{B}$, by the equation

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) . \quad (1.1.1)$$

This force is called the Lorentz force. We shall use equation (1.1.1) as the defining equation for electric field strength $\mathbf{E}$ and for magnetic induction $\mathbf{B}$.

Figure 1.1.1. Movement of charge q placed at a source point $\mathbf{r}'$ with velocity $\mathbf{v}$ with respect to an observer placed at point P in $\mathbf{r}$. Distance between the charge and observer is $|\mathbf{R}| = |\mathbf{r} - \mathbf{r}'|$.



The Lorentz force law (1.1.1) gave us fundamental definitions about the vectors of electric and magnetic field. One important consequence of that definition equation is the fact that *characteristics of the space described by vectors $\mathbf{E}$ and $\mathbf{B}$ exist regardless of whether the test charge q has been put into that space or not.*

1.2. FIELD SOURCES: CHARGE AND CURRENTS

In the study of fields, one has to bear in mind the fact that the fields are caused by sources which may be inside or outside the region of interest. We shall analyse the *macroscopic theory of classical electromagnetism* by the introduction of “smooth” distributions

of charge and current in space as sources of electromagnetic fields. We usually consider electrical charges and their arranged motion that forms current as primal sources of a field. Since electrical charges are always bound to a material, it is correct to consider charged material (electrons, nuclei of atoms, ions etc.) as primal sources of an electromagnetic field.

1.2.1. Microscopic and macroscopic theory

When electrical engineers analyse fields in a space containing a material in everyday practice, they are usually uninterested in changes in the electromagnetic field in inter-atomic space. The study of these phenomena belongs to the area of microscopic or *quantum field theory*. Time and space averaged values of a field within very short time intervals and volumes compared to our measuring equipment, but great in atomic dimensions are of greater interest. We refer to such an approximative approach as macroscopic field theory. *Macroscopic field theory enables an approximation approach to reality.*

In mathematical expressions in electromagnetism, we shall often come across volume integrals of the form $\int \rho dV$ and time derivatives of the form $\partial \rho / \partial t$, where ρ stands for the density of the field sources. These operations result from the limits

$$\lim_{\Delta V \rightarrow 0} \sum_n (\rho_n \Delta V_n) = \int_V \rho dV \quad ; \quad V = \sum_n \Delta V_n ,$$

$$\lim_{\Delta t \rightarrow 0} \frac{\Delta \rho}{\Delta t} = \frac{\partial \rho}{\partial t} .$$
(1.2.1a,b)

In a macroscopic approach, we shall assume that dV and dt are very small non-zero quantities.

We are generally not going to be interested in the behaviour of particular microscopic particles. The theory of their motion belongs to quantum theory. We shall be exclusively interested in some mean effects, the ones that we shall determine experimentally within macroscopic dimensions. Of course, there is no sharp division between macroscopic and quantum theory. A final approval of validity can only be achieved by obtaining the results of macroscopic theory that are in accordance with experiments.

1.2.2. Charge and current densities

As we have mentioned before, in the study of macroscopic theory we shall use continuous distributions of charge and current. Generally, we shall assume that all sources which occur in reality have a continuous distribution in space and that they can be represented by continuous functions of both space and time coordinates.

We shall introduce volume charge density ρ , or simply *charge density* ρ at a point in space P . This will be defined as the ratio of the total charge ΔQ in the vicinity of point P and of volume ΔV of that vicinity, taking into account the limit when this volume converges to a point P

$$\rho = \lim_{\Delta V \rightarrow 0} \frac{\Delta Q}{\Delta V} = \frac{dQ}{dV} . \quad (1.2.2)$$

Charge density is measured in *coulombs per cubic metre* (C/m^3), which are equal to *ampere-seconds per cubic metre*. Within a macroscopic approach, charge density $\rho(\mathbf{r}, t)$ is a continuous function of spatial coordinates and time.

We define the current I (current strength) which flows through a conductor as the limit of the ratio of total charge ΔQ flowing through surface S of area ΔS through a time interval Δt where Δt limits to zero

$$I = \lim_{\Delta t \rightarrow 0} \frac{\Delta Q}{\Delta t} = \frac{dQ}{dt} . \quad (1.2.3)$$

Current strength is measured in *amperes* (A) and one ampere is equivalent to one *coulomb per second*. The sign is conventionally adopted as positive for a current flowing in the direction of motion of positive charge and negative for a current flowing in the direction of motion of negative charge.

The current density vector $\mathbf{J}$ or current per unit of surface is defined as a vector in the direction of the positive current with a magnitude equal to the ratio between the current strength ΔI flowing through elemental surface ΔS perpendicular to the direction of the current where ΔS limits to zero

$$\mathbf{J} = \mathbf{a}_v \left[\lim_{\Delta S \rightarrow 0} \frac{\Delta I}{\Delta S} \right] = \mathbf{a}_v \frac{dI}{dS} , \quad (1.2.4)$$

where $\mathbf{a}_v$ is a unit vector in the direction of the positive current flow. The unit for $|\mathbf{J}|$, in accordance with equation (1.2.4), is the *ampere per square metre* (A/m^2). In macroscopic theory, current density $\mathbf{J}(\mathbf{r}, t)$ is a continuous vector function of both space and time coordinates.

On the basis of the definition of a current as an arranged motion of charge, one may expect that current density is proportional to charge density. But that is not always the case. It is correct only for particular types of current known as *convection currents*. There are a lot of examples: currents in a vacuum tube or in a cathode tube are convection currents.

The direct proportionality between $\mathbf{J}$ and ρ for convection currents is obvious. Taking into consideration (1.2.4), one obtains the density of convection current

$$\mathbf{J} = \mathbf{a}_v \rho v = \rho \mathbf{v} . \quad (1.2.5)$$

Motion of charge in matter leads us to the introduction of *conduction currents*. These currents are the result of the flow of electrons and/or ions in various sorts of matter such as metals, semiconductors, electrolytes and gaseous plasma. In contrast to convection currents, the current density $\mathbf{J}$ of conduction currents is independent of the total charge density ρ . It has been proven that in reality, the charge density ρ equals zero in the case of a stationary flow of direct current (current density $\mathbf{J}$ differs from zero) through homogeneous conductors. The mean velocity of moveable charge is directly proportional to the

field $\mathbf{E}$ in the conductor. As a consequence, the density of conductive currents obeys *Ohm's law* (Georg Simon Ohm, 1789 – 1854)

$$\mathbf{J} = \kappa \mathbf{E} , \quad (1.2.6)$$

here κ stands for *the electrical conductivity of the material* at the point where we are measuring $\mathbf{J}$ and $\mathbf{E}$. The unit of conductivity is *the siemens per metre* (S/m).

At the end of this unit, we shall introduce integrals for calculating the current strength and amount of charge. Starting from the definition expressions (1.2.3) and (1.2.4) one can relate the current $I(t)$ through the surface S directly to the current density $\mathbf{J}(\mathbf{r}, t)$ by the integral

$$I(t) = \int_S \mathbf{J}(\mathbf{r}, t) \cdot \mathbf{n} \, dS , \quad (1.2.7)$$

where $\mathbf{n}$ denotes the unit vector normal to surface S . Similarly one can express the amount of charge Q in enclosed volume V as

$$Q(t) = \int_V \rho(\mathbf{r}, t) \, dV . \quad (1.2.8)$$

1.2.3. Continuity equation

It has been observed that in all physical experiments, the total amount of charge remains constant, and so the claim of conservation of charge forms a fundamental postulate of theoretical physics. It states that the amount of charge flowing from some closed space equals the rate at which charge decreases within this space

$$\nabla \cdot (\rho \mathbf{v}) + \frac{\partial \rho}{\partial t} = 0 . \quad (1.2.9)$$

Analogous to the corresponding equation in hydrodynamics, we name the partial differential equation (1.2.9) *the continuity equation*.

In reality, the continuity equation (1.2.9) is valid for any type of current expressed in the form

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 , \quad (1.2.10)$$

since the product $\rho \mathbf{v}$ is equal to the convection current density $\mathbf{J}$.

1.2.4. Singular source densities

All physical fields known in nature are always finite and continuous. They are caused by sources having finite volume densities. In approximate macroscopic electromagnetic theory it is often mathematically useful to idealise real volume densities of current and

charge into surface or line densities, or express them as point sources. Such idealisations result in an infinite, singular volume density of current and charge. These unreal source densities produce discontinuous and sometimes infinite (singular) fields. We shall analyse such non-physical behaviour of an electromagnetic field produced by singular charges and currents later in the unit on fields in matter and also when we come to analyse boundary conditions.

In any case, one should bear this in mind and be careful when one transforms fields calculated in such a way from macroscopic theory into reality. Idealisation of real charge and current into point, line and surface sources enables not only necessary mathematical simplicity, but motivates the introduction of appropriate physical notions that describe and represent real fields as well. Due to these reasons, we shall analyse the nature of such hypothetical sources that we shall use within the scope of an approximate macroscopic approach to electromagnetic theory in detail.

1.2.4.1. Surface charge density and surface current density

It was found that discontinuities in the components of electromagnetic fields arise on surfaces with charge and currents distributed within a thin layer, meaning in the form of surface charge and surface currents, which we define as follows:

Surface charge is defined as the charge distributed in a layer with a thickness equal to zero, and thus we define the surface charge density σ on a geometrical surface S as

$$\sigma = \lim_{\Delta S \rightarrow 0} \frac{\Delta Q}{\Delta S} = \frac{dQ}{dS} , \quad (1.2.11)$$

where ΔQ is the total charge on an elementary surface ΔS . The unit of σ is *the coulomb per square metre* (C/m²).

Let us imagine that a very small cylinder with volume $\Delta V = \Delta S \Delta l$ (ΔS is the area of the base and Δl is the height of the cylinder) is positioned on a geometrical surface (with thickness equal to zero). If the charge density ρ in the cylinder is known, then the total charge contained in the cylinder may be expressed as $\Delta Q = \rho \Delta V = \rho \Delta S \Delta l$. If we now suppose that the height of the cylinder Δl limits to zero, meaning the cylinder degenerates into a surface element with area ΔS , then the finite quantity of charge ΔQ contained in volume ΔV has to move onto a surface of area ΔS . Consequently, the volume charge density ρ within a layer with zero thickness has to become infinite, and we say that ρ is singular on the surface. To express this connection we use

$$\sigma = \lim_{\substack{\rho \rightarrow \infty \\ \Delta l \rightarrow 0}} (\rho \Delta l) . \quad (1.2.12)$$

The total charge on a surface of area S is calculated with help of equation (1.2.8) as

$$Q = \int_S \sigma dS . \quad (1.2.13)$$

The surface current is defined as the current flowing along a surface layer of zero thickness, and the surface current density $\mathbf{K}$ is defined as

$$K = \lim_{\Delta s \rightarrow 0} \left[\mathbf{a}_l \frac{\Delta I}{\Delta s} \right] = \mathbf{a}_l \frac{dI}{ds} , \quad (1.2.14)$$

where ΔI is the amount of current flowing along the surface layer S of zero thickness through an element of length Δs positioned perpendicular to the current flow ΔI , and $\mathbf{a}_l$ is a unit vector in the direction of current flow. The unit of $|\mathbf{K}|$ is *the ampere per metre* (A/m). Analogous to expression (1.2.5), surface current density can be linked with surface charge density σ that flows along the surface S with velocity $\mathbf{v} = \mathbf{a}_l v$ by

$$\mathbf{K} = \sigma \mathbf{v} . \quad (1.2.15)$$

And here we can see that the corresponding current density $\mathbf{J} = \rho \mathbf{v}$ is infinite, which means singular on the surface S in a dimension normal to the surface, since the charge density ρ is singular as well. Accordingly, we can connect $\mathbf{K}$ and $\mathbf{J}$ using the expression

$$\mathbf{K} = \lim_{\substack{J \rightarrow \infty \\ \Delta l \rightarrow 0}} (J \Delta l) , \quad (1.2.16)$$

where Δl is the thickness of the surface layer S that limits to zero.

In reality we find surface charge on the surfaces of conductors. When we put a volume charge with density ρ into a conductor at the same time, we establish an electric field $\mathbf{E}$ and a conduction current $\mathbf{J} = \kappa \mathbf{E}$ begins to flow. This current carries the volume charge onto an external surface of the conductor.

Surface currents become physically meaningful if we use them to approximate an alternating current flow in good conductors, especially when the skin effect occurs. In such cases, the largest part of the current flows close to the surface of the conductor, particularly at high frequencies. We shall show in Section 8.3.4. that inside a conductor, the current density decreases exponentially with normal distance from its outer surface. Therefore, we often use the term *current sheet* to describe the surface charge density $\mathbf{K}$.

It is helpful to approximate good conductors, such as metals, with perfect conductors which possess infinite conductivity. In perfect conductors ($\kappa = \infty$), the notion of surface charge density becomes exact. The total current flowing through such a conductor can be expressed as

$$I = \oint_c (\mathbf{K} \cdot \mathbf{a}_l) ds . \quad (1.2.17)$$

1.2.4.2. Densities of a line charge and line current

For the sake of simplicity in analysing macroscopic theory, we often find it convenient to study electrically charged conductors or current-carrying conductors that have an infinitesimally small cross-section. Densities of charge and current in these lines are two-dimensionally singular. We shall denote the resulting final density of a line charge at the point $\mathbf{r}_l$ with $\lambda(\mathbf{r}_l)$, that satisfies

$$Q = \int \lambda(\mathbf{r}_l) dl . \quad (1.2.18)$$

The unit of λ is *coulomb per metre* (C/m).

When a line charge with density $\lambda(\mathbf{r}_l)$ moves along a line with velocity $\mathbf{v}(\mathbf{r}_l)$, this movement forms a *line current* with direction $\mathbf{a}_l$

$$i\mathbf{a}_l = \lambda(\mathbf{r}_l) \mathbf{v}(\mathbf{r}_l) , \quad (1.2.19)$$

where $\mathbf{a}_l$ is a unit vector tangential to the line at point $\mathbf{r}_l$. The unit for line current is *the ampere*. From another point of view, we can regard a line current as the limit of the volume current distribution whose cross-section limits to zero and whose density $\mathbf{J}$ becomes infinite.

A current with density $\mathbf{J} = Jd\mathbf{l}/dl$ flows through an element of a line with length dl and with cross-section dS . The current through the cross-section is $i = \mathbf{J} \cdot \mathbf{n} dS$, where $\mathbf{n}$ is a unit vector normal to the cross-section. Therefore, $i d\mathbf{l} = (\mathbf{J} \cdot \mathbf{n}) dS d\mathbf{l}$, and because $\mathbf{J}$ and $d\mathbf{l}$ are parallel, we can write $i d\mathbf{l} = (d\mathbf{l} \cdot \mathbf{n} dS) \mathbf{J} = J dV$, so

$$\mathbf{J} dV \equiv i d\mathbf{l} . \quad (1.2.20)$$

Hence, we can idealise a current distribution as being a line current by satisfying this equivalence.

1.2.4.3. Point sources

At great distances from a total charge $q \neq 0$ distributed within a space, we can associate the field with a point charge q placed in the middle of the charge distribution $\mathbf{r}'$. Such idealisation results in an infinite, i.e. singular, charge density at the point $\mathbf{r}'$, and we can express it with the help of the *Dirac δ -function* as

$$\rho(\mathbf{r}) = q\delta(\mathbf{r} - \mathbf{r}') , \quad (1.2.21)$$

and the total charge in volume V that contains the point $\mathbf{r}'$ is

$$\int_V \rho(\mathbf{r}) dV = \int_V q\delta(\mathbf{r} - \mathbf{r}') dV = q , \quad (1.2.22)$$

since

$$\int_V f(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}') dV = \begin{cases} f(\mathbf{r}') & ; \quad \mathbf{r}' \in V , \\ 0 & ; \quad \mathbf{r}' \notin V . \end{cases}$$

We call sources with such impulse densities as $\delta(\mathbf{r} - \mathbf{r}')$ discrete, singular or point sources. They are mathematically convenient, and in spite of being singularities, we shall remember that they represent finite quantities placed in a very small volume which is nonetheless not equal to zero. The point charge model is frequently used in electromagnetism to describe the behaviour of particles that are small compared to the distance between them.

1.2.5. Discontinuities in a field

As was mentioned before, the existence of surface charge and surface current leads to discontinuities in some field components on the surface. Such non-physical discontinuities are caused by non-physical singularities in the source densities.

In this section, we shall describe a general problem associated with these discontinuities in fields that are caused by singular sources. In the interest of generality, we shall analyse a field $\mathbf{F}$ that can be any field vector. We shall denote the singular density of a vector field source as $\mathbf{K}$, and the singular density of a scalar field source as σ .

1.2.5.1. Field discontinuities caused by the surface density of a scalar source

Let us consider the scalar source of a field $\mathbf{F}$ having a singular density ρ on surface S as shown in Figure 1.2.1. so that $\rho dV = \sigma dS$ holds.

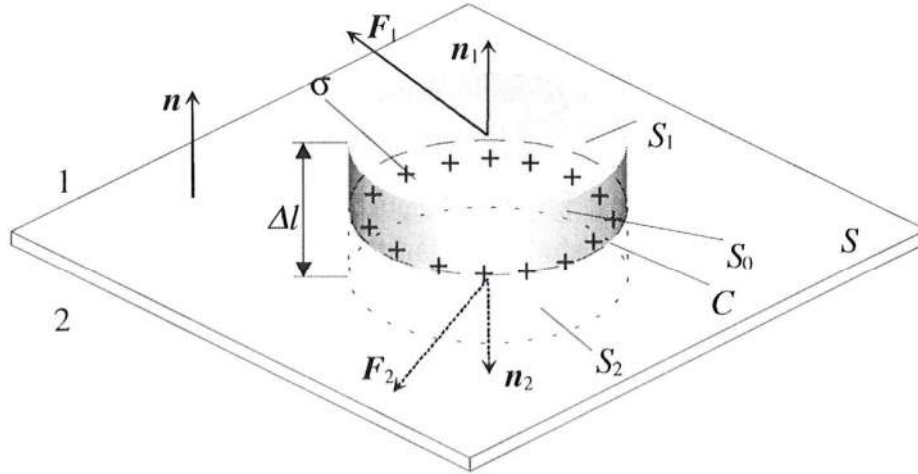


Figure 1.2.1. Discontinuity in the field $\mathbf{F}$ on a surface S that carries surface scalar source σ

Here, the surface density σ is finite and defined by equation (1.2.12). Surface S divides the whole space into two regions so that the direction of the normal $\mathbf{n}$ points from region 2 to region 1. A small cylinder with volume ΔV and height Δl was built on S_0 , with bases S_1 i S_2 parallel to S_0 . If we apply the divergence theorem to the field $\mathbf{F}$ inside the volume of the cylinder ΔV , we get

$$\int_{\Delta V} \nabla \cdot \mathbf{F} dV = \oint_{S_{\Delta V}} \mathbf{F} \cdot \mathbf{n} dS = \int_{S_1} \mathbf{F} \cdot \mathbf{n} dS + \int_{S_2} \mathbf{F} \cdot \mathbf{n} dS + \int_{S_{\text{curved surface}}} \mathbf{F} \cdot \mathbf{n} dS .$$

Since the last integral vanishes as $\Delta l \rightarrow 0$, it follows

$$\lim_{\Delta l \rightarrow 0} \int_{\Delta V} \nabla \cdot \mathbf{F} dV = \lim_{\Delta l \rightarrow 0} \left[\int_{S_1} \mathbf{F}_1 \cdot \mathbf{n}_1 dS + \int_{S_2} \mathbf{F}_2 \cdot \mathbf{n}_2 dS \right],$$

where $\mathbf{F}_1$ and $\mathbf{F}_2$ are discontinuous values of $\mathbf{F}$ at the same point on S , but in the regions 1 and 2, and $\mathbf{n}_1$ and $\mathbf{n}_2$ are unit normals in these two regions.

In Unit 1.3, we shall prove that the left-hand side of this equation equals the scalar source field $\int \rho dV$ in ΔV , and

$$\lim_{\Delta l \rightarrow 0} \int_{\Delta V} \nabla \cdot \mathbf{F} dV = \lim_{\substack{\Delta l \rightarrow 0 \\ \rho \rightarrow \infty}} \int_{\Delta V} \rho dV = \int_{S_0} \sigma dS.$$

In the limit $\Delta l \rightarrow 0$, we have

$$\lim_{\Delta l \rightarrow 0} \left[\int_{S_1} \mathbf{F}_1 \cdot \mathbf{n}_1 dS + \int_{S_2} \mathbf{F}_2 \cdot \mathbf{n}_2 dS \right] = \int_{S_0} \mathbf{n} \cdot (\mathbf{F}_1 - \mathbf{F}_2) dS,$$

and since surfaces S_1 and S_2 become identical to S_0 , the normals become $\mathbf{n}_1 = -\mathbf{n}_2 = \mathbf{n}$. In the end, we get

$$\mathbf{n} \cdot (\mathbf{F}_1 - \mathbf{F}_2) = \sigma. \quad (1.2.23)$$

Therefore, if the density of the scalar source of the vector field $\mathbf{F}$ is singular on a surface, then the normal component of the field possesses a discontinuity at any point of the surface equal to the scalar source density at that point.

1.2.5.2. Field discontinuities caused by the surface density of a vector source

Referring to Figure 1.2.1, we shall analyse the density of the vector source $\mathbf{J}$ of a field $\mathbf{F}$. Let $\mathbf{J}$ be singular in a dimension perpendicular to the surface S , which means let $\mathbf{J}dV = \mathbf{K}dS$, while $\mathbf{K}$ is finite and lies on S . By application of the curl (Gauss') theorem to the field $\mathbf{F}$ inside a cylinder with volume ΔV , we get

$$\int_{\Delta V} \nabla \times \mathbf{F} dV = \oint_{S_{\Delta V}} \mathbf{n} \times \mathbf{F} dS.$$

In the following Unit 1.3 we shall prove that the left-hand side of this equation equals the vector source of the field

$$\int_{\Delta V} \nabla \times \mathbf{F} dV = \int_{\Delta V} \mathbf{J} dV = \int_{\Delta S} \int_{\Delta l} (\mathbf{J} dl) dS.$$

When $\Delta l \rightarrow 0$, the limit of the integral is

$$\lim_{\Delta l \rightarrow 0} \int_{\Delta l} \mathbf{J} dl \rightarrow \mathbf{K},$$

and we finally obtain

$$\int_{S_0} \mathbf{K} dS = \lim_{\Delta l \rightarrow 0} \oint_{S_{\Delta V}} \mathbf{n} \times \mathbf{F} dS .$$

Again we can express the surface integral on the right-hand side of this equation as the sum of three integrals over S_1 , S_2 and over the curved surface of the cylinder. When $\Delta l \rightarrow 0$, the value of the integral over the curved surface limits to zero. From Figure 1.2.1, we can also see that

$$\lim_{\Delta l \rightarrow 0} \int_{S_1} \mathbf{n}_1 \times \mathbf{F}_1 dS = \int_{S_0} \mathbf{n} \times \mathbf{F}_1 dS ,$$

$$\lim_{\Delta l \rightarrow 0} \int_{S_2} \mathbf{n}_2 \times \mathbf{F}_2 dS = \int_{S_0} -\mathbf{n} \times \mathbf{F}_2 dS ,$$

and so we have

$$\int_{S_0} \mathbf{K} dS = \int_{S_0} \mathbf{n} \times (\mathbf{F}_1 - \mathbf{F}_2) dS .$$

Since this equation is valid as $S_0 \rightarrow 0$ as well, and since $\mathbf{n} \times \mathbf{F}_n \equiv 0$ it follows that

$$\mathbf{K} = \mathbf{n} \times (\mathbf{F}_1 - \mathbf{F}_2) = \mathbf{n} \times (\mathbf{F}_{1n} + \mathbf{F}_{1t} - \mathbf{F}_{2n} - \mathbf{F}_{2t}) = \mathbf{n} \times (\mathbf{F}_{1t} - \mathbf{F}_{2t}) . \quad (1.2.24)$$

This equation states that the tangential component of the field $\mathbf{F}$ on surface S possesses a discontinuity equal to the surface density $\mathbf{K}$ of the vector source of the field.

Densities of the surface scalar and vector sources σ and $\mathbf{K}$ on the surface that divides the space of interest into two regions will often be given or assumed. Using the triple cross product $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$ we get $\mathbf{n} \times [(\mathbf{F}_1 - \mathbf{F}_2) \times \mathbf{n}] = (\mathbf{n} \cdot \mathbf{n})(\mathbf{F}_1 - \mathbf{F}_2) - [\mathbf{n} \cdot (\mathbf{F}_1 - \mathbf{F}_2)]\mathbf{n}$. If we take into account that $\mathbf{n} \cdot \mathbf{n} = 1$, $(\mathbf{F}_1 - \mathbf{F}_2) \times \mathbf{n} = -\mathbf{K}$ and $\mathbf{n} \cdot (\mathbf{F}_1 - \mathbf{F}_2) = \sigma$, we get

$$\mathbf{F}_1 - \mathbf{F}_2 = \mathbf{K} \times \mathbf{n} + \mathbf{n} \sigma . \quad (1.2.25)$$

This is a general result that directly describes the discontinuity in the field $\mathbf{F}$ as being the sum of the normal and tangential components, and using

$$\mathbf{F}_{1t} - \mathbf{F}_{2t} = \mathbf{K} \times \mathbf{n} , \quad (1.2.24a,b)$$

$$\mathbf{F}_{1n} - \mathbf{F}_{2n} = \mathbf{n} \sigma ,$$

we can express the tangential and normal discontinuities dependent on the vector and scalar source density.

1.3. MAXWELL'S EQUATIONS IN A VACUUM

In order to approach the study of electromagnetic fields and their properties directly, we shall start this unit with Maxwell's equations that have been founded on many experimental facts. Maxwell's equations form the basis of classical electromagnetic theory.

Electromagnetic theory is founded on a few basic postulates and experiments. It is a surprising fact that all experimental observations can be formulated in a few fundamental laws: Coulombs', Ampere's (André Marie Ampère, 1775 – 1836), Biot-Savart (Jean Baptiste Biot, 1774 – 1862 and Félix Savart, 1791 – 1841), Faraday's and others. These experimentally stated laws connect classical electromagnetic theory with reality. By adding fundamental definitions, notions and some postulates, as well as by accepting a macroscopic approach and by introducing the notion of a field, Maxwell built a complete and unique *classical macroscopic electromagnetic theory*. Its mathematical expressions are called field equations or Maxwell's equations and they contain all the natural laws of electromagnetism.

We shall rearrange the experimental laws of Coulomb, Ampere and Faraday into expressions for scalar and vector densities of sources of the fields $\mathbf{E}$ and $\mathbf{B}$, in such a way that the resulting fields form the force determined by the Lorentz law.

The generalisation of Coulomb's law for the force between two point charges that are at rest with respect to one another leads us directly to the conclusion that charge is a scalar source of electric field. Therefore, we can write $\nabla \cdot \mathbf{E}(\mathbf{r}) = \rho(\mathbf{r})$, where $\rho(\mathbf{r})$ is the electric charge density at a point where we observe $\mathbf{E}$. Since we measure $\mathbf{E}$ in *volts per metre*, and ρ in *coulombs per cubic metre*, we must introduce the dimensional constant for free space $\epsilon_0 = (8.85418788 \pm 0.00000007) \cdot 10^{-12}$ As/Vm or *farads per metre* (F/m). We call it *permittivity of free space*, or *permittivity of a vacuum*. Finally, as a consequence of Coulomb's law, it follows that

$$\epsilon_0 \nabla \cdot \mathbf{E} = \rho_u, \quad (1.3.1)$$

an equation that connect the electric field $\mathbf{E}$ with scalar sources, eg with the total charge density ρ_u . In free space only free charge ρ_s may exist and thus $\rho_u = \rho_s$.

If we want to complete the definition of an electric field $\mathbf{E}$ connected with its sources, we have to prescribe the curl of the electric field together with its divergence, as prescribed by equation (1.3.1). Experiments indicate that a change in magnetic field produces an electric field. This fact is expressed by *Faraday's law* in integral form

$$\oint_{C_S} \mathbf{E} \cdot d\mathbf{l} = - \frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{n} dS, \quad (1.3.2)$$

where C_S is a curve that encloses surface S . The positive direction of the normal $\mathbf{n}$ points in the direction of progress of a right-hand screw turned in the direction of $d\mathbf{l}$. Application of *Stokes's theorem* to the left-hand side of equation (1.3.2) results in

$$\int_S \nabla \times \mathbf{E} \cdot \mathbf{n} dS = - \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} dS,$$

and since this equation must be valid for any surface S , it follows that

$$\nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t}. \quad (1.3.3)$$

It is important to note here that we have a “source” which is neither charge nor current.

Because of this, we call charges and currents “primary” sources of an electromagnetic field. Finally, we can state that the prescription of the scalar and vector sources in accordance with equations (1.3.1) and (1.3.3), together with the assumption that the field $\mathbf{E}$ together with its derivatives vanishes at infinity, uniquely define the field $\mathbf{E}$.

Let us take a look at the sources of the field $\mathbf{B}$. From Faraday's law of induction (1.3.3), it follows that $\nabla \cdot \mathbf{B} = 0$. This can be derived using the fact that the divergence of any curl is identically equal to zero

$$\nabla(\nabla \times \mathbf{E}) = -\nabla \cdot \frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial}{\partial t}(\nabla \cdot \mathbf{B}) \equiv 0 .$$

This expression shows that $\nabla \cdot \mathbf{B}$ is always independent of time, which means that $\nabla \cdot \mathbf{B} = 0$ (this is not a necessary consequence, but it is the most natural one). The equation

$$\nabla \cdot \mathbf{B} = 0 , \quad (1.3.4)$$

states that there are no scalar sources of the field $\mathbf{B}$. In other words we can say that, thus far, insulated magnetic charges have not been observed in nature.

Prescription of the quantity $\nabla \times \mathbf{B}$ is based on Ampere's experiments. From *Ampere's law* generalised for dynamic fields in free space, it thus follows that

$$\oint_{C_s} \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_S \mathbf{J}_u \cdot \mathbf{n} dS , \quad (1.3.5)$$

where $\mathbf{J}_u$ is the (total) density of all currents flowing through the surface S enclosed by the closed curve C_s . The dimensional constant in SI system is $\mu_0 = 4\pi \cdot 10^{-7}$ Vs/Am or *henries per metre* (H/m) in free space. We call it the *permeability in a vacuum*. If we apply Stokes's theorem to the left-hand side of equation (1.3.5), we get

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}_u . \quad (1.3.6)$$

The expression (1.3.6) states that the density of the total current $\mathbf{J}_u$ is the vector source of the field $\mathbf{B}$. There is no matter in free space and only free charge ρ_s may exist. If these charges move with velocity $\mathbf{v}$, they form a convection current with density $\mathbf{J}_s = \rho_s \mathbf{v}$.

Maxwell noticed an incompleteness and lack of harmony in Ampere's law and stated that another term had to be added to the current $\mathbf{J}_s$ in order to get the total current $\mathbf{J}_u$ that would be in accordance with the equations

$$\epsilon_0 \nabla \cdot \mathbf{E} = \rho_u ,$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}_u ,$$

and with the continuity equation. Since $\mathbf{J}_u$ is the curl of the vector, it follows that $\nabla \cdot \mathbf{J}_u \equiv 0$. But, from equations (1.2.10) and (1.3.1) it follows that

$$\nabla \cdot \mathbf{J}_s = -\frac{\partial \rho_s}{\partial t} = -\epsilon_0 \nabla \cdot \frac{\partial \mathbf{E}}{\partial t} ,$$

which means

$$\nabla \cdot \left(\mathbf{J}_s + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) = 0 ,$$

and so the expression within the brackets must be equal to the curl of the vector. Maxwell stated that the density of the total current in free space should be taken as

$$\mathbf{J}_u = \mathbf{J}_s + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} . \quad (1.3.7)$$

Thus, he obtained a new vector source of the field $\mathbf{B}$. We shall show that the introduction of the term $\epsilon_0 \partial \mathbf{E} / \partial t$ into this source anticipates the propagation of an electromagnetic field with a finite velocity. This fact was not known until Maxwell's time, but later was experimentally confirmed. This discovery based on mathematical harmony is one of the greatest achievements in the history of science.

1.3.1. Maxwell's equations in differential form

A coupled electric and magnetic field, i.e. an electromagnetic field resulting from a given set of charges moving in free space, can be expressed concisely using four differential equations. These equations summarise the fundamental laws of electromagnetism. They are:

1. *Coulomb's law (Gauss' law):*

$$\nabla \cdot \mathbf{E} = \frac{\rho_s}{\epsilon_0} , \quad (1.3.8a)$$

2. *Ampere's (generalised) law:*

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J}_s + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) , \quad (1.3.8b)$$

3. *Faraday's law of induction:*

$$\nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t} , \quad (1.3.8c)$$

4. *Gauss' law (non-existence of free magnetic charge):*

$$\nabla \cdot \mathbf{B} = 0 . \quad (1.3.8d)$$

These four equations are known as *Maxwell's equations in differential form* for a vacuum.

These forms of the equations are valid for the study of electromagnetic fields in systems at rest. The fact is, relativistic corrections are rarely necessary in engineering work, ie. in classical electromagnetic theory. This allows us to analyse electromagnetic fields in systems that are moving with low velocities using similar equations with additional terms.

Not all Maxwell's equations are independent. We showed that with the application of the divergence operation to equation (1.3.8c), we get equation (1.3.8d). In the same way, we can easily derive equation (1.3.8a) from equation (1.3.8b) and the continuity equation (1.2.10). Therefore we can say that Maxwell's equations (1.3.9b) and (1.3.9c) are independent (we often call them the *main Maxwell's equations*), in addition to the continuity equation, while equations (1.3.8a) and (1.3.8d) are derived (we call them *subsidiary Maxwell's equations*).

It is worth mentioning that the synthesis of all electromagnetic laws into one complete set of equations would not be possible without the term $\epsilon_0 \partial \mathbf{E} / \partial t$ for free space. Its invention and introduction into electromagnetic theory were the essence of *Maxwell's improvement*. He had foreseen that light is an electromagnetic phenomenon and that one could produce electromagnetic waves of all frequencies.

The term $\epsilon_0 \partial \mathbf{E} / \partial t$ was historically called the *displacement current*. Maxwell introduced it for two reasons: firstly, the additional term in current density was necessary to achieve mathematical harmony between Ampere's law and other laws of electromagnetism, and secondly, at this time, the prevailing opinion was that free space was a kind of material media called *aether*. In reality, nothing moves in free space, and as such, the notion "displacement current" is a misrepresentation. This name, originating from the discarded aether theory, has only a historical meaning and we thus use it as such.

1.3.2. Maxwell's equations in integral form

The system of Maxwell's equations (1.3.8a – d) is a set of expressions written for a point in free space. These linear differential equations simultaneously connect space and time derivatives of the fields $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ at every point of space and at every moment of time with one another and with the charge density $\rho_s(\mathbf{r}, t)$ and the current density $\mathbf{J}_s(\mathbf{r}, t)$ positioned at the same point $\mathbf{r}$ at the same moment in time t . We notice that these equations do not directly link the fields $\mathbf{E}$ and $\mathbf{B}$ at a chosen point with the sources at other points in space.

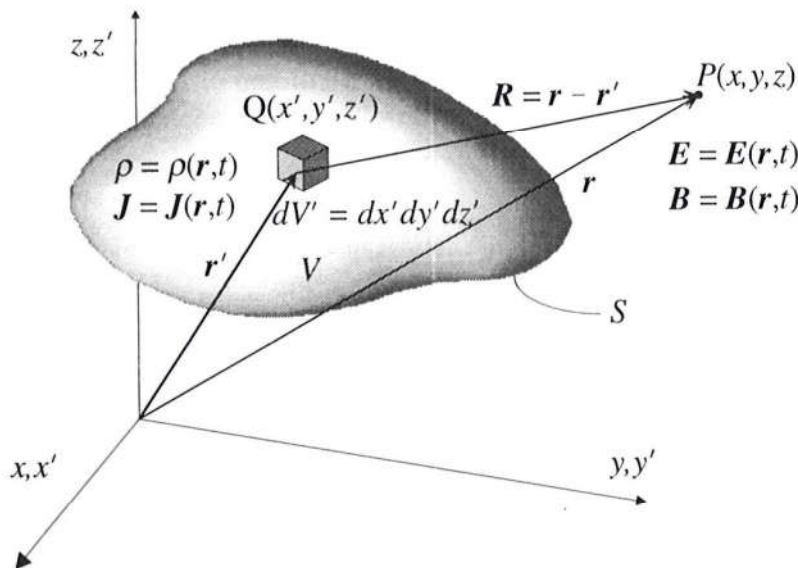


Figure 1.3.1. Calculation of the field by means of Maxwell's equations in integral form. $P(x, y, z)$ is the point at which we observe the field and $Q(x', y', z')$ is the point at which an elemental field source is placed.

Equally important integral expressions give the field distribution inside a whole region and often facilitate an easier physical interpretation than the one we could achieve using differential equations. The integral form of Maxwell's equations connect the behaviour of the fields $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ at a given point $\mathbf{r}$ with the sources $\rho(\mathbf{r}',t')$ and $\mathbf{J}(\mathbf{r}',t')$ placed everywhere in $\mathbf{r}'$ within a given region, as shown in Figure 1.3.1.

By application of Stokes's theorem and the divergence theorem, one can transform the system of Maxwell's differential equations (1.3.9a – d) into a system of integral equations in free space

$$\begin{aligned}\oint_S \epsilon_0 \mathbf{E} \cdot \mathbf{n} dS &= \int_V \rho_s dV = Q , \\ \oint_{C_S} \mathbf{B} \cdot d\mathbf{l} &= \int_S \mu_0 \mathbf{J}_s \cdot \mathbf{n} dS + \frac{d}{dt} \int_S \mu_0 \epsilon_0 \mathbf{E} \cdot \mathbf{n} dS , \\ \oint_{C_S} \mathbf{E} \cdot d\mathbf{l} &= - \frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{n} dS , \\ \oint_S \mathbf{B} \cdot \mathbf{n} dS &= 0 .\end{aligned}\tag{1.3.9a – d}$$

These expressions are applied to a closed curve C_S that encloses an open surface S , and to a closed surface S that encloses volume V in free space. With the help of the divergence theorem, we can express the postulate of charge conservation as the continuity equation (1.2.10) in integral form

$$\oint_S \mathbf{J} \cdot \mathbf{n} dS = - \frac{d}{dt} \int_V \rho dV = - \frac{dQ}{dt} .\tag{1.3.10}$$

The unit normal vector $\mathbf{n}$ on surface S is positive if it is directed outward from the closed surface S , or if it is directed in the direction of the right-hand screw out of the open surface S . As was the case with the differential equations, this equation can substitute for the two subsidiary equations and we can use the system of Maxwell's equations in integral form constituting only three independent equations (1.3.10b), (1.3.10c) and (1.3.11).

The general properties of electromagnetic fields in a given region of free space can be determined by means of the system of Maxwell's equations, either in differential (1.3.9a – d), or in integral (1.3.10a – d) form, without knowing or having any interest in the state outside of that region.

In order to determine the distribution of the field caused by a given distribution of sources in a space with a defined shape, it is necessary to prescribe all necessary *boundary and initial conditions*. They prescribe the distribution of all sources and fields along boundary surfaces that bound and enclose a given region at an initial moment in time. These conditions ensure the extraction of a particular and unique solution from the general solution of the differential equation system.

Finally, we can say that Maxwell anticipated a lot of natural phenomena that were proved later (like the electromagnetic nature of light) by accepting Faraday's idea of a

field and by introducing the concept of displacement current. By usage of this, he built a unique basis for the whole area of classical electromagnetic field theory. Maxwell's equations play the same role in field theory that Kirchhoff's laws do in the theory of circuits and networks.

1.4. AN ELECTROMAGNETIC FIELD IN MATERIALS

Within this unit, we shall supplement the study of electromagnetism with the macroscopic behaviour of fields in the presence of conductors, dielectrics and magnetic materials. We shall connect some macroscopic features of materials with their microscopic sources using simple classical models that ensure acceptable approximated solutions of essentially quantum mechanics problems.

Although one should not accept the planetary model of an atom too literally, we shall use it as a basis for understanding electric phenomena in materials. Therefore, we start with the fact that all matter is composed of atoms that contain subatomic particles. Some particles possess positive or negative charge, and some possess magnetic dipole moments due to orbital motion and spin. In light of this simple model, it is not surprising that we can describe certain behaviour of a material as being the source of an electromagnetic field. It brings us to the introduction of additional sources characteristic to a material to the laws of fields in free space.

The response of a material to the influence of external fields consists of a manifestation of secondary fields comparable to the applied external fields. These phenomena are equivalent to a distribution of charge and current acting as they would in free space from a macroscopic point of view. In fact, we have an interesting situation in which equivalent distributions of charge and current relate to the total fields in a material (these fields include externally applied fields and secondary fields), while secondary fields relate to equivalent charge and forces. Following this logic, one would find it appropriate to introduce new vector fields and derive a new set of Maxwell's equation with attached constitutive relationships.

The resulting electromagnetic fields in a material are very complex in their exact form. Fortunately, detailed knowledge of these microscopic phenomena is not necessary for a lot of engineering applications. Our macroscopic apparatus and devices are simply too big and insensitive to such small detail. In fact, macroscopic tests are unable to measure real fields in a material. In the best case scenario, we can measure only fields constituted in free space in the vicinity of the material. We shall use the term macroscopic to denote the fields produced outside the material, but in the vicinity of the material. The distance from the material has to be large enough in relation to the dimension of atoms inside the material, so that we may detect only the mean effects of all atoms in the material.

Accordingly, we shall present here a suitable macroscopic model of a material on the basis of which we can correctly estimate fields outside the material. Our models may give expressions for the fields inside materials. These expressions should be regarded as mathematically suitable, but physically unrealistic quantities. They are needed to develop a theory of macroscopic fields in the vicinity of materials.

1.4.1. A macroscopic model of a conductive material

We can classify materials in terms of their response to an applied electric field as being either *conductors*, *semiconductors* or *insulators*. According to the classical planetary model, an atom consists of a fixed and positively charged nucleus surrounded by a diffuse electronic cloud that has an equal quantity of negative charge. Most of the electrons are firmly tied to the nucleus of an atom and we call them *bound electrons*. These bound electrons can change their position locally within a microscopic region, but are unable to leave the atom under the influence of an applied electric field.

Free electrons are not within this class. They are always in a state of thermally unarranged motion, leaving one atom and moving to another one. When there is no external electric field their movement is random. The mean thermal velocity of free electrons from a macroscopic point of view is equal to zero and there is no current flow.

If we apply an electric field, an additional velocity that superimposes on the unarranged velocities occurs due to Coulomb's force. This velocity causes the flow of the mean position of electrons in a direction opposite to the direction of the electric field. We call this process *current flow*.

Materials having a great number of free electrons are called *electrical conductors*. In some materials only a small or negligible number of electrons can take part in current flow and we call these *insulators*.

A special type of material in which current flow involves not only electrons but another kind of charge carrier known as "holes" is in the area between conductors and insulators. We call such materials *semiconductors*.

1.4.2. Density of a conductive current in a material. Ohm's law

The relationship between current density and electric field strength in a material depends on the nature of the material. Let us consider two extreme cases: a current in a gas of a low concentration and a current in a metal.

For a gas whose molecular cross-section is $8 \cdot 10^{-20} \text{ m}^2$, "the mean free path" is around 400 m, which is much greater than the dimensions of the vacuum tube itself. Therefore, we consider that in such a media, electrons move from the cathode to the anode without any collisions with other particles. The motion of an electron is determined by the Lorentz force and Newton's law (acceleration of the electron is proportional to the electric field strength E).

In solid materials, the mean free path of an electron is within microscopic dimensions at around 10^{-7} m . With any macroscopic movement an electron frequently collides with particles of the material arranged in crystal grids. Due to the great number of collisions, the mean velocity of the electrons in solid materials linearly depends on the electric field strength within a wide range of interest. When the linear relationship between the velocity of flow v and the electric field strength E is established, we can write $v = \mu E$, where μ is

a property of a material called “mobility”. With a charge density of electrons $\rho_s = -Nq_e$, we can write, according to equation (1.2.5)

$$\mathbf{J}_s = \rho_s \mathbf{v} = -Nq_e \mathbf{v} = -Nq_e \mu \mathbf{E} = \kappa \mathbf{E} , \quad (1.4.1)$$

where N is the concentration of electrons, q_e is the charge of the electron, and κ is a constant called *the electrical conductivity* of the material. Relationship (1.4.1) is called Ohm’s law in elementary form. Materials in which this relationship is valid have $\kappa = -q_e N \mu$ irrespective of the magnitude and direction of $\mathbf{E}$, and are known as *linear isotropic conductors*. For some conductors every component of $\mathbf{J}_s$ can be dependent on all components of $\mathbf{E}$. In such cases $\mathbf{J}_s$ and $\mathbf{E}$ are, in general, not in the same direction, and the conductor is *anisotropic*.

We can see that in both of these examples, *free charge* with density ρ_s constitutes the current. It is capable of moving within a medium, unlike *bound charge* whose density shall be denoted as ρ_p . We should differentiate between *internal free charge* (these are free electrons in metals) and *external free charge* (this is space charge in vacuum tubes).

The currents caused by the motion of free charge are often called real currents, as opposed to polarization and magnetization currents that are caused by the motion of bound charge. Therefore, *real current* is a convection current caused by the motion of external free charge ρ_s in a vacuum tube as well as the conduction current caused by the motion of internal free charge ρ_s in a metal. *Conduction currents* are caused by the motion of free electrons and ions in a material, and are affected by an electric field, as is the case of metals and electrolytes.

1.4.3. A conductor in an electric field

Our intention is to define the restrictions that are imposed on static electric and dynamic fields, as well as on the conductive current densities that can exist in a conductive material. First, we introduce the assumption, which is in fact an additional postulate of field theory, that the behaviour of an electric and magnetic field in a conductive material is governed by the fundamental field equations in free space (1.3.8a – d) with the addition of two new source densities: the density of conductive currents $\mathbf{J}_s = \kappa \mathbf{E}$ and the corresponding density of free electric charge ρ_s that result from the law of conservation of charge (1.2.10)

$$\nabla \cdot \mathbf{J}_s + \frac{\partial \rho_s}{\partial t} = 0 . \quad (1.4.2)$$

A static electric field is a stationary field caused by charge in a state of rest. Since all charges are stationary, $\mathbf{v} = 0$, and $\mathbf{J} = 0$. Therefore we can conclude that an *electric field cannot exist in a conductive material* because in such a material $\mathbf{J}_s$ is algebraically coupled with $\mathbf{E}$ in such a way that $\mathbf{J}_s = 0$ only when $\mathbf{E} = 0$. On the other hand, since $\mathbf{E} = 0$, we have $\nabla \cdot \mathbf{E} = \rho_s / \epsilon_0 = 0$ as well, and we can conclude that there is no charge within the conductor under electrostatic conditions.

We shall prove this statement using the continuity equation (1.4.2) and Maxwell’s divergence equation (1.3.8a) to show that every free charge placed within a linear isotro-

pic conductor disappears (becomes equal to zero) very soon, even when dynamic fields are present. So, substitution of (1.4.1) and (1.3.8a) into equation (1.4.2) result in

$$\nabla \mathbf{J}_s = \nabla \cdot (\kappa \mathbf{E}) = \kappa \nabla \cdot \mathbf{E} + \mathbf{E} \cdot \nabla \kappa = \frac{\kappa}{\epsilon_0} \rho_s + \mathbf{E} \cdot \nabla \kappa = - \frac{\partial \rho_s}{\partial t} ,$$

and we get a differential equation for the distribution of free charge within a conductor

$$\frac{\partial \rho_s}{\partial t} + \frac{\kappa}{\epsilon_0} \rho_s = - \mathbf{E} \cdot \nabla \kappa , \quad (1.4.3)$$

in which we can regard the free charge density ρ_s as the response of the conductor to the source function $\mathbf{E} \cdot \nabla \kappa$. Inside the conductor in a stationary field we have $\mathbf{E} \equiv 0$, which is true for homogeneous ($\kappa = \text{const.}$) and non-homogeneous ($\kappa = \kappa(\mathbf{r})$) conductors as well, and the right-hand side of equation (1.4.3) equals zero.

In a dynamic field we can have $\mathbf{E} \neq 0$ inside the conductor, and the right-hand side of the equation (1.4.3) equals zero only in the case of homogeneous conductors, because then $\nabla \kappa = 0$. Therefore, the density of free charge at any point inside the conductor is defined only by the component independent of any source from the homogeneous differential equation

$$\frac{\partial \rho_s}{\partial t} + \frac{\kappa}{\epsilon_0} \rho_s = 0 . \quad (1.4.4)$$

To calculate the density ρ_s we integrate this expression and obtain

$$\rho_s = \rho_{s0} e^{-(\kappa/\epsilon_0)t} = \rho_{s0} e^{-t/T} , \quad (1.4.5)$$

where we define

$$T = \frac{\epsilon_0}{\kappa} . \quad (1.4.6)$$

This is exact proof that the density of free charge ρ_s inside a conductor decreases exponentially with time constant T , from an initial value ρ_{s0} to zero. The time constant $T = \epsilon_0/\kappa$ (permittivity of free space divided by the conductivity of the conductor) is measured in *seconds* (s) and we call it the *time constant of charge relaxation in the conductor*. In practice we can take it that after a period of $3T$ to $5T$, the charge will have completely disappeared inside the conductor. The order of magnitude of the relaxation constant varies from 10^{-19} s for copper, to 10^{-6} s for distilled water, to 10^{+6} s for quartz.

We can see that T for metals is extremely small. Therefore, we can suppose that each density of free charge inside a conductor vanishes and then appears almost immediately at its surface. Thus, we may suppose that surface charge on the surface of a conductor immediately follows all time variations of the electric field that caused it, as long as these time changes are large compared to the relaxation constant. This is not only the case in good conductors. Indeed, in distilled water free charge vanishes with a time constant of about 1 μ s, and within a few microseconds approaches zero.

It is convenient to suppose that conductivity can be infinite ($\kappa = \infty$) within a macroscopic approach to electromagnetic theory. In this way we can introduce the notion of the *perfect conductor* that does not actually exist in nature, but which superconductors are very much alike to. In a perfect conductor $T = 0$, and a free charge cannot remain in its interior at all. This charge immediately appears on the surface of a perfect conductor. It must always be inside such a material (in a static and in a dynamic field) that $\mathbf{E} \equiv 0$.

Therefore, there is no time dependent magnetic field inside a perfect conductor since $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t = 0$. Finally, *in a perfect conductor, there is no free charge, and neither time dependent (dynamic) magnetic field nor any (static and dynamic) electric field can exist.* An electromagnetic field cannot exist inside a perfect conductor and cannot penetrate into it.

Let us consider the equation (1.4.5) again in the case where a homogeneous conductor with finite dimensions enters an electrostatic field $\mathbf{E}$ in free space. In this case, $\nabla \kappa$ on the surface of the conductor (or on the boundary conductor-free space) is infinite and perpendicular to the boundary. If the field $\mathbf{E}$ possesses a normal component on the boundary, the right-hand side of equation (1.4.5) is then infinite on the surface of the conductor as well. Obviously, this results in an infinite volume charge density on the boundary surface. Since an infinite charge density is physically impossible, we have a material with conductivity κ and surface charge with finite density σ on one side of the mathematical boundary surface, and free space on the other side of the surface.

This sudden jump at the boundary, or discontinuity in space, is acceptable from the macroscopic point of view. Since free charge exists on the conductor, we shall denote this surface charge density as $\sigma_s(\mathbf{r}_s)$, where $\mathbf{r}_s$ is the position vector of a point on the boundary surface S . We call this charge on the surface of a conductor *induced surface charge*.

Since σ_s is the only charge on the conductor, (inside the conductor there can be no charge), we can represent the influence of the conductive body on the electrostatic field $\mathbf{E}$ with a corresponding distribution of surface charge with density $\sigma_s(\mathbf{r}_s)$ on the surface of the conductor S . This surface charge annihilates the electric field inside the conductor and changes the outer field $\mathbf{E}$ in its surroundings in free space. We say that the induced surface charge produces its own secondary electric field that superimposes on the externally applied field $\mathbf{E}$. Finally, we must once again warn that the existence of the conductive current $\mathbf{J}_s$ inside a conductive material ($\kappa \neq 0$), does not necessarily imply the existence of a total charge with a density different from zero in its interior. The total charge can, but nonetheless will not, be equal to zero in a conductive material while conductive currents are flowing through it.

1.4.4. Polarization of materials. The vector of polarization $\mathbf{P}$

In many cases, we come across the analysis of electromagnetic fields in materials whose response to an electric field cannot be neglected. In previous units, we focussed our attention on free charge and currents during the study of a field in conductors. Now we shall focus it on bound charge within a material. We find such charges in *dielectric materials*. The aim of this unit is to determine how we can substitute microscopic movements of bound charge in the presence of an externally applied electric field using macro-

scopic distributions of *polarized charge and currents*.

We have already said that in the presence of an electromagnetic field in a material, distributions of atomic charge and currents are distorted from their neutral state. These distortions affect the applied field and we can simulate them by equivalent distributions of charge and currents. In order to find out this equivalence, we shall introduce the *dipole moment* of an atom and *polarization* to represent the density of dipole moments of atoms and molecules in the material.

Figure 1.4.1. shows simple models of an atom of a material in a neutral state and an atom whose symmetry of bound charge distribution is distorted by an applied external field.

We say that the material with a charge distribution disturbed from its normal state is *electrically polarized*. In order to determine a measure of the polarization of such materials, which we call dielectrics, we shall define the dipole moment of a microscopic electric dipole belonging to an atom.

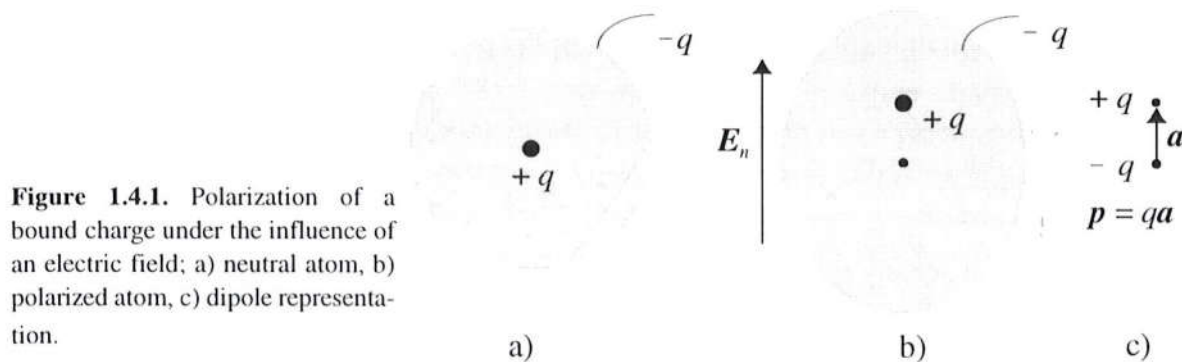


Figure 1.4.1. Polarization of a bound charge under the influence of an electric field; a) neutral atom, b) polarized atom, c) dipole representation.

Two displaced charge distributions in a polarized atom are not able to neutralise each other totally. In fact, for an observer outside the atom one can substitute the displaced charge using an electric dipole whose dipole moment p is defined as

$$p = qa, \quad (1.4.7)$$

where q is the quantity of charge of two point charges, equal in magnitude and opposite in sign, displaced by a . The displacement a is a vector directed from negative to positive charge as shown in Figure 1.4.1. The dipole moment is measured in *coulomb metres* (Cm). Since a is of atomic dimensions, p is very small.

We may distinguish between three types of polarization of materials. Many dielectric materials are neutral and do not show any effects of polarization if there is no applied external field. We say that there is no permanent polarization in the molecular structure of such materials. These are *non-polar molecules*. When an external electric field is applied to such non-polar molecules, processes similar to the afore described ones arise, and the electric charge in all atoms is displaced. We call this process *electronic polarization*.

Some dielectric materials possess polarized molecular structures when there is no external field. We say that such dielectrics have *polar molecules*. Polarization of individual atoms and molecules is oriented randomly and therefore the material is not polarized from a macroscopic point of view. If we apply an external electric field, it acts on the moment of polar molecules and tries to line them up from an initial random position to a partially coherent state in the direction of the applied field. This second type of polariza-

tion is known as *orientation polarization*. If we enable some materials, which we call “electrets”, to solidify in the external field, they become permanently polarized in the direction of the field. Some other materials, known as “ferroelectric materials”, possess *spontaneous permanent polarization*.

The third type of polarization, called *ionic polarization*, results from a separation of positive and negative ions in molecules that are held together by ionic bonds formed by the exchange of an electron from one atom to another within the molecule. All three kinds of polarization can take place simultaneously within a material.

Although the actual physical mechanisms of these three types of polarization in a material are basically different, their common macroscopic behaviour as sources of an electromagnetic field outside the material is similar. The simplest macroscopic model that we can use is to assume a random spatial distribution of dipoles, which varies smoothly and continuously from one point to another point in the material. Let us define the macroscopic vector field $\mathbf{P}(\mathbf{r}, t)$, called *polarization density* (vector of polarization, or more simply polarization), in such a way that it represents the equivalent smooth distribution of electric dipoles in unit volume at every point of the polarized material

$$\mathbf{P}(\mathbf{r}) = \lim_{\substack{\Delta V \rightarrow 0 \\ \text{around } \mathbf{r}}} \frac{\sum \mathbf{p}_i}{\Delta V} . \quad (1.4.8)$$

The polarization density $\mathbf{P}$ is measured in *coulombs per square metre* (C/m^2). It is defined in such a way that the total dipole moment of a volume dV is $d\mathbf{p} = \mathbf{P}dV$. $\mathbf{P}$ is obviously the *dipole moment of the material per unit volume*. Polarization is a measure of the distortion of bound charge distribution in a material from a macroscopic point of view. In the next section, we shall show that the polarization of a dielectric produces a (secondary) field that opposes the applied external field and changes its distribution both inside and outside of the dielectric material.

1.4.5. Dielectric in an electric field

Polarization takes place in dielectric materials under the influence of an applied electric field. We defined polarization using the polarization vector $\mathbf{P}$, which is the dipole moment of unit volume of a material or number of dipoles in unit volume $\mathbf{P} = N\mathbf{p}$ (N is number of molecules in unit volume). When we put a dielectric into the electric field $\mathbf{E}_n$, the induced polarization produces a secondary electric field $\mathbf{E}_s$, which decreases the applied field $\mathbf{E}_n$, which in turn causes a change in the polarization vector $\mathbf{P}$, etc.

When this adapting procedure is completed, a stationary state is reached, the sum of the original applied field $\mathbf{E}_n$ and the secondary field $\mathbf{E}_s$ must produce the polarization $\mathbf{P}$ that results in the secondary field. We can illustrate this state by a feedback loop, as shown in Figure 1.4.2, and express it using the equation

$$\mathbf{E} = \mathbf{E}_n + \mathbf{E}_s ,$$

where $\mathbf{E}$ represents the real (resultant, total) field in the dielectric.

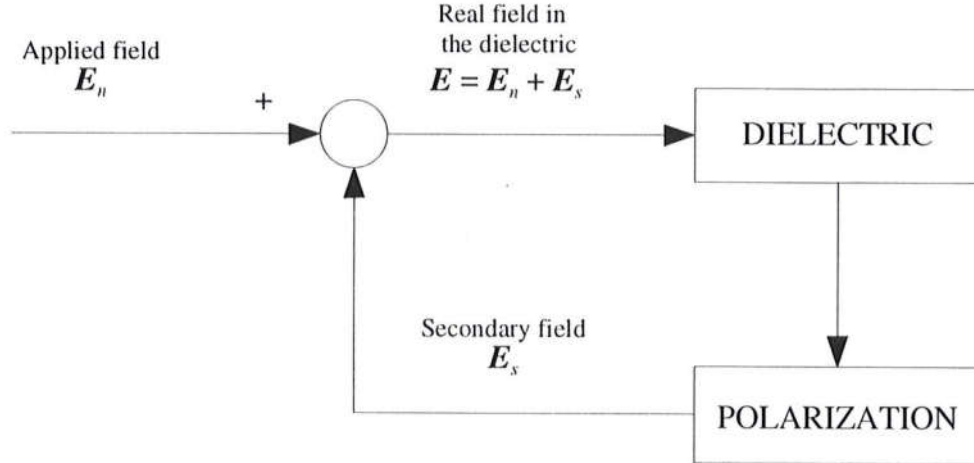


Figure 1.4.2. Feedback loop that explains polarization in a dielectric

The effect of polarization can be simply replaced by a distribution of dipoles whose density is defined by equation (1.4.8). Let us put a material in a time dependent external electric field in order to simulate the effect of the dielectric in an electromagnetic field with the help of equivalent distributions of charge and currents. The displacement $\mathbf{a}_i$ from the centre of the negative to centre of the positive charges $\pm q_i$ in an i -th atom of the material is time dependent and linked with the velocity of the charge by $\mathbf{v}_i = \partial \mathbf{a}_i / \partial t$. Movement of the charges $\pm q_i$ forms a *current element* $q_i \mathbf{v}_i$. If the volume of an i -th atom is δV_i , and if the density of the polarized charge in this volume is $\pm \rho_{pi}$, then we have $q_i \mathbf{v}_i = \rho_{pi} \delta V_i \partial \mathbf{a}_i / \partial t$, a current element linked to the i -th oscillating dipole of the polarized material. If we denote the current density of the i -th dipole as $\mathbf{J}_{pi} = \rho_{pi} \mathbf{v}_i = \rho_{pi} \partial \mathbf{a}_i / \partial t$ and $\mathbf{p}_i = q_i \mathbf{a}_i$, then $\mathbf{J}_{pi} = \partial \mathbf{p}_i / \partial t$, since q_i is constant for the i -th atom. If we transform this equivalence into a macroscopic form by the summation of the microscopic elemental currents over a macroscopically small volume dV , we get

$$\mathbf{J}_p = \frac{\partial}{\partial t} \frac{\sum_i \mathbf{p}_i}{dV} = \frac{\partial \mathbf{P}}{\partial t}, \quad (1.4.9)$$

where $\mathbf{J}_p$ is the macroscopic *polarization current density*. Since the polarized (bound) charge is conserved in every atom of the material, we can use the continuity equation

$$\nabla \cdot \mathbf{J}_p + \frac{\partial \rho_p}{\partial t} = 0. \quad (1.4.10)$$

If we substitute $\mathbf{J}_p = \partial \mathbf{P} / \partial t$ and integrate this expression we obtain

$$\rho_p = -\nabla \cdot \mathbf{P} \quad \text{in } V, \quad (1.4.11)$$

where ρ_p is macroscopic *density of the equivalent polarized charge in the material*. The total polarized charge Q_p in the volume of interest V enclosed by surface S , is, according to (1.2.8) and (1.4.11), and with help of the divergence theorem equal to

$$Q_p = \int_V \rho_p dV = - \int_V \nabla \cdot \mathbf{P} dV = - \oint_S \mathbf{P} \cdot \mathbf{n} dS , \quad (1.4.12)$$

where $\mathbf{n}$ is a unit vector normal to the surface S directed outwards. From equation (1.4.12) it follows that there is no density of polarized charge in a homogeneous volume distribution of charge, $\rho_p = - \nabla \cdot \mathbf{P} = 0$, because $\mathbf{P} = \text{const}$. Therefore, a polarized surface charge will appear with density

$$\sigma_p = - \mathbf{n} \cdot \mathbf{P} \quad \text{on } S , \quad (1.4.13)$$

on a boundary of homogeneous distribution of dipoles (boundary surface S of the homogeneous dielectric placed in free space), which means at points where $\mathbf{P}(\mathbf{r})$ has discontinuities.

Finally, we should point out the fact that in a material at rest, polarized surface currents are not physically observed. If for a material that is moving with velocity $\mathbf{v}$, a surface layer of charge σ_p is sufficient to produce an equivalent surface current $\mathbf{K}_p = \sigma_p \mathbf{v}$, then in stationary systems we shall always have that $\mathbf{K}_p = 0$. On the surface of the dielectric in free space, according to the definition of polarized currents $\mathbf{J}_p$ and charge σ_p , the expression

$$\mathbf{n} \cdot \mathbf{J}_p = - \frac{\partial \sigma_p}{\partial t} , \quad (1.4.14)$$

should be satisfied. Here $\mathbf{n}$ is the normal to the surface of the dielectric, pointing from the dielectric to free space.

1.4.6. Magnetization of materials

It has been experimentally observed that some materials when put in a magnetic field, influence the field and disturb it. This response of the material to an externally applied field is caused by microscopic elementary magnetic dipoles, formed by the motion of bound charge in a material. The resulting change, which is expressed as microscopic *magnetic dipole moments*, is known as *magnetization*. Such materials are called magnetic materials.

Let us define the magnetic dipole moment. Current i flowing through loop C enclosing surface S , produces at larger distances the same effect as a magnetic needle whose north pole is oriented in the direction of the positive normal $\mathbf{n}$ to the surface S , as shown in Figure 1.4.3. Since the magnetic needle has two poles (north and south) displaced by a small distance (compared to the distance from an observer), we call it the magnetic dipole, and we measure its strength by way of its magnetic moment $\mathbf{m}$. The magnetic moment of a current loop is defined as

$$\mathbf{m} = \mathbf{n} i S , \quad (1.4.15)$$

and we measure it in *amperes square metre* (Am^2).

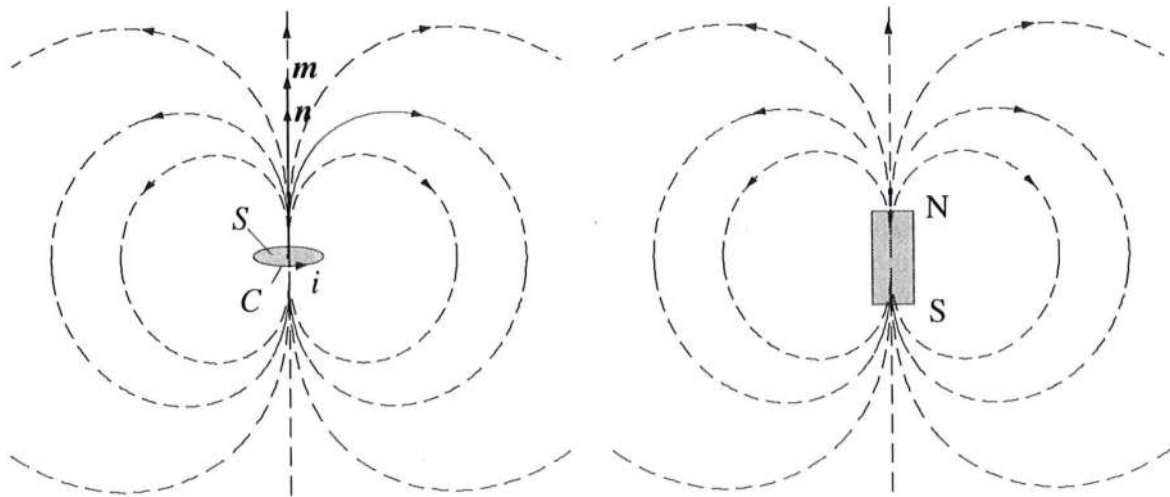


Figure 1.4.3. Magnetic field of a current loop and magnetic needle

There are four main types of magnetic effects in a material: *ferromagnetism*, *ferrimagnetism*, *paramagnetism* and *diamagnetism*. The first three types of magnetism are mostly caused by permanent magnetic dipoles of electrons that result from their intrinsic spin around their own axes. We can attribute only a small part to the current caused by the orbital motion of electrons around the nucleus. Diamagnetism is completely caused by these orbital motions. It results from changes induced in the magnetic dipoles of orbital electrons when an external magnetic field is applied.

In many materials, the total magnetic moment of every atom is equal to zero in the absence of an external magnetic field. An applied magnetic field causes an induction of a magnetic moment or *magnetization* of a material. This induced magnetization is opposite to the applied field, and a decrease in magnetic flux density relative to the applied value appears. We call such materials *diamagnetic materials*.

Paramagnetism, ferromagnetism and ferrimagnetism differ only in the way of linking of permanent dipole moments. In paramagnetic materials, individual atoms possess magnetic moments, even in the absence of an external magnetic field. These permanent magnetic moments are randomly oriented, so that the total magnetization equals zero from a macroscopic point of view. The applied magnetic field lines up these permanent dipoles in a partially directed arrangement along the field, thus inducing a total magnetization that results in an increase in the applied field.

In ferromagnetic and ferrimagnetic materials, forces between neighbouring materials are strong enough in relation to the random effects of thermic motions to orient dipoles in the same direction within some small regions known as domains. The resultant magnetic field in the vicinity of the material will depend on the relative size of these domains and on the orientation of the molecular dipoles inside them. It can be different from zero, even in the absence of an external magnetic field, below a critical temperature known as Curie's temperature. In this case, the material is generally called a *permanent magnet*.

The structure of the domains can be altered by an external magnetic field. Under its influence there is an increase in the volume of the domains in which molecular dipoles possess components parallel to the external field, while the volume of other domains decreases. This change in domain structure is reflected in an increase in total macroscopic

magnetization in the direction of the applied field.

If the applied field is strong enough, the motion of boundaries of the region is united with the rotation of the domains, and the material is magnetized till *saturation*. The material maintains a certain level of magnetization even after the external field stops acting. In fact, an opposite magnetic field needs to be applied in order to annihilate the total magnetization. This phenomenon, where the momentary magnetization state of the material depends on its previous magnetic history, is known as hysteresis. Contrary to diamagnetic and paramagnetic materials, magnetization of ferromagnetic materials is a non-linear phenomenon with respect to the applied field.

The molecular dipoles inside every domain in ferromagnetic materials are parallel, while strong interatomic connections in ferrimagnetic materials force some molecular dipoles in every domain to be antiparallel to the main direction of the magnetization within the domain. This creates a difference between ferromagnetism and ferrimagnetism. A subgroup of ferrimagnetic materials, called ferrites, is of great importance to engineering, since these materials possess significantly smaller electrical conductivity compared to ferromagnetic materials, while their magnetic characteristics are comparable to those of ferromagnetic materials.

For the sake of clarity, we have begun an analysis of magnetism with this description of microscopic characteristics of magnetic materials. Knowledge of the microscopic theory of magnetism is not necessary for determining the macroscopic characteristics of magnetic materials. As was in the analogous case of electric polarization, a macroscopic description of the behaviour of magnetic materials can be founded on only one new vector, the *macroscopic density of magnetization* $\mathbf{M}$ (vector of magnetization, or more simply magnetization), defined by magnetization effects observed outside the material. We shall assume that there is in existence a smoothly changing macroscopic density of equivalent dipole moments. Therefore, we define the vector of magnetization density $\mathbf{M}$ as an equivalent continuous distribution of magnetic dipoles in unit volume, at every point of the magnetized material

$$\mathbf{M}(\mathbf{r}) = \lim_{\substack{\Delta V \rightarrow 0 \\ \text{around } \mathbf{r}}} \frac{\sum_{\Delta V} \mathbf{m}_i}{\Delta V} . \quad (1.4.16)$$

Magnetization density $\mathbf{M}$ is the dipole moment per unit volume, and the total dipole moment in volume dV is $d\mathbf{m} = \mathbf{M}dV$. The unit for $\mathbf{M}$ is *ampere per metre* (A/m).

1.4.7. A model of amperian currents in a magnetic material

There are two fundamentally different theoretical approaches to the analysis of the behaviour of a magnetic material in an electromagnetic field, or in other words for the determination of densities of various equivalent macroscopic sources from the magnetization vector $\mathbf{M}$. One approach operates with a model of an amperian current $\mathbf{J}_a$, and the other with a model of a magnetic charge ρ_m and current $\mathbf{J}_m$. We shall analyse and compare both approaches, beginning with the model of amperian currents.

Similar to electrical polarization in Section 1.4.5, we can represent the magnetization of a magnetic material placed in an applied field $\mathbf{B}_n$ using a feedback loop, or with the action of a secondary field $\mathbf{B}_s$ in the material, in such a way that the real field in the magnetic material is $\mathbf{B} = \mathbf{B}_n + \mathbf{B}_s$. In the same way as we took into consideration the electric polarization of a dielectric by means of polarization charge ρ_p , we shall take the effects of electron spins and their orbital motion into consideration with the help of an equivalent macroscopic electric current with density

$$\begin{aligned}\mathbf{J}_a &= \nabla \times \mathbf{M} , \\ \nabla \cdot \mathbf{J}_a &= 0 .\end{aligned}\tag{1.4.17a,b}$$

called the *density of amperian current*. We measure it in *amperes per square metre* (A/m^2). Equation (1.4.19a) is applicable only when $\nabla \times \mathbf{M}$ is finite (non-singular). But, on the boundary surface of a magnetic material S , discontinuities in $\mathbf{M}$ happen, and we have to be very careful when using this equation. The vector $\mathbf{J}_a$ on the surface of the magnetic material becomes singular, and we introduce the notion of a layer of *amperian surface current* $\mathbf{K}_a$, with density

$$\mathbf{K}_a = \mathbf{M} \times \mathbf{n} \quad \text{on } S ,\tag{1.4.18}$$

where $\mathbf{n}$ is the normal to the surface S that encloses the magnetic material in volume V . We measure the amperian current sheet $\mathbf{K}_a$ in *ampere per metre* (A/m). Finally, we may conclude that the magnetic dipole moments, caused by the magnetization of the magnetic material placed in the magnetic field, are generally caused by:

- amperian surface currents defined by equation (1.4.23) on the boundary surfaces of the magnetic material;
- amperian volume currents with a density defined by equation (1.4.20) inside the magnetic material, in such a way that the total volume current that flows through any cross-sectional surface in the material is exactly opposite to the total surface current flowing through the loop that encircles this surface.

We have found that these amperian currents disturb the applied magnetic field inside and outside the magnetic material.

1.4.8. A model of magnetic charge and current

An elementary current loop is not the only acceptable source that produces a dipole field in free space. Such a field can produce two point charges of opposite sign and equal quantity as well.

From mathematical characteristics of Maxwell's equations (1.3.8a – d) for an electromagnetic field in free space, such as similarity and anti-symmetry with respect to the vectors $\mathbf{E}$ and $\mathbf{B}$, we may intuitively expect that analogous magnetic dipoles as field sources are possible, if a magnetic copy of the electric charge could be established. A *magnetic source, being a copy of electric charge, is magnetic charge* ρ_m .

Before we proceed with a survey of magnetic charge, let us formulate some fundamental postulates. First, an insulated particle of magnetic charge has not been experimen-

tally observed in nature. Gauss' law (1.3.8d) for a magnetic field states that *there is no magnetic charge*. Secondly, although there has not been any experimental proof of the existence of magnetic charges or monopoles so far, research of monopoles always starts afresh when new energetic areas in high energy physics are discovered, or in the case where new sources of materials, such as rocks from the Moon, appear.

This is mostly due to the earlier theoretical work of Dirac. He argued that only through the existence of magnetic monopoles in the Universe can we be assured of an explanation for the discrete nature of electric charge, which is the deepest mystery of the physical world.

On the other hand, the invariance of Maxwell's equations that is achieved by a dual transformation shows that when we speak about particles which possess electrical charge and no magnetic charge, it is only a matter of convention. The only important fact is that all particles possess the same ratio of magnetic and electric charge. If they do, we can perform a dual transformation with the choice that magnetic charge and currents equate to zero ($\rho_m = 0$ and $\mathbf{J}_m = 0$). In this case, we obtain Maxwell's equation in a well known form, which we shall permanently use in this book.

Therefore, we have to take the concept of magnetic charge as pure fiction of mathematical theory that has been introduced to simplify the explanation of magnetization of materials in an electromagnetic field.

We can define the magnetic charge q_m in a magnetized material analogously to the corresponding bound electric charge in a polarized material. Similar to the definition of the electric dipole $\mathbf{p} = q\mathbf{a}$, we shall define the *magnetic dipole moment* as

$$\mathbf{j} = \mu_0 \mathbf{m} = q_m \mathbf{a} , \quad (1.4.19)$$

where permeability of a vacuum $\mu_0 = 4\pi \cdot 10^{-7}$ (H/m) is included only because $\mathbf{j} = \mu_0 \mathbf{m}$ is traditionally taken to be analogous to $\mathbf{p}$. The dimension of q_m is *the weber* (Wb), which is equal to *volt second*. The unit for $\mathbf{j}$ is *the weber metre* (Wbm).

It is important to note that the unit for q_m is the weber and not the coulomb, and although magnetic charge is in many ways similar to electric charge, it is not in any way equal to q . To be more precise, we can say that q_m acts as the dual of q , and while these dual quantities are mathematically similar, they are physically completely different.

Continuing to draw conclusions based on the analogy between q_m and q , and between $\mathbf{j} = \mu_0 \mathbf{m}$ and $\mathbf{p}$, we can express the total macroscopic distribution of sources in a model with magnetic charge by an equivalent set of magnetic charge density ρ_m

$$\rho_m \equiv -\nabla \cdot (\mu_0 \mathbf{M}) , \quad (1.4.20)$$

and magnetic current density $\mathbf{J}_m$

$$\mathbf{J}_m \equiv + \frac{\partial}{\partial t} (\mu_0 \mathbf{M}) . \quad (1.4.21)$$

according to equations (1.4.9) and (1.4.11). Here, ρ_m represents the density of volume magnetic charge (in Wb/m³), and $\mathbf{J}_m$ represents the density of the magnetic current (in V/m²). The magnetic charge must satisfy the postulate of conservation of charge

$$\nabla \cdot \mathbf{J}_m = \nabla \cdot \left(\frac{\partial}{\partial t} \mu_0 \mathbf{M} \right) = \frac{\partial}{\partial t} (\nabla \cdot \mu_0 \mathbf{M}) = -\frac{\partial \rho_m}{\partial t} . \quad (1.4.22)$$

In order to fulfil the prescription of equivalent source densities using magnetic charge, we should imply the prescription of density of magnetic surface charge σ_m on the boundary surface of a magnetic material in free space

$$\sigma_m = -\mathbf{n} \cdot \mu_0 \mathbf{M} , \quad (1.4.23)$$

where the normal $\mathbf{n}$ on the boundary surface is oriented from the magnetic material into free space. Finally, the density of the magnetic surface charge must also satisfy the postulate of the conservation of charge, and on the boundary surface of the magnetic material we obtain

$$\mathbf{n} \cdot \mathbf{J}_m = -\frac{\partial \sigma_m}{\partial t} . \quad (1.4.24)$$

As in the case of electric polarization, no *magnetic surface current* $\mathbf{K}_m$ (in V/m) exists in a material at rest.

1.4.9. Field vectors $\mathbf{D}$ and $\mathbf{H}$. Electric and magnetic fluxes

We found out that, in comparison with Maxwell's equations in free space, we have to introduce new sources of the field into the laws that govern the fields in materials. These are the vectors of polarization $\mathbf{P}$ and magnetization $\mathbf{M}$ that represent the macroscopic response to an applied field.

A further step was the representation of the interaction between a material and an electromagnetic field defined by the vectors $\mathbf{E}$ and $\mathbf{B}$ with the help of sources: free charge ρ_s and currents $\mathbf{J}_s$ (conductive and convective), equivalent bound (in a material) polarization charge ρ_p and currents $\mathbf{J}_p$, and amperian currents $\mathbf{J}_a$ or magnetic charges ρ_m and currents $\mathbf{J}_m$. All these spatially distributed densities of charge and currents can often occur as surface densities of charge and currents on boundary surfaces.

To complete the idea about the solution of an electromagnetic field in which a material is set, we should remind that at the base of our approach is the postulate that the principle of linear superposition is valid. This means that if independent sources (ρ_s and $\mathbf{J}_s$), and particular dependent sources ($\mathbf{P}$ and $\mathbf{M}$, or ρ_p , $\mathbf{J}_p$, $\mathbf{J}_a$, ρ_m , $\mathbf{J}_m$) representing the influence of a material are given, then the total really existing electromagnetic field with vectors $\mathbf{E}$ and $\mathbf{B}$ can be calculated from Maxwell's equations in free space.

In this section we shall introduce vector fields $\mathbf{D}$ and $\mathbf{H}$, which are very important in a macroscopic approach to electromagnetic fields. They will enable the solution of the field including material media, without considering the various phenomena that actually happen in the materials.

We define a new vector $\mathbf{D}$, known as the *vector of electric induction, density of displacement flux* or *vector of electric displacement*, as

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} . \quad (1.4.25)$$

This vector satisfies the equation

$$\nabla \cdot \mathbf{D} = \rho_s . \quad (1.4.26)$$

We can see that the unit for $\mathbf{D}$ is the same as for $\epsilon_0 \mathbf{E}$ and $\mathbf{P}$, i.e. *coulomb per square metre* (C/m^2), and therefore $\mathbf{D}$ is *the vector of electric flux density*. Although the definition equation (1.4.25) is conceptually appropriate, it cannot really be regarded as the operational definition. Therefore, we often replace the vector $\mathbf{P}$ by introducing a new quantity called *permittivity*. We have mentioned before that $\mathbf{P}$ usually depends on $\mathbf{E}$ in a complex and unknown way. In practical circumstances it has been determined empirically that polarization $\mathbf{P}(\mathbf{r})$ is linearly coupled with the electric field $\mathbf{E}(\mathbf{r})$

$$\mathbf{P} = \epsilon_0 \tilde{\chi}_e \mathbf{E} . \quad (1.4.27)$$

For isotropic dielectrics the tensor $\tilde{\chi}_e$, which is called the *electrical susceptibility* of a dielectric material, is scalar and we obtain

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E} . \quad (1.4.28)$$

With the help of (1.4.27) we get

$$\mathbf{D} = \epsilon_0 (1 + \chi_e) \mathbf{E} = \epsilon_0 \epsilon_r \mathbf{E} = \epsilon \mathbf{E} . \quad (1.4.29)$$

In these new *constitutive relationships* (1.4.27 – 29) vectors $\mathbf{E}$ and $\mathbf{D}$ are directly connected with the help of new quantities $\epsilon_r = 1 + \chi_e$ and $\epsilon = \epsilon_r \epsilon_0$. We call ϵ *the absolute permittivity* (it is measured in *farads per metre* (F/m)), and ϵ_r , which is a dimensionless quantity, *the relative permittivity*, or simply *the permittivity*, or *the dielectric constant*. We can see that expression (1.4.29) is valid only for linear dielectrics, while the definition equation (1.4.25) is valid generally. We can express $\mathbf{P}$ as a function of $\mathbf{D}$:

$$\mathbf{P} = \epsilon_0 \chi_e \frac{\mathbf{D}}{\epsilon} = \frac{\chi_e}{1 + \chi_e} \mathbf{D} . \quad (1.4.30)$$

Starting from different ways of defining the vectors $\mathbf{E}$ and $\mathbf{D}$, we come across an interesting way of comprehending their sources. *It is obvious that the source of the field $\mathbf{E}$ is charge* (both free and bound). On the other hand, from (1.4.26) it follows that the *source of the field $\mathbf{D}$ is only free charge*.

The notion *flux* is of great importance in electromagnetic theory. Faraday originally invented the flux that arises from free charge, which is independent of the material. He experimentally proved that the charge induced on a metal surface is independent of the medium. Maxwell later introduced the vector $\mathbf{D}$ and named it “displacement”, which explains the term “density of displacement flux”.

The idea of flux is needed for the explanation of transfer of electrical influence from one point of space to another. The flux density at any point in space can be mathematically defined as the amount of flux through unit surface perpendicular to the direction of the flux lines

$$\mathbf{D} = \mathbf{n} \frac{d\Phi_e}{dS} , \quad (1.4.31)$$

where $\mathbf{D}$ is the flux density on an elemental surface dS , and $d\Phi_e$ is the amount of the flux that is defined by $d\Phi_e = \mathbf{D} \cdot \mathbf{n} dS$. The total flux through a surface with area S is determined by a surface integral over the surface S

$$\Phi_e = \int_S \mathbf{D} \cdot \mathbf{n} dS . \quad (1.4.32)$$

In the SI system, electric flux is measured in *coulombs* (C). If we imagine a closed surface around volume V that contains a given charge distribution ρ with a total amount Q , then, according to (1.3.9a) and (1.4.32), the total electric flux through the closed surface S is

$$\Phi_{eu} = \oint_S \mathbf{D} \cdot \mathbf{n} dS = \int_V \rho dV = Q . \quad (1.4.33)$$

This is an integral form of Gauss' law. The differential form of Gauss' law is expressed by equation (1.4.26). The physical notion of both forms is, naturally, the same: *there is no charge without flux, and there is no line of flux ($\mathbf{D}$ -line) that begins and ends anywhere but on a (free) charge.*

We shall define another new vector of a magnetic field $\mathbf{H}$, known as the *vector of magnetic field strength*, as

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} , \quad (1.4.34)$$

and which in static fields satisfies the equation

$$\nabla \times \mathbf{H} = \mathbf{J}_s . \quad (1.4.35)$$

In Unit 1.5, we shall develop the corresponding expression for a dynamic electromagnetic field. The unit of $\mathbf{H}$ is the same as for $\mathbf{B}/\mu_0$ and $\mathbf{M}$: the *ampere per metre* (A/m). The validity of the definition equation (1.4.34) is general; it is valid for both static and dynamic fields. In many magnetic materials a linear relationship between the magnetic field $\mathbf{H}$ and induced magnetization $\mathbf{M}$ exists

$$\mathbf{M} = \tilde{\chi}_m \mathbf{H} , \quad (1.4.36)$$

where $\tilde{\chi}_m$ is the *tensor of the magnetic susceptibility*. If we restrict ourselves only to *isotropic materials* where χ_m is the scalar operator, we get $\mathbf{M} = \chi_m \mathbf{H}$. For such materials χ_m is called the *magnetic susceptibility*. It can be a positive or negative number. For paramagnetic materials, the magnetic susceptibility is positive, and for diamagnetic materials the magnetic susceptibility is negative. For linear materials we can express $\mathbf{H}$ as

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0(1 + \chi_m)} = \frac{\mathbf{B}}{\mu_0\mu_r} = \frac{\mathbf{B}}{\mu} , \quad (1.4.37)$$

where we define $\mu_r = 1 + \chi_m$ and $\mu_r = \mu_0\mu_r$. Quantity μ_r is known as the *relative permeabil-*

ity of the magnetic material, and μ is the *permeability (absolute)* of the magnetic material. We see that μ_r is a dimensionless quantity and that the constitutive relationship (1.4.37) holds only for linear magnetic materials, for which we can take μ as being constant, while the defining relationship (1.4.34) is valid generally. The unit of permeability is the *henry per metre* (H/m).

The calculation of magnetic fields is much more complex compared to the calculation of electric fields in dielectrics, because one often has to take into account the non-linear character of μ , while dielectrics generally possess constant permittivity. Such non-linear materials are ferromagnetic materials. They can be magnetized in the absence of an external field. The relationship between $\mathbf{B}$ and $\mathbf{H}$ in a ferromagnetic material is non-linear, and depends on, among other things, the magnetic history of the material.

The relationship between $\mathbf{B}$ and $\mathbf{H}$ in a ferromagnetic material can be given in tabular form, or graphically in the form of a *magnetization curve*. There are in existence materials with μ_r less than one, while there is no dielectric with ϵ_r less than one. Finally, we may express $\mathbf{M}$ as a function of $\mathbf{B}$

$$\mathbf{M} = \chi_m \frac{\mathbf{B}}{\mu} = \frac{\chi_m}{1 + \chi_m} \frac{\mathbf{B}}{\mu_0} . \quad (1.4.38)$$

Similar to electric flux, we shall introduce the notion of *magnetic flux* Φ_m , as the surface integral of the flux density $\mathbf{B}$

$$\Phi_m = \int_S \mathbf{B} \cdot \mathbf{n} dS , \quad (1.4.39)$$

over a finite and open surface with area S . The total magnetic flux through every closed surface is, according to equation (1.3.9d), equal to zero.

1.4.10. Materials and constitutive relationships

Although Maxwell did not include the vectors of polarization and magnetization explicitly in his equations, he was certain that magnetization and polarization were induced by an externally applied field, and that relationships

$$\begin{aligned} \mathbf{D} &= \mathbf{D}[\mathbf{E}, \mathbf{B}] , \\ \mathbf{H} &= \mathbf{H}[\mathbf{E}, \mathbf{B}] , \end{aligned} \quad (1.4.40a,b)$$

exist. These relationships are known as constitutive relationships. We can add *the generalized Ohm's law* for conductive materials

$$\mathbf{J} = \mathbf{J}[\mathbf{E}, \mathbf{B}] , \quad (1.4.41)$$

to them. This law also belongs to the class of constitutive relationships. We have used the square brackets here to emphasise that these relationships are not always simple, and that they can depend on the history of a material (hysteresis), and that they can be multi-valued, non-linear, anisotropic, etc.

There exists a huge diversity in electrical and magnetic properties of materials, especially in crystal solids. The study of these properties is more a subject for solid state physics. We shall only describe some basic features of these macroscopic properties here.

1.4.10.1. Conductors

Beginning with conductors, one can say that the density of free conductive current J_s can depend on E in different ways. The most important case is the flow of free current in materials under isotropic conditions. In this case, as was mentioned before, the current density at any temperature is a unique function of the electric field, $J = \kappa E$, where conductivity κ is a unique parameter that can be determined experimentally. Generally κ can depend on both the magnitude and direction of E . In a special, but at the same time a very frequent case, κ is independent of E and J_s is directly proportional to E .

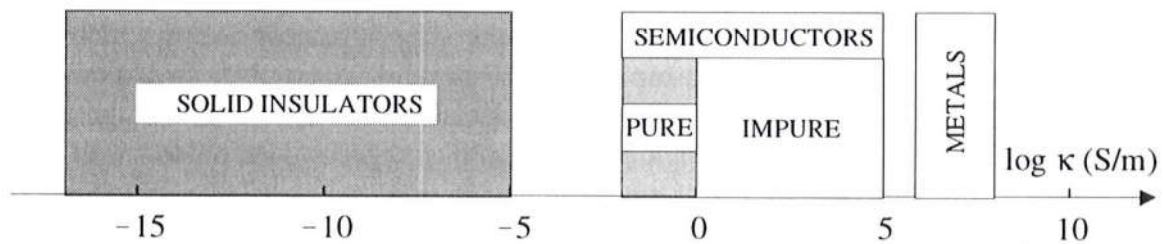


Figure 1.4.4. Areas of conductivity in various types of materials

In Figure 1.4.4. the order of magnitude of conductivity for conductors, semiconductors and insulators is presented. The constant value of conductivity does not imply that the conduction current density is proportional to the applied electric field for all values of J and E . However, there is a wide range of values of current density for metals whose conductivity is constant.

1.4.10.2. Insulators

In many important cases M and P are functions of the field in a material. Polarization P and magnetization M are functions of the electric and magnetic field, $P = P(E)$ and $M = M(H)$. We shall begin with dielectric materials (insulators).

The relationship $P = P(E)$ implies a great diversity in the physical properties of dielectric materials. In some materials, although P is directly proportional to E , the value of P at any given moment is not necessarily uniquely defined by the value of E at the same moment. This important phenomenon, in which P is a function of the “history” of E , is known as *hysteresis*.

In *gyroelectric materials*, P of the material is a function of the past values of E , as well. In such materials the function $P = P(E)$, under certain conditions, approaches the form of a linear differential equation with time derivatives. Finally, if P is independent of the history of E , then the relationship $P = P(E)$ reduces to a single-valued function of E . All afore mentioned materials can be classified as *isotropic* and *anisotropic*.

For isotropic materials, the relationship between $\mathbf{P}$ and $\mathbf{E}$ does not depend on the orientation of $\mathbf{E}$ with respect to the material. In anisotropic materials, the directions of $\mathbf{P}$, $\mathbf{E}$ and $\mathbf{D}$ are not the same. A lot of crystal structures are anisotropic materials. In order to describe the dependance of $\mathbf{P}$ on $\mathbf{E}$ in anisotropic materials, we generally need nine separate scalar expressions that can be represented in the form of a 3×3 matrix or tensor, where each of the nine members of the matrix connects one of the three components of $\mathbf{P}$ with one of three components of $\mathbf{E}$.

The permittivity $\epsilon = \epsilon_0(1 + \chi_e)$ is constant (*the dielectric constant*) and independent of the magnitude of $\mathbf{E}$ in a linear isotropic material. Permittivity can be a function of position $\epsilon = \epsilon(\mathbf{r})$, and be independent of $\mathbf{E}$ in a linear, isotropic, but non-homogeneous material. If this is not the case, the material is homogeneous.

1.4.10.3. Magnetic materials

We may classify magnetic materials in a similar way. It has been shown that inside homogeneous materials, the relationship $\mathbf{M} = \mathbf{M}(\mathbf{H})$ is valid. Materials with $\mathbf{M}$ dependent on the history of magnetization show hysteresis, or else they are gyromagnetic materials. We may differentiate between isotropic and anisotropic magnetic materials as well.

1.4.10.4. Hysteresis

A phenomenon that causes a shift of $\mathbf{B}$ behind $\mathbf{H}$ in such a way that the magnetization curve for ascending and descending fields is not the same is known as hysteresis. The loop that represents the magnetization curve is called *the hysteresis loop*, and one is shown in Figure 1.4.5. If a material is in saturation at the ends of the hysteresis, we call the loop *saturated*, or *the main hysteresis loop*.

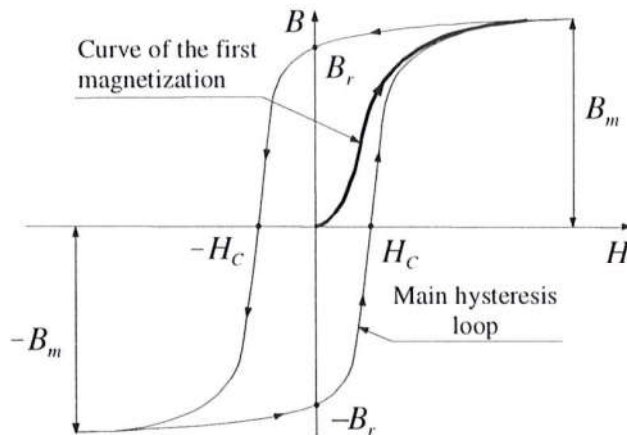


Figure 1.4.5. Hysteresis loop

The residual flux density B_r in the saturated loop is called *the retentivity*, and *the coercive force* H_c on this loop is called *the coercivity*. Thus, the retention of a material is the maximum possible residual flux density, and the coercivity of a material is the maximum possible coercive force. Therefore, there is no point outside the saturated hysteresis loop that can be on a B - H diagram, but all points inside the loop are possible.

In soft materials which can be easily magnetized, the hysteresis loop is thin with a small area enclosed by the loop. The hysteresis loop of hard magnetic materials is wide, compared to soft magnetic materials, with a great area enclosed within the loop.

The permeability $\mu = \mu(1 + \chi_m)$ is constant independent of the magnitude of $\mathbf{H}$ in a linear isotropic material. In a non-homogeneous linear isotropic material the permeability is a function of position, $\mu = \mu(\mathbf{r})$.

For diamagnetic and paramagnetic materials with relative permeability $\mu_r = 1 + \chi_m$ slightly differing from one, it is more suitable to give data for their magnetic susceptibility $\chi_m = \mu_r - 1$.

1.4.10.5. Ferromagnetic materials

The constitutive relationship which connects $\mathbf{B}$ and $\mathbf{H}$ for ferromagnetic materials is non-linear, and the hysteresis loop shown in Figure 1.4.5. is typical for such a material. Therefore, there is no unique value for μ , or μ_r for ferromagnetic materials. Accordingly, we shall focus our attention on the definition of the notion of permeability for non-linear ferromagnetic materials.

Let us take a look at the hysteresis loop shown in Figure 1.4.6.a) again. The corresponding diagram for $\mu = \mu(\mathbf{H})$ is shown in Figure 1.4.6.b). It is obvious that if $\mathbf{H} = 0$, μ becomes infinite. When $\mathbf{B} = 0$ then $\mu = 0$, and under such conditions the concept of permeability loses its sense. Therefore, *usage of μ has to be associated with a state in which it makes sense, meaning with a curve of first magnetization.*

One should note that the notion of maximum permeability denotes a specific maximum permeability, and that it is associated with a curve of first magnetization, and not to a hysteresis loop, or any other magnetization curve for that matter.

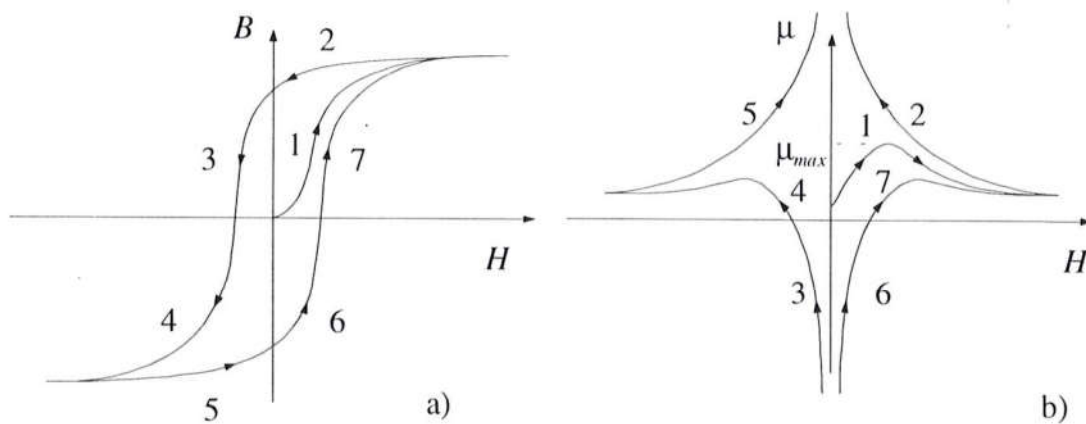


Figure 1.4.6. a) Hysteresis loop b) Curves of permeability

The second type of magnetization curve with which the permeability μ can be associated is a *normal magnetization curve*. This curve is the locus of points in a series of hysteresis loops obtained by cyclic permutations of $\mathbf{H}$ via consecutive small shifts. This is shown in Figure 1.4.7.

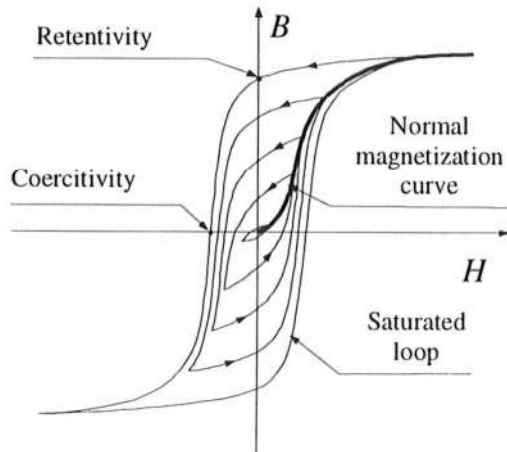


Figure 1.4.7. Normal magnetization curve and hysteresis loops

This curve is useful because it is recurring, and it is characteristic for specific types of ferromagnetic materials. *A normal magnetization curve is very similar to the curve of first magnetization in its shape.*

After this short explanation of permeability, we can say that the constitutive relationship (1.4.39) can be used for fields in non-linear ferromagnetic materials as well, only with $\mu = \mu(H)$.

1.4.10.6. Permanent magnets

Materials suitable for permanent magnets are characterized by quantities that are determined from part of the hysteresis loop in the second quadrant of the B - H diagram, as shown in Figure 1.4.7.

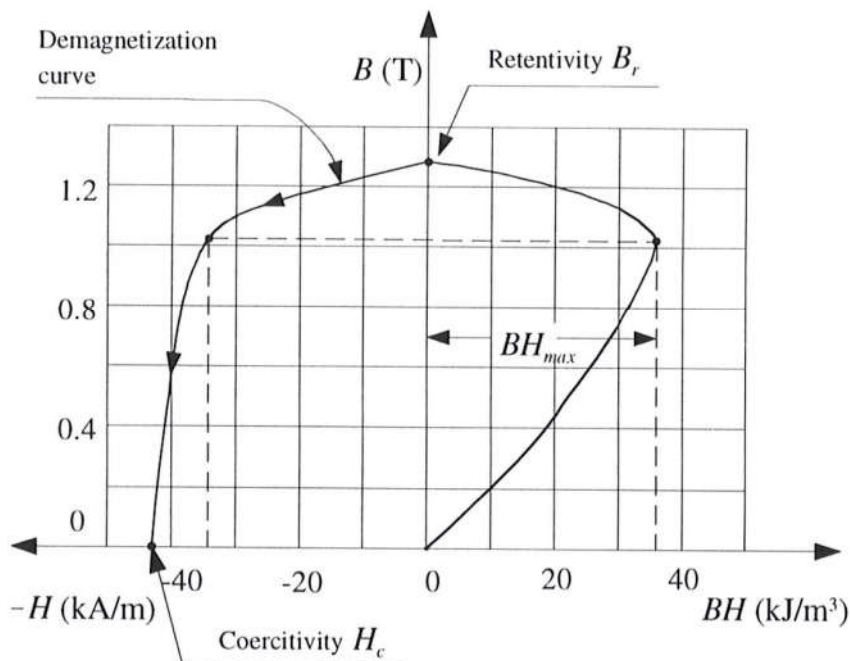


Figure 1.4.7. Curves of the demagnetization and of the product (BH)

1.5. MAXWELL'S EQUATIONS FOR FIELDS IN A MATERIAL

In this unit we shall extend Maxwell's equations in such a way as to incorporate the behaviour of electromagnetic fields in a material at rest. On the basis of analysis carried out in Unit 1.4. we have accepted the point of view that a material is basically an aggregate of elementary charge and currents, merged together in free space.

Accordingly, we postulate that electromagnetic fields in the presence of materials are governed by the fundamental laws of fields in free space, extended by the inclusion of equivalent macroscopic distributions of charge and currents within a material as if these sources were placed in free space.

We have showed that the macroscopic effects of a polarized material can be expressed by way of the vector of polarization density $\mathbf{P}$, and that space and time changes of $\mathbf{P}$ are equivalent to a distribution of polarized charge density $\rho_p = -\nabla \cdot \mathbf{P}$ and polarized current density $\mathbf{J}_p = \partial \mathbf{P} / \partial t$. The fundamental field laws in dielectric materials can be taken as being the field laws in free space with sources ρ_p and $\mathbf{J}_p$ added to free charge ρ_s and currents $\mathbf{J}_s$.

Similarly, we can express the macroscopic effects of a magnetized material by way of the vector of magnetization density $\mathbf{M}$. In Sections 1.4.7. and 1.4.8, we showed that there exist two different models for the representation of equivalent magnetic source densities in a material based on essentially different physical explanations of the nature of magnetic dipole moments.

The first one is the *model of the amperian current*, which is based on the universally accepted physical model of the magnetic dipole as an elementary current loop. An equivalent macroscopic density of magnetic source is in the form of an amperian current density $\mathbf{J}_a = \nabla \times \mathbf{M}$. The fundamental laws for fields in magnetic materials then follow from the laws of free space, with the addition of $\mathbf{J}_a$ to the free current density $\mathbf{J}_s$.

The second way of representing magnetization is based on a *magnetic charge model*. Although the existence of magnetic charge has not yet been proved, it is not contrary to the fundamental laws to use a model with a pair of opposite magnetic charges inside a material as the representation of magnetic dipole moments. We can use the idea of a magnetic charge only as a mathematical notion analogous to polarized electric charges substituting for equivalent magnetic dipole moments. The corresponding densities of magnetic sources can be represented by a magnetic charge density $\rho_m = -\nabla \cdot (\mu_0 \mathbf{M})$ and magnetic current density $\mathbf{J}_m = \partial (\mu_0 \mathbf{M}) / \partial t$. Macroscopic laws for fields in magnetic materials are obtained by the inclusion of sources ρ_m and $\mathbf{J}_m$ into the field equations together with free sources ρ_s and $\mathbf{J}_s$.

Both approaches to field equations in magnetic materials reduce to the same equations in free space. Although in these two models different quantities for the representation of magnetic field strength inside the material ($\mathbf{H}_a$ for the model with amperian current and $\mathbf{H}$ for the model with magnetic charge) arise, they differ only in the magnitude of $\mathbf{M}$, $\mathbf{H}_a = \mathbf{H} + \mathbf{M}$.

Complete parity between these two approaches is obtained if we define the magnetic flux density $\mathbf{B}$ as $\mathbf{B} = \mu_0 \mathbf{H}_a$ in the first approach, and as $\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M})$ in the second

approach. As a consequence, both approaches reduce to a single set of laws for the macroscopic electromagnetic field inside a materials if we express them in terms of four field vectors $\mathbf{E}$, $\mathbf{B}$, $\mathbf{D}$ and $\mathbf{H}$.

1.5.1. Maxwell's equations for the model using amperian current

To set up the problem, we shall prescribe the existence of free charge ρ_s and currents $\mathbf{J}_s$ that satisfy the continuity equation $\nabla \cdot \mathbf{J}_s = -\partial \rho_s / \partial t$ in a given space with volume V filled with a homogeneous isotropic material of known characteristics. The existence of polarized charge $\rho_p = -\nabla \cdot \mathbf{P}$ and currents $\mathbf{J}_p = \partial \mathbf{P} / \partial t$ in V which satisfy the equation of conservation of polarized charge $\nabla \cdot \mathbf{J}_p = -\partial \rho_p / \partial t$ is prescribed too. Finally, in the given volume V , bound amperian currents with density $\mathbf{J}_a = \nabla \times \mathbf{M}$ exist, which by definition possess no sources. They satisfy the continuity equation $\nabla \cdot \mathbf{J}_a = 0$.

1.5.1.1. Equations

On the basis of Units 1.3. and 1.4, we shall postulate that the vectors of electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ at every point of space in volume V filled with the homogeneous isotropic material, are the subject of Maxwell's equations

$$\begin{aligned}\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} , \\ \nabla \times \mathbf{B} &= \mu_0 \mathbf{J}_u = \mu_0 \left(\mathbf{J}_s + \mathbf{J}_p + \mathbf{J}_a + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) = \mu_0 \left(\mathbf{J}_s + \frac{\partial \mathbf{P}}{\partial t} + \nabla \times \mathbf{M} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) , \\ \nabla \cdot \mathbf{E} &= \frac{1}{\epsilon_0} \rho_u = \frac{1}{\epsilon_0} (\rho_s + \rho_p) = \frac{1}{\epsilon_0} (\rho_s - \nabla \cdot \mathbf{P}) , \\ \nabla \cdot \mathbf{B} &= 0 .\end{aligned}\tag{1.5.1a - d}$$

The vectors $\mathbf{D}$, $\mathbf{B}$ and $\mathbf{J}_s$ have to satisfy the defining and constitutive relationships in the material that connects them with $\mathbf{E}$ and $\mathbf{H}_a$, and with $\mathbf{H}$. Therefore, in V we get

$$\begin{aligned}\mathbf{D} &= \epsilon_0 \mathbf{E} + \mathbf{P} , \\ \mathbf{B} &= \mu_0 \mathbf{H}_a = \mu_0 (\mathbf{H} + \mathbf{M}) , \\ \mathbf{J}_s &= \kappa \mathbf{E} .\end{aligned}\tag{1.5.2a - c}$$

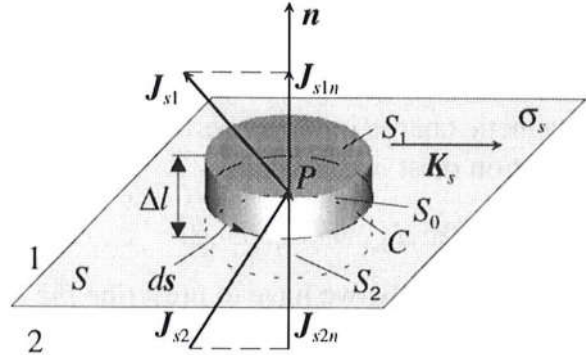
1.5.1.2. Boundary conditions

From analysis presented in Section 1.2.5, we know that discontinuities in the components of the field vectors occur at points on the boundary surface S between two homogeneous isotropic materials. The analysis in Unit 1.4. showed that on the boundary surfaces between material media, singular surface densities of equivalent bound charge and cur-

rents arise.

Let us prescribe the existence of free surface charge with density σ_s , and free surface currents with density $\mathbf{K}_s$, on the boundary surface S between two different materials, as shown in Figure 1.5.1. We shall prove that on the boundary surface S , the law of conservation of free charge should be satisfied.

Figure 1.5.1. With proof of conservation of free charge on the boundary surface S between two materials



Let us apply the continuity equation in integral form to an elementary cylinder with basis S_1 and S_2 of area ΔS , and height Δl , as shown in Figure 1.5.1. The volume of the cylinder is ΔV . If the height of the cylinder tends to zero $\Delta l \rightarrow 0$, then the cylinder degenerates into a surface S_0 of area ΔS on the boundary surface S , and we thus get

$$\lim_{\Delta l \rightarrow 0} \oint_S \mathbf{J} \cdot \mathbf{n} dS = - \lim_{\Delta l \rightarrow 0} \left(\frac{d}{dt} \int_{\Delta V} \rho_s dV \right),$$

which results in

$$\mathbf{n} \cdot (\mathbf{J}_{s1} - \mathbf{J}_{s2}) \Delta S + \oint_C \mathbf{K}_s \cdot (d\mathbf{s} \times \mathbf{n}) = - \frac{\partial \sigma_s}{\partial t} \Delta S.$$

If we introduce a two-dimensional divergence operator ∇_s defined by

$$\lim_{\Delta S \rightarrow 0} \frac{\oint_C \mathbf{K}_s \cdot (d\mathbf{s} \times \mathbf{n})}{\Delta S} = \nabla_s \cdot \mathbf{K}_s, \quad (1.5.3)$$

we can finally express the law of the conservation of free charge on a boundary surface in the form

$$\mathbf{n} \cdot (\mathbf{J}_{s1} - \mathbf{J}_{s2}) + \nabla_s \cdot \mathbf{K}_s = - \frac{\partial \sigma_s}{\partial t}. \quad (1.5.4)$$

Let us prescribe, using equations (1.2.24) and (1.4.13), the existence of bound surface polarized charge on the boundary surface S between two materials of density

$$\sigma_p = -\mathbf{n} \cdot (\mathbf{P}_1 - \mathbf{P}_2) \quad \text{on } S, \quad (1.5.5)$$

which satisfies the continuity equation on the boundary surface S

$$\mathbf{n} \cdot (\mathbf{J}_{p1} - \mathbf{J}_{p2}) = -\frac{\partial \sigma_p}{\partial t} . \quad (1.5.6)$$

Similarly, with the help of equations (1.2.25) and (1.4.18), we can prescribe the existence of bound surface amperian currents of magnetization of density

$$\mathbf{K}_a = \mathbf{n} \times (\mathbf{M}_1 - \mathbf{M}_2) , \quad (1.5.7)$$

on the boundary surface S between two homogeneous materials. Since there is no surface magnetic charge in the amperian current model, on the boundary surface S the continuity equation must be satisfied:

$$\mathbf{n} \cdot (\mathbf{J}_{a1} - \mathbf{J}_{a2}) = 0 . \quad (1.5.8)$$

And finally, we have to prescribe *the boundary conditions* for normal and tangential components of the field vectors $\mathbf{E}$ and $\mathbf{B}$, using equations (1.2.23) and (1.2.24)

$$\begin{aligned} \mathbf{n} \times (\mathbf{E}_1 - \mathbf{E}_2) &= 0 , \\ \mathbf{n} \times \frac{1}{\mu_0} (\mathbf{B}_1 - \mathbf{B}_2) &= \mathbf{K}_s + \mathbf{K}_a = \mathbf{K}_s + \mathbf{n} \times (\mathbf{M}_1 - \mathbf{M}_2) , \\ \mathbf{n} \cdot \epsilon_0 (\mathbf{E}_1 - \mathbf{E}_2) &= \sigma_s + \sigma_p = \sigma_s - \mathbf{n} \cdot (\mathbf{P}_1 - \mathbf{P}_2) , \\ \mathbf{n} \cdot (\mathbf{B}_1 - \mathbf{B}_2) &= 0 . \end{aligned} \quad (1.5.9a - d)$$

1.5.2. Maxwell's equations for a model with magnetic charge

We shall prescribe the existence of free charge ρ_s and currents $\mathbf{J}_s$ in a space of volume V filled by a homogeneous isotropic material with known characteristics. The prescribed free charge and currents must satisfy the continuity equation. Bound polarized charge with volume density $\rho_p = -\nabla \cdot \mathbf{P}$ and polarized currents with density $\mathbf{J}_p = \partial \mathbf{P} / \partial t$, exist there too. And these bound sources must satisfy the continuity equation. Finally, in the model of a magnetic material with magnetic charge we prescribe the existence of (fictional) bound magnetic charge and currents defined by equations $\rho_m = -\nabla \cdot (\mu_0 \mathbf{M})$ and $\mathbf{J}_m = \partial (\mu_0 \mathbf{M}) / \partial t$. These sources must satisfy the continuity equation as well.

1.5.2.1. Equations

As in the previous unit, we shall postulate here that the vectors of electric and magnetic field strength $\mathbf{E}$ and $\mathbf{H}$ at every point in a space of volume V , are the subjects of Maxwell's equations

$$\begin{aligned}
\nabla \times \mathbf{E} &= -\mathbf{J}_m - \frac{\partial}{\partial t}(\mu_0 \mathbf{H}) = -\frac{\partial}{\partial t}(\mu_0 \mathbf{M}) - \frac{\partial}{\partial t}(\mu_0 \mathbf{H}) , \\
\nabla \times \mathbf{H} &= \mathbf{J}_s + \mathbf{J}_p + \frac{\partial}{\partial t}(\epsilon_0 \mathbf{E}) = \mathbf{J}_s + \frac{\partial \mathbf{P}}{\partial t} + \frac{\partial}{\partial t}(\epsilon_0 \mathbf{E}) , \\
\nabla \cdot \mathbf{E} &= \frac{1}{\epsilon_0}(\rho_s + \rho_p) = \frac{1}{\epsilon_0}(\rho_s - \nabla \cdot \mathbf{P}) , \\
\nabla \cdot \mathbf{H} &= \frac{1}{\mu_0} \rho_m = -\nabla \cdot \mathbf{M} .
\end{aligned}
\tag{1.5.10a - d}$$

When there are no free charge and currents ($\rho_s = 0$, $\mathbf{J}_s = 0$), equations (1.5.10a - d) are anti-symmetric. The constitutive relationships are

$$\begin{aligned}
\mathbf{D} &= \epsilon_0 \mathbf{E} + \mathbf{P} , \\
\mathbf{B} &= \mu_0 (\mathbf{H} + \mathbf{M}) , \\
\mathbf{J}_s &= \kappa \mathbf{E} .
\end{aligned}
\tag{1.5.11a - c}$$

1.5.2.2. Boundary conditions

On the boundary surface S that separates two homogeneous isotropic materials, we shall prescribe the existence of free surface charge and currents σ_s and $\mathbf{K}_s$, which satisfy the continuity equation on the surface S . A bound polarized charge of density $\sigma_p = -\mathbf{n} \cdot (\mathbf{P}_1 - \mathbf{P}_2)$ defined by equation (1.5.5) which satisfies the continuity equation (1.5.6) will be prevalent on the boundary surface. For models using magnetic charge we can, on the basis of equation (1.4.23), prescribe the existence of a bound surface charge of magnetization of density

$$\sigma_m = -\mathbf{n} \cdot \mu_0 (\mathbf{M}_1 - \mathbf{M}_2) , \tag{1.5.12}$$

on the boundary surface S between two homogeneous isotropic materials. This charge, analogous to equation (1.4.24), must satisfy the continuity equation. Therefore, on the boundary surface S between two materials we get

$$\mathbf{n} \cdot (\mathbf{J}_{m1} - \mathbf{J}_{m2}) = -\frac{\partial \sigma_m}{\partial t} . \tag{1.5.13}$$

Finally, we need to prescribe the boundary conditions for normal and tangential components of the vectors of electric and magnetic field strength $\mathbf{E}$ and $\mathbf{H}$, using equations (1.2.23) and (1.2.24). These conditions should hold on the boundary surface S between two homogeneous and isotropic materials

$$\begin{aligned}
\mathbf{n} \times (\mathbf{E}_1 - \mathbf{E}_2) &= 0 , \\
\mathbf{n} \times (\mathbf{H}_1 - \mathbf{H}_2) &= \mathbf{K}_s , \\
\mathbf{n} \cdot \epsilon_0 (\mathbf{E}_1 - \mathbf{E}_2) &= \sigma_s + \sigma_p = \sigma_s - \mathbf{n} \cdot (\mathbf{P}_1 - \mathbf{P}_2) , \\
\mathbf{n} \cdot \mu_0 (\mathbf{H}_1 - \mathbf{H}_2) &= \sigma_m = -\mathbf{n} \cdot \mu_0 (\mathbf{M}_1 - \mathbf{M}_2) .
\end{aligned} \tag{1.5.14a - d}$$

1.5.3. Maxwell's equations for field vectors $\mathbf{E}$, $\mathbf{B}$, $\mathbf{D}$ and $\mathbf{H}$

The dependent equivalent source densities ρ_p , $\mathbf{J}_p$, $\mathbf{J}_a$, ρ_m and $\mathbf{J}_m$ are not explicitly included in Maxwell's original formulation of his electromagnetic theory. In Section 1.4.9 we demonstrated the importance of the introduction of two new vector fields $\mathbf{D}$ and $\mathbf{H}$ and their addition to Maxwell's formulation. By the addition of these two new vectors defined by equations (1.4.25) and (1.4.34) into the fundamental field laws that were expressed by sets of Maxwell's equations (1.5.1a - d) and (1.5.10a - d), we shall develop a much simpler form of Maxwell's equations. They will enable the solution of electromagnetic fields in materials, without explicit involvement of various phenomena that take place in the material. The expression of Maxwell's equations in the form

$$\begin{aligned}
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} , \\
\nabla \times \mathbf{H} &= \mathbf{J}_s + \frac{\partial \mathbf{D}}{\partial t} , \\
\nabla \cdot \mathbf{D} &= \rho_s , \\
\nabla \cdot \mathbf{B} &= 0 ,
\end{aligned} \tag{1.5.15a - d}$$

is the basis for a macroscopic description of electromagnetic fields in a material at rest. It should be noted that in *naturally-occurring problems*, the free charge ρ_s and currents $\mathbf{J}_s$ must satisfy the continuity equation

$$\nabla \cdot \mathbf{J}_s = -\frac{\partial \rho_s}{\partial t} . \tag{1.5.15c}$$

The corresponding boundary conditions on a boundary surface S between two homogeneous isotropic materials, according to equations (1.2.23) and (1.2.24), may be simply expressed as

$$\begin{aligned}
\mathbf{n} \times (\mathbf{E}_1 - \mathbf{E}_2) &= 0 , \\
\mathbf{n} \times (\mathbf{H}_1 - \mathbf{H}_2) &= \mathbf{K}_s , \\
\mathbf{n} \cdot (\mathbf{D}_1 - \mathbf{D}_2) &= \sigma_s , \\
\mathbf{n} \cdot (\mathbf{B}_1 - \mathbf{B}_2) &= 0 .
\end{aligned} \tag{1.5.16a - d}$$

An important characteristic of the sets of equations (1.5.15a – d) and (1.5.16a – d) is that they are completely free of polarized and magnetized sources, which means that there are no dependent quantities: $\mathbf{P}$, $\mathbf{M}$, ρ_p , $\mathbf{J}_p$, ρ_m , $\mathbf{J}_m$ and $\mathbf{J}_a$. In fact, by introducing the permittivity ϵ , permeability μ and conductivity κ as physical macroscopic quantities that can be determined by measurements, it is possible to express Maxwell's equations in a form that directly connects them with experimental observations and thus enables their experimental approval.

There is a conspicuous similarity of these laws for fields in a material with the original laws for fields in free space. The only difference in their form is that for materials, we introduce the permittivity ϵ and permeability μ instead of the corresponding values ϵ_0 and μ_0 for free space. The constitutive relationships for homogeneous isotropic materials, according to equations (1.2.7), are

$$\begin{aligned}\mathbf{D} &= \epsilon \mathbf{E} , \\ \mathbf{B} &= \mu \mathbf{H} , \\ \mathbf{J}_s &= \kappa \mathbf{E} .\end{aligned}\tag{1.5.17a – c}$$

Maxwell's equations (1.5.15a – d), together with the constitutive relationships (1.5.17a – c), form a complete system of equations which uniquely define electromagnetic fields, meaning the vectors $\mathbf{E}$, $\mathbf{B}$, $\mathbf{D}$ and $\mathbf{H}$, for prescribed sources ρ_s and $\mathbf{J}_s$ in a material with experimentally determined values of ϵ , μ and κ , with given boundary and initial conditions.

Let us mention that vectors $\mathbf{E}$ and $\mathbf{B}$ are defined by the Lorentz force law (1.1.1), and that equations (1.4.25) and (1.4.34) define vectors $\mathbf{D}$ and $\mathbf{H}$. These three definition equations, together with Maxwell's equations and constitutive relationships, form a basis for the study of electromagnetic fields in materials.

It is sufficient only to prescribe the free charge and currents ρ_s and $\mathbf{J}_s$, which we often call real charge and currents. Only these sources determine the fields characterized by field strength vectors $\mathbf{E}$ and $\mathbf{H}$ and the corresponding flux density vectors $\mathbf{D}$ and $\mathbf{B}$.

In Unit 1.3, we explained the notion of displacement currents in free space, as introduced by Maxwell and represented by the term $\epsilon_0 \partial \mathbf{E} / \partial t$ in equation (1.3.8b). We need to extend the notion of displacement currents to materials. From different forms of Maxwell's equations (1.5.1a – d), (1.5.10a – d) and (1.5.15a – d) for fields in a material, it follows that the displacement current density is given by

$$\frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_p + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \frac{\partial \mathbf{P}}{\partial t} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} ,\tag{1.5.21}$$

and is measured in *amperes per square metre* (A/m^2). Since in free space $\mathbf{P} = 0$ and $\mathbf{J}_p = 0$, the displacement current density is simply $\partial \mathbf{D} / \partial t = \epsilon_0 \partial \mathbf{E} / \partial t$. We can say that the total electric current $\mathbf{J}_u$, which represents the source of magnetic field $\mathbf{B}$ consists of $\mathbf{J}_u = \mathbf{J}_s + \mathbf{J}_p + \mathbf{J}_a + \epsilon_0 \partial \mathbf{E} / \partial t = \mathbf{J}_s + \mathbf{J}_a + \partial \mathbf{D} / \partial t$.

PROBLEMS

- 1.1. Set up operational definitions of the field quantities $\mathbf{E}$ and $\mathbf{B}$ using the Lorentz force law!
(The word “operational” means that the definition prescribes the way of measuring the quantity and assures a unique connection between theory and reality.)
- 1.2. Prove the direct proportionality between the convection current density $\mathbf{J}$ and charge density ρ !
- 1.3. Express Ohm’s law (1.4.1) in integral form for use in circuit theory!
- 1.4. Prove the relationship between the density of amperian current and magnetization $\mathbf{J}_a = \nabla \times \mathbf{M}$!
- 1.5. Show that inside any homogeneous material, no free electric charge may exist!
- 1.6. Prove that bound polarized charge cannot exist inside a homogeneous material!

2. ELECTROMAGNETIC POTENTIALS

In the previous chapter we described the different forms of Maxwell's equations for solving electromagnetic fields. We represented the fields using principal field vectors $\mathbf{E}$ and $\mathbf{B}$, and subsidiary field vectors $\mathbf{D}$ and $\mathbf{H}$. In this chapter we shall introduce new supplementary functions known as *electromagnetic potentials* to represent the field. These mathematical functions, called vector and scalar potentials, will be introduced in accordance with Maxwell's equations. In addition, the usage of potentials will significantly simplify and thus make easier the calculation and analysis of electromagnetic fields.

2.1. VECTOR AND SCALAR POTENTIAL FUNCTIONS

Generally speaking, we are interested in finding *supplementary functions* that make the representation and calculation of electromagnetic fields easier. The various forms of the system of Maxwell's equations for fields in free space and in a material at rest are in fact sets of mutually related first order partial differential equations. Therefore, in order to calculate an electromagnetic field, it is necessary to solve a complete set of equations simultaneously. The simultaneous solution of such large systems of equations is a very difficult, and sometimes unsolvable, problem.

Consequently, the need for introducing a new class of function for representing electromagnetic fields is obvious. In particular, we want to replace the vector functions of field strength $\mathbf{E}$ and $\mathbf{H}$, and inductions $\mathbf{B}$ and $\mathbf{D}$. The aim of the defining and introduction of new potential functions into electromagnetic field theory is to significantly lower the number of equations that have to be solved in the mathematical model of the field, and to uncouple equations in the system to make possible their separate solution. In this unit, we shall reach this aim by introducing two classes of potential functions known as *potentials* and *antipotentials*.

2.1.1. Potentials and antipotentials in a source-free dielectric

We shall start the study of potentials by the analysis of an electromagnetic field in an *unbounded space* filled with a *material at rest*. There is no free charge and currents within this material ($\rho_s = 0$, $\mathbf{J}_s = 0$). We assume that the material is linear, isotropic and homogeneous and lossless. Thus, it is a dielectric with constant constitutive parameters ϵ and μ (and with $\kappa = 0$). The behaviour of an electromagnetic field in such a material is determined by the homogeneous part of Maxwell's equation system (1.5.15a – d)

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 ,$$

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = 0 , \quad (2.1.1a - d)$$

$$\nabla \cdot \mathbf{D} = 0 ,$$

$$\nabla \cdot \mathbf{B} = 0 .$$

2.1.1.1. Vector and scalar potentials

According to equation (2.1.1d), a magnetic field is always *source-free* (solenoidal). Therefore, one can represent the vector $\mathbf{B}$ as the curl of another vector $\mathbf{A}$

$$\mathbf{B} = \nabla \times \mathbf{A} . \quad (2.1.2)$$

The vector field $\mathbf{A}$, being a vector potential function of an electromagnetic field, is called the *magnetic vector potential*. The vector identity $\nabla \cdot \nabla \times \mathbf{A} \equiv 0$ ensures the validity of the subsidiary Maxwell's equation (2.1.1d), and if we substitute for $\mathbf{B}$ in equation (2.1.1a) with $\mathbf{B} = \nabla \times \mathbf{A}$, we obtain

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0 .$$

This expression shows that vector $(\mathbf{E} + \partial \mathbf{A} / \partial t)$ is an *irrotational* (conservative) vector field. Using the vector identity $\nabla \times (\nabla \phi) \equiv 0$, where $\phi = \phi(\mathbf{r}, t)$ is a scalar function, one may always express it as the gradient of the scalar function ϕ , and we can thus write

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \phi .$$

The negative sign for the gradient is a result of the definition of the potential ϕ . Conventionally, we desire that, in a static field, vector $\mathbf{E}$ points from a greater to a lower potential. According to this equation, we can express an electric field $\mathbf{E}$ with the help of the *magnetic vector potential* $\mathbf{A}$ and *electric scalar potential* ϕ by the expression

$$\mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t} , \quad (2.1.3)$$

which directly satisfies equation (2.1.1a). According to equations (2.1.2) and (2.1.3), the unit for vector potential $\mathbf{A}$ is *weber per metre* (Wb/m), and the unit for scalar potential ϕ is *volt* (V).

Equations (2.1.2) and (2.1.3) yield fields $\mathbf{E}$ and $\mathbf{B}$ expressed in terms of two potentials ϕ and $\mathbf{A}$ in this system. These potentials are two completely arbitrary and independent functions of spatial coordinates and time.

Field vectors $\mathbf{B}$ and $\mathbf{E}$ are connected to these potentials via equations (2.1.2) and (2.1.3), and we calculate them using space and time derivatives of these potentials. Therefore, we can conclude that at every point of space where $\mathbf{E}$ and $\mathbf{B}$ are finite, potentials ϕ

and $\mathbf{A}$ must be continuous. This condition makes them more suitable for calculating the fields than quantities $\mathbf{E}$ and $\mathbf{B}$, which can be discontinuous on the boundaries between different materials and at places where the densities of charge and current are singular.

2.1.1.2. Wave equations

The wave equations for ϕ and $\mathbf{A}$ follow from the substitution of equations (2.1.2) and (2.1.3) into equation (2.1.1b) multiplied by μ , and into equation (2.1.1c) divided by ϵ . So we get

$$\begin{aligned}\Delta\mathbf{A} - \mu\epsilon\frac{\partial^2\mathbf{A}}{\partial t^2} &= \nabla\left(\nabla\cdot\mathbf{A} + \mu\epsilon\frac{\partial\phi}{\partial t}\right), \\ \Delta\phi + \frac{\partial}{\partial t}(\nabla\cdot\mathbf{A}) &= 0,\end{aligned}\tag{2.1.4a,b}$$

which are a pair of coupled linear second-order differential equations for the potentials ϕ and $\mathbf{A}$, which can be uncoupled by a suitable choice of potentials. The scalar Laplacian operator has been introduced into equations (2.1.4a,b) as $\Delta\phi \equiv \nabla\cdot(\nabla\phi)$, and the vector Laplacian operator is determined from the vector identity $\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla\cdot\mathbf{A}) - \Delta\mathbf{A}$.

In Unit 2.2, we shall show that vector and scalar potentials are not unique for one electromagnetic field. On the other hand, all partial solutions of the system of equations (2.1.4a,b) for ϕ and $\mathbf{A}$, subjected to the same boundary and initial conditions, result in the same vectors of electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$. These partial solutions for potentials will differ for an arbitrary function Ψ .

This liberty in choice of potentials allows us to prescribe an additional condition that both the vector and scalar potentials should fulfil besides the condition that they must exactly determine the desired field vectors $\mathbf{E}$ and $\mathbf{B}$. This *additional condition* involves the prescribing of the divergence of the vector potential. Such prescription is always possible because the curl and divergence of any vector field (such as $\mathbf{A}$) are totally independent characteristics of a field. So far we have defined only $\nabla \times \mathbf{A}$ using equation (2.1.2).

The additional condition is generally chosen in such a way as to uncouple and simplify equations (2.1.4a,b), which have to be solved in order to calculate the potentials. In non-conductive materials (dielectrics) it can be determined that we can uncouple potentials ϕ and $\mathbf{A}$ in equations (2.1.4a,b) if we prescribe that the divergence of $\mathbf{A}$ satisfies

$$\nabla\cdot\mathbf{A} + \mu\epsilon\frac{\partial\phi}{\partial t} = 0.\tag{2.1.5}$$

This condition is known as the *Lorentz condition*. The substitution of the condition (2.1.5) into equations (2.1.4a,b) results in symmetrical wave equations

$$\begin{aligned}\Delta\mathbf{A} - \mu\epsilon\frac{\partial^2\mathbf{A}}{\partial t^2} &= 0, \\ \Delta\phi - \mu\epsilon\frac{\partial^2\phi}{\partial t^2} &= 0.\end{aligned}\tag{2.1.6a,b}$$

The Lorentz condition enables the expression of the fundamental laws of electromagnetic fields in a lossless material with constitutive parameters ϵ and μ ($\kappa = 0$) in the form of a similar pair of uncoupled, linear, differential, homogeneous second-order wave equations for vector and scalar potential $\mathbf{A}$ and ϕ .

2.1.1.3. Antipotentials

The expressions (2.1.2) and (2.1.3) for vectors $\mathbf{B}$ and $\mathbf{E}$ are not general, but only partial solutions of the system of homogeneous equations (2.1.1a – d). As this system is antisymmetrical, it can be identically satisfied by new solutions

$$\begin{aligned} \mathbf{D} &= -\nabla \times \mathbf{A}^* , \\ \mathbf{H} &= -\nabla \phi^* - \frac{\partial \mathbf{A}^*}{\partial t} , \end{aligned} \quad (2.1.7a,b)$$

and we can obtain the fields $\mathbf{E}$ and $\mathbf{B}$ with the help of new potentials, which we shall call *antipotentials*. Fields $\mathbf{E}$ and $\mathbf{B}$ are determined by antipotentials using the expressions

$$\begin{aligned} \mathbf{E} &= -\frac{1}{\epsilon} \nabla \times \mathbf{A}^* , \\ \mathbf{B} &= -\mu \left(\nabla \phi^* + \frac{\partial \mathbf{A}^*}{\partial t} \right) . \end{aligned} \quad (2.1.7c,d)$$

Everything we have said about the potentials ϕ and $\mathbf{A}$ is valid for antipotentials ϕ^* and $\mathbf{A}^*$ as well. The antipotentials are the subjects of the following expressions

$$\begin{aligned} \Delta \mathbf{A}^* - \mu \epsilon \frac{\partial^2 \mathbf{A}^*}{\partial t^2} &= 0 , \\ \Delta \phi^* - \mu \epsilon \frac{\partial^2 \phi^*}{\partial t^2} &= 0 , \\ \nabla \cdot \mathbf{A}^* + \mu \epsilon \frac{\partial \phi^*}{\partial t} &= 0 . \end{aligned} \quad (2.1.8a - c)$$

We determine antipotentials ϕ^* and $\mathbf{A}^*$ by solving of the pair of homogeneous wave equations (2.1.8a,b) with corresponding boundary and initial conditions. The unit of electric vector potential $\mathbf{A}^*$ is *coulomb per metre* (C/m), and the unit of magnetic scalar potential ϕ^* is *the ampere* (A).

Finally, we can write the general solution of the system of Maxwell's equations (2.1.1.a – d) for a field in a linear homogeneous dielectric in the absence of free charge and currents, expressed using potentials and antipotentials, as

$$\begin{aligned}
\mathbf{B} &= \nabla \times \mathbf{A} - \mu \frac{\partial \mathbf{A}^*}{\partial t} - \mu \nabla \varphi^* , \\
\mathbf{E} &= -\nabla \varphi - \frac{\partial \mathbf{A}}{\partial t} - \frac{1}{\epsilon} \nabla \times \mathbf{A}^* .
\end{aligned}
\tag{2.1.9a,b}$$

These general solutions of an electromagnetic field are determined by the pair of vectors $\mathbf{E}$ and $\mathbf{B}$, which are determined from equations (2.1.6a,b) and (2.1.8a,b) in terms of potentials and antipotentials.

We can explain functions φ and $\mathbf{A}$ as potentials corresponding to free electric charge and currents distributed outside the region of interest, and functions φ^* and $\mathbf{A}^*$ as potentials (antipotentials) corresponding to equivalent (fictitious) magnetic charge and currents distributed outside the region of interest.

It is usual that, whenever possible, we choose φ^* and $\mathbf{A}^*$ to be equal to zero, and thus represent the field with only one pair of potentials φ and $\mathbf{A}$ using equations (2.1.2) and (2.1.3). In some cases it is convenient to use the pair of antipotentials φ^* and $\mathbf{A}^*$, in others to use only one of them. This is the case for quasi-static and static magnetic fields.

2.1.2. Potentials in a conductive source-free material

In this section, we shall focus our attention on potentials in an unbounded space filled with a linear homogeneous isotropic conductive material. In such a material with given constitutive parameters ϵ , μ and κ , conductive currents may flow because the conductivity is not equal to zero. According to Ohm's law (1.2.6), conductive currents can flow in a material in which an electromagnetic field exists, even in the case where there is no free charge within the material.

If we assume the existence of some initial distribution of free charge ρ_{s0} , this will vanish exponentially with time with a time constant of charge relaxation $T = \epsilon/\kappa$. For good conductors the relaxation time is extremely short, and we can suppose that the density of free charge inside a conductor is always equal to zero ($\rho_s \equiv 0$).

The phenomenon of very quickly vanishing free charge inside a conductor and its transfer to the conductor's surface leads us to the introduction of the notion of *transient current*. So we can represent a free (real) current as if it were composed of two parts, one being stationary, the other being transient. A current is called stationary if there is no gathering of charge at any point of the field. The criterion that stationary current density $\mathbf{J}_{STAT}$ must satisfy is

$$\nabla \cdot \mathbf{J}_{STAT} = 0 , \tag{2.1.10}$$

The transient part of the real currents originates from the initial distribution of free charge, and the continuity equation $\nabla \cdot \mathbf{J}_{TRAN} = -\partial \rho_s / \partial t$ must be satisfied. Thus, in a conductive material the density of the real conductive current may be expressed as being

$$\mathbf{J}_s = \mathbf{J}_{TRAN} + \mathbf{J}_{STAT} = \mathbf{J}_{s0} e^{-t/T} + \kappa \mathbf{E} , \quad (2.1.11)$$

where $\mathbf{J}_{s0}$ is the density of the initial transient part of the conductive current that stems from the initial distribution of free charge. They are connected via equation

$$\nabla \cdot \mathbf{J}_{s0}(\mathbf{r}) = \frac{\rho_{s0}}{T} = \frac{\kappa}{\epsilon} \rho_{s0}(\mathbf{r}) . \quad (2.1.12)$$

Equation (2.1.11), according to the expression for relaxation of charge $\rho = \rho_{s0} e^{(-t/T)}$, shows that in good conductors, the transient part vanishes very quickly, and the total real current can be represented by the stationary part as $\mathbf{J}_s = \kappa \mathbf{E}$.

Therefore, in order to calculate fields in good conductors, we shall use the system of Maxwell's equation (1.5.15a – d), with substituted constitutive relationships (1.5.17a – c), along with the fact that $\rho_s = 0$. In such a way, we may obtain a system of homogeneous equations:

$$\begin{aligned} \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0 , & \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} - \kappa \mathbf{E} &= 0 , \\ \nabla \cdot \mathbf{D} &= 0 , & \nabla \cdot \mathbf{B} &= 0 . \end{aligned} \quad (2.1.13a - d)$$

In the first alternative, we introduce the potentials $\mathbf{A}$ and ϕ according to equations (2.1.2) and (2.1.3), meaning $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{E} = -\nabla \phi - \partial \mathbf{A} / \partial t$.

$$\begin{aligned} \Delta \mathbf{A} - \mu \epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} - \mu \kappa \frac{\partial \mathbf{A}}{\partial t} &= \nabla \left(\nabla \cdot \mathbf{A} + \mu \epsilon \frac{\partial \phi}{\partial t} + \mu \kappa \phi \right) , \\ \Delta \phi + \frac{\partial}{\partial t} (\nabla \cdot \mathbf{A}) &= 0 . \end{aligned} \quad (2.1.14a,b)$$

We define the Lorentz condition for a conductive material as

$$\nabla \cdot \mathbf{A} + \mu \epsilon \frac{\partial \phi}{\partial t} + \mu \kappa \phi = 0 , \quad (2.1.15)$$

and obtain wave equations for potentials $\mathbf{A}$ and ϕ

$$\begin{aligned} \Delta \mathbf{A} - \mu \epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} - \mu \kappa \frac{\partial \mathbf{A}}{\partial t} &= 0 , \\ \Delta \phi - \mu \epsilon \frac{\partial^2 \phi}{\partial t^2} - \mu \kappa \frac{\partial \phi}{\partial t} &= 0 . \end{aligned} \quad (2.1.16a,b)$$

This is a pair of uncoupled homogeneous dampened wave equations for potentials $\mathbf{A}$ and ϕ that represent the electromagnetic field in a lossy conductive material. They differ from the pair of equations (2.1.6a,b) because they possess the additional terms $\mu \kappa \partial \mathbf{A} / \partial t$ and $\mu \kappa \partial \phi / \partial t$. These terms represent the damping of the electromagnetic wave in a lossy material ($\kappa \neq 0$).

Both pairs of equations (2.1.6a,b) and (2.1.16a,b) show that an electromagnetic field in nature, as defined by the system of Maxwell's equations, possesses a wavelike character.

2.1.3. Potentials and antipotentials of a given distribution of independent sources

There are some cases in applied electromagnetic field theory where it is convenient to calculate fields and potentials from a given distribution of independent sources. By the term independent sources we mean sources with their distribution unaffected by fields governed by the system of Maxwell's equations that we are solving.

2.1.3.1. Conductive material

We shall assume that a field exists in an unbounded space filled with a linear homogeneous isotropic material with given constitutive parameters ϵ , μ and κ , and generally differing from zero. The distribution of independent sources is defined by a prescribed independent distribution of free charge, which we denote as ρ_s' to emphasize its independence from the calculated field. We shall assume that the density of free current $\mathbf{J}_s$ consists of two parts: one that is to be calculated and determined by Ohm's law as $\kappa\mathbf{E}$, and the other part denoted as $\mathbf{J}_s'$, independent of the field that is to be calculated. We can write it as $\mathbf{J}_s' = \kappa\mathbf{E}'$, where the electric field $\mathbf{E}'$ is not included in Maxwell's equations, contrary to the field $\mathbf{E}$, which is to be calculated from Maxwell's equations. In other words, $\mathbf{J}_s'$ is the part of the prescribed current caused by external electromotive forces. It does not contain any part of the current caused by the electric fields within the conductive material, and we get

$$\mathbf{J}_s = \kappa\mathbf{E} + \mathbf{J}_s' = \kappa(\mathbf{E} + \mathbf{E}') . \quad (2.1.17)$$

In many cases $\mathbf{J}_s'$ is an idealisation similar to current source in circuit theory. Now we need to introduce the Lorentz condition (2.1.15) for a conductive material into the system of Maxwell's equations (1.5.15a – d) together with given independent sources ρ_s' and $\mathbf{J}_s'$, and potentials $\mathbf{A}$ and ϕ determined by equations (2.1.2) and (2.1.3). Since we have defined $\mathbf{J}_s$ by (2.1.17) and that the remaining constitutive relationships are $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$, we obtain the wave equations in terms of potentials

$$\begin{aligned} \Delta\mathbf{A} - \mu\epsilon\frac{\partial^2\mathbf{A}}{\partial t^2} - \mu\kappa\frac{\partial\mathbf{A}}{\partial t} &= -\mu\mathbf{J}_s' , \\ \Delta\phi - \mu\epsilon\frac{\partial^2\phi}{\partial t^2} - \mu\kappa\frac{\partial\phi}{\partial t} &= -\frac{\rho_s'}{\epsilon} . \end{aligned} \quad (2.1.18a,b)$$

These are *non-homogeneous dampened wave equations*, with whose help we can calculate the potentials $\mathbf{A}$ and ϕ caused by independent sources given by their densities ρ_s' and $\mathbf{J}_s'$. Partial solutions of these equations can be expressed as integrals over given distributions of charge and currents, while their complementary solutions, which are solutions of the

homogeneous parts of the equations, are obviously wave-like. We should bear in mind that independent sources ρ_s' and $\mathbf{J}_s'$ are produced by external causes.

2.1.3.2. Non-conductive material

If an unbounded space is filled with a dielectric ($\kappa = 0$), it can be easily shown that in such a case we can calculate the potentials from the pair of *non-homogeneous wave equations*

$$\begin{aligned}\Delta A - \mu\epsilon \frac{\partial^2 A}{\partial t^2} &= -\mu \mathbf{J}_s' , \\ \Delta \varphi - \mu\epsilon \frac{\partial^2 \varphi}{\partial t^2} &= -\frac{\rho_s'}{\epsilon} ,\end{aligned}\tag{2.1.19a,b}$$

for a given density distribution of independent sources ρ_s' and $\mathbf{J}_s'$. It is useful to remember that the Lorentz condition for a dielectric is given by equation (2.1.5). We shall show that the continuity equation is included in the Lorentz condition (2.1.5).

$$\nabla \cdot \mathbf{A} = -\mu\epsilon \frac{\partial \varphi}{\partial t} .$$

If we apply a Laplacian operator Δ on both sides of this equation and substitute ΔA and $\Delta \varphi$ from equations (2.1.19a,b) we obtain

$$\mu\epsilon \frac{\partial^2}{\partial t^2} \left(\nabla \cdot \mathbf{A} + \mu\epsilon \frac{\partial \varphi}{\partial t} \right) = \mu \left(\nabla \cdot \mathbf{J}_s' + \frac{\partial \rho_s'}{\partial t} \right) .$$

Therefore, in applying the Lorentz condition (2.1.5), we must include that $\nabla \cdot \mathbf{J}_s' + \partial \rho_s' / \partial t = 0$, which is the continuity equation. Since the continuity equation must be satisfied in *naturally-occurring problems* of physical distributions of charge and current, it is convenient to use the Lorentz condition for the decoupling of the wave equations for $\mathbf{A}$ and for φ . Although we have the mathematical freedom to prescribe any distribution of independent sources, it is obvious that in naturally-occurring problems, we need to prescribe the independent sources in such a way that they satisfy the continuity equation.

2.1.3.3. Static charge and currents

In the cases of charge at rest and direct currents and their corresponding static fields, there is no time dependence, which means that in equations for fields and potentials, all terms with time derivatives vanish. So we determine the electric field by means of the static electric potential, according to equation (2.1.3), as

$$\mathbf{E} = -\nabla \varphi .\tag{2.1.20}$$

This expression is characteristic of a *conservative field*. The vector magnetic potential is prescribed by equations

$$\begin{aligned}\nabla \times \mathbf{A} &= \mathbf{B} , \\ \nabla \cdot \mathbf{A} &= 0 ,\end{aligned}\tag{2.1.21a,b}$$

where the last equation represents *Coulomb's additional condition*. In such a case the wave equations for potentials (2.1.19a,b) of given independent sources ρ_s' and $\mathbf{J}_s'$ (which must satisfy the static continuity equation for physical charge and currents), reduce to a pair of non-homogeneous differential equations

$$\begin{aligned}\Delta \mathbf{A} &= -\mu \mathbf{J}_s' , \\ \Delta \phi &= -\frac{\rho_s'}{\epsilon} ,\end{aligned}\tag{2.1.22a,b}$$

which are known as *the vector and scalar Poisson's equation*. These equations serve to determine the potentials of given sources in static fields. When considering a field in a source-free space, equations (2.1.22a,b) become homogeneous second-order partial equations of the form

$$\begin{aligned}\Delta \mathbf{A} &= 0 , \\ \Delta \phi &= 0 .\end{aligned}\tag{2.1.23a,b}$$

These fundamental equations for static fields are the simplest equations, and we call them the *vector and scalar Laplace equations*.

2.1.3.4. Magnetic charge and currents

There are some cases when it is convenient to calculate electromagnetic fields with the help of antipotentials ϕ^* and $\mathbf{A}^*$, which we determine from their prescribed sources. In this case, we are analysing a field in an unbounded space filled with a non-conductive, linear material with constitutive parameters ϵ and μ ($\kappa = 0$). A fictional distribution of independent magnetic charge ρ_m' and independent magnetic currents $\mathbf{J}_m'$ is given. We assume that there is no electric charge and current. Using the similarity to the system (1.5.10a,d) we then obtain Maxwell's equations for such a field

$$\begin{aligned}\nabla \times \mathbf{E} &= -\mathbf{J}_m' - \frac{\partial \mathbf{B}}{\partial t} , \\ \nabla \times \mathbf{H} &= \frac{\partial \mathbf{D}}{\partial t} , \\ \nabla \cdot \mathbf{D} &= 0 , \\ \nabla \cdot \mathbf{B} &= \rho_m' .\end{aligned}\tag{2.1.24a - d}$$

The valid constitutive relationships in this case are $\mathbf{D} = \epsilon \mathbf{E}$ and $\mathbf{B} = \mu \mathbf{H}$. We introduce antipotentials ϕ^* and $\mathbf{A}^*$ such that the field vectors $\mathbf{D}$ and $\mathbf{H}$ are determined by equations (2.1.7a,b), while the corresponding Lorentz condition is given by equation (2.1.8c). After

substitution into the system of equations (2.1.24a – d) we obtain non-homogeneous wave equations of the form

$$\begin{aligned}\Delta A^* - \mu\epsilon \frac{\partial^2 A^*}{\partial t^2} &= -\epsilon J_m', \\ \Delta \varphi^* - \mu\epsilon \frac{\partial^2 \varphi^*}{\partial t^2} &= -\frac{\rho_m'}{\mu}.\end{aligned}\tag{2.1.25a,b}$$

It is clear that one can add all the partial solutions of the homogeneous parts of the equations (2.1.26a,b) and (2.1.8a,b) to the partial solutions of these equations obtained by integration over the given independent sources ρ_m' and J_m' .

Finally, we should mention that the *density of the independent magnetic current* J_m' is a *fictitious quantity* analogous to voltage source in circuit theory.

2.2. GAUGE TRANSFORMS

After careful examination of equations (2.1.2) and (2.1.3), we find that the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ remain unaffected if we transform the potential $\mathbf{A}$ using the expression

$$\mathbf{A}_{new} = \mathbf{A} + \nabla\psi, \tag{2.2.1a}$$

and the potential φ with the expression

$$\varphi_{new} = \varphi - \frac{\partial\psi}{\partial t}, \tag{2.2.1b}$$

where ψ is an arbitrarily chosen scalar function of spatial coordinates and time. This means that every law of electromagnetism expressed with the help of electromagnetic potentials $\mathbf{A}$ and φ must remain unchanged after the transformations given by equations (2.2.1a,b). The physical law that is invariant with respect to such transformations is called the *gauge-invariant law*, and such transformations are known as *gauge transforms*.

In other words, the fields are over-determined by the prescription of these potentials, and one pair of fields possesses many pairs of potentials. By choice of the function ψ , we can choose, in a mathematically appropriate way, an infinite number of potentials $\mathbf{A}$ and φ for the same field given by vectors $\mathbf{B}$ and $\mathbf{E}$.

Within this choice of possible potentials there are two specially significant groups of potentials, known as *Lorentz potentials* and *Coulomb's potentials*. Both can be obtained by the prescription of additional conditions. These potentials are very suitable because they result in mathematical models, those being corresponding systems of potential equations whose results are in complete agreement with reality and experimentally observed facts. In addition, the Lorentz gauge is suitable because it introduces total symmetry between the scalar and vector potential by ensuring that both potentials satisfy the wave equation of the same form.

Coulomb's gauge is suitable because it results in the most complete decoupling of potentials $\mathbf{A}$ and ϕ and their so-called transverse and longitudinal fields. In some special cases, one may choose that ϕ vanishes, so that the electromagnetic field can be represented using only one potential $\mathbf{A}$. One great advantage of Coulomb's gauge is its simplicity.

2.2.1. The Lorentz gauge

Let us assume that an electromagnetic field exists in a linear source-free conductive material. Let the potentials $\mathbf{A}$ and ϕ satisfy the wave equations (2.1.16a,b), and the Lorentz gauge (2.1.15). By applying the gauge transform (2.2.1a,b), we introduce new potentials $\mathbf{A}_{new}$ and ϕ_{new} . We want the new potentials to satisfy the same wave equations (2.1.16a,b) and Lorentz condition (2.1.15)

$$\nabla \cdot \mathbf{A}_{new} + \mu\epsilon \frac{\partial \phi_{new}}{\partial t} + \mu\kappa \phi_{new} = 0 . \quad (2.2.2)$$

Substitution of equations (2.2.1a,b) into this equation results in

$$\Delta\psi - \mu\epsilon \frac{\partial^2 \psi}{\partial t^2} - \mu\kappa \frac{\partial \psi}{\partial t} = - \left(\nabla \cdot \mathbf{A} + \mu\epsilon \frac{\partial \phi}{\partial t} + \mu\kappa \phi \right) = 0 . \quad (2.2.3)$$

If we assure that the gauge function ψ satisfies the wave equation (2.2.3), we can be assured that the new potentials $\mathbf{A}_{new}$ and ϕ_{new} satisfy wave equations (2.1.16a,b) as well, and that the Lorentz condition (2.2.2) remains preserved.

All potentials in this restricted class are said to belong to the Lorentz gauge. Indeed, in all the sections of the previous Unit 2.1, we analysed electromagnetic potentials for fields in various materials, with or without prescribed independent sources, which all belong to the Lorentz gauge.

2.2.2. Coulomb's gauge

We shall study a new class of potentials belonging to *Coulomb's* or *transverse gauging*. It is the simplest gauging determined by the simple additional condition

$$\nabla \cdot \mathbf{A} = 0 , \quad (2.2.4)$$

which is called Coulomb's condition, and which is introduced into equations for electromagnetic potentials.

The Coulomb's or transverse gauge is often used when there are no sources in the region of interest, which means that independent free charge ρ_s' and independent free currents $\mathbf{J}_s'$ are equal to zero.

2.2.2.1. Non-conductive material

The fields in a non-conductive source-free material are determined by Maxwell's homogeneous equations (2.1.1a – d). The potentials introduced by the expressions (2.1.2) and (2.1.3) satisfy the equations (2.1.4a,b). If we introduce Coulomb's gauge (2.2.4) and substitute the gauge transformations (2.2.1a,b), it follows that the gauge function satisfies Laplace's equation

$$\Delta\psi = 0 . \quad (2.2.5)$$

Potentials are determined from the transformed equations (2.1.4a,b), which are

$$\Delta\mathbf{A} - \mu\epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = \mu\epsilon \nabla \left(\frac{\partial\phi}{\partial t} \right) , \quad (2.2.6a,b)$$

$$\Delta\phi = 0 .$$

The vector potential in this case is determined from the non-homogeneous wave equation, and the scalar potential is determined from Laplace's equation.

Although Laplace's equation is simpler, this pair of equations is not simple enough to utilize all the advantages of Coulomb's gauge. One can always choose trivial solutions for the gauge function ψ , as well as for the scalar potential ϕ (they both satisfy the same Laplace's equation), in such a way that the scalar potential vanishes ($\phi = 0$). If this is the case, electromagnetic fields can always be represented by a single vector potential, which is defined as

$$\begin{aligned} \nabla \times \mathbf{A} &= \mathbf{B} , \\ \nabla \cdot \mathbf{A} &= 0 . \end{aligned} \quad (2.2.7a,b)$$

Here, we determine the vector potential from given boundary and initial conditions, and the homogeneous wave equation

$$\Delta\mathbf{A} - \mu\epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = 0 . \quad (2.2.8)$$

Taking into account that $\phi = 0$, we obtain the simple expressions

$$\begin{aligned} \mathbf{B} &= \nabla \times \mathbf{A} , \\ \mathbf{E} &= -\frac{\partial \mathbf{A}}{\partial t} , \end{aligned} \quad (2.2.10a,b)$$

which determine the field vectors.

2.2.2.2. Conductive material

Similarly, in a linear conductive source-free material we can apply Coulomb's gauging to equations (2.1.14a,b) and introduce the gauge transforms (2.2.1a,b) in order to prove that the gauge function satisfies Laplace's equation $\Delta\psi = 0$, and that the potentials A and ϕ satisfy the equations

$$\Delta A - \mu\epsilon \frac{\partial^2 A}{\partial t^2} - \mu\kappa \frac{\partial A}{\partial t} = \nabla \left(\mu\epsilon \frac{\partial \phi}{\partial t} + \mu\kappa \phi \right) , \quad (2.2.11a,b)$$

$$\Delta \phi = 0 .$$

This system of equations can be essentially simplified if one chooses the trivial solution $\phi = 0$. In this case, the electromagnetic field in a conductive material is determined using only one potential, which we determine from the damped wave equation

$$\Delta A - \mu\epsilon \frac{\partial^2 A}{\partial t^2} - \mu\kappa \frac{\partial A}{\partial t} = 0 , \quad (2.2.12)$$

and the field vectors $\mathbf{B}$ and $\mathbf{E}$ are defined by equations (2.2.10a,b).

2.2.2.3. A field with independent sources

We can apply Coulomb's gauge, which means that $\nabla \cdot \mathbf{A} = 0$, even in cases where independent sources exist in the region of interest. We shall analyse the field in a conductive homogeneous linear material with given constitutive parameters ϵ , μ , κ and with a given independent free charge ρ_s' and currents $\mathbf{J}_s'$. If we define the current using equation (2.1.17), meaning $\mathbf{J}_s = \kappa \mathbf{E} + \mathbf{J}_s'$, the field is determined by the system of Maxwell's equations (1.5.15a – d). After the introduction of potentials using expressions (2.1.2) and (2.1.3), we obtain the following equations

$$\Delta A - \mu\epsilon \frac{\partial^2 A}{\partial t^2} - \mu\kappa \frac{\partial A}{\partial t} = -\mu \mathbf{J}_s' + \nabla \left(\nabla \cdot \mathbf{A} + \mu\epsilon \frac{\partial \phi}{\partial t} + \mu\kappa \phi \right) ,$$

$$\Delta \phi + \frac{\partial}{\partial t} (\nabla \cdot \mathbf{A}) = -\frac{\rho_s'}{\epsilon} .$$

If we apply Coulomb's gauge (2.2.4), we can always choose the gauge function ψ in such a way that we get

$$\Delta A - \mu\epsilon \frac{\partial^2 A}{\partial t^2} - \mu\kappa \frac{\partial A}{\partial t} = -\mu \mathbf{J}_s' + \nabla \left(\mu\epsilon \frac{\partial \phi}{\partial t} + \mu\kappa \phi \right) , \quad (2.2.13a,b)$$

$$\Delta \phi = -\frac{\rho_s'}{\epsilon} .$$

The scalar potential satisfies Poisson's equation (2.2.13b). Its solution, similar to Sections

2.4.1, is

$$\varphi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon} \int_V \frac{\rho_s'(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} dV' . \quad (2.2.14)$$

The scalar potential is *the instantaneous Coulomb potential* caused by the density of the independent free charge $\rho_s'(\mathbf{r}, t)$. Hence the name “Coulomb’s gauge”.

Poisson’s equation is typical for static fields. However, here we do not impose such a limitation, ρ_s' is a function of time and the scalar potential follows changes of charge without any retardation in space, because Poisson’s equation does not contain the term $[\mu\epsilon\partial^2\varphi/\partial t^2]$ causing retardation of the potential. In any case, it is easier to solve Poisson’s rather than the wave equation.

When equation (2.2.14) is solved, we get the potential φ caused by the distribution of independent charge ρ_s' . Afterwards, we can calculate the potential $\mathbf{A}$ from the non-homogeneous damped wave equation (2.2.13a). When the potentials φ and $\mathbf{A}$ are calculated, we determine the fields $\mathbf{B}$ and $\mathbf{E}$ from equations (2.1.2,3). The “current” term on the right-hand side of equation (2.2.13a) which contains the potential φ , can be calculated from (2.2.14). Since this term involves the gradient operator, its curl is equal to zero. This leads us to the conclusion that we can omit the corresponding part relating to current density. Therefore, we shall denote the independent current density as the sum of two parts

$$\mathbf{J}_s' = \mathbf{J}_t' + \mathbf{J}_l' , \quad (2.2.15)$$

where $\mathbf{J}_t'$ is called the transverse or solenoidal current and possesses $\nabla \cdot \mathbf{J}_t' = 0$ (source-free), and $\mathbf{J}_l'$ is called the longitudinal or irrotational current and possesses $\nabla \times \mathbf{J}_l' = 0$. With the help of the continuity equation

$$\nabla \cdot \mathbf{J}_s' = \nabla \cdot \mathbf{J}_l' = -\frac{\partial \rho_s'}{\partial t} , \quad (2.2.16)$$

which links the longitudinal part of the current $\mathbf{J}_l'$ to any change in independent free charge ρ_s' if a problem is a naturally-occurring one, and from equation (2.2.13b) it follows that

$$\mathbf{J}_l' = \epsilon \nabla \left(\frac{\partial \varphi}{\partial t} \right) . \quad (2.2.17)$$

If a material is non-conductive, then the terms that contain κ in equation (2.2.13a) are equal to zero. Therefore, from equations

$$\begin{aligned} \Delta \mathbf{A} - \mu \epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} &= -\mu \mathbf{J}_t' , \\ \Delta \varphi &= -\frac{\rho_s'}{\epsilon} . \end{aligned} \quad (2.2.18a,b)$$

we can calculate the potentials in a dielectric with a given distribution of independent

sources ρ_s' and $\mathbf{J}_s'$. The source of $\mathbf{A}$ in the wave equation is the transverse current $\mathbf{J}_t'$, which is not connected with any change in independent free charge ρ_s' . Due to this, Coulomb's gauge is called the *transverse gauge*. We also often call it the *radiation gauge*, because transverse radiation fields are determined using only the vector potential, while the instantaneous Coulomb potential affects only near fields.

Finally, we can conclude that if there is no free charge ρ_s' , and if the total current is transverse $\mathbf{J}_t'$, then we can choose ϕ as being zero and calculate the field only with the help of the vector potential $\mathbf{A}$. If free charge ρ_s' does exist, then we calculate its potential ϕ from Poisson's equation. Free charge does not affect the vector potential $\mathbf{A}$, which we calculate with the help of equation (2.2.18a) from the distribution of free (transverse) current.

Before we move on, some peculiarities of Coulomb's gauge should be pointed out. It is well known that electromagnetic waves propagate with a finite velocity. Equations (2.2.18b) and (2.2.14) show that the scalar potential propagates instantaneously throughout the whole of space. Additionally, the vector potential satisfies the wave equation (2.2.18a), which includes a finite velocity of propagation $c = 1/(\mu\epsilon)^{1/2}$. At first glance it is not easy to explain away such obviously non-physical behaviour. An answer can be found in the fact that it is the fields that are the primary physical quantities of interest, rather than the potentials. A further observation is that the transverse current $\mathbf{J}_t'$ is determined by integration over all space.

2.3. HERTZ POTENTIALS

In the previous Units 2.1. and 2.2, we saw that electromagnetic fields can be solved in a way where we first determine the vector and scalar potential and then obtain field vectors by differentiation. Due to potentials satisfying the Lorentz condition (2.1.5), we can suppose that the field is determined only by three functions. In fact, Heinrich Rudolf Hertz (1857 – 1894) showed that an electromagnetic field, under certain conditions, can be represented by only one vector function.

We shall analyse an electromagnetic field in a non-conductive linear, isotropic and homogeneous material with a given distribution of independent sources. In Section 2.1.3 we showed that the Lorentz condition (2.1.5) is a direct consequence of the continuity equation.

Since physical charge and currents cannot be prescribed separately, it is convenient to represent them with one vector function in the calculation of fields. This vector function should be chosen in such a way that the continuity equation will always be satisfied. Therefore, we introduce the vector $\mathbf{P}_s'(\mathbf{r}, t)$, so that the density of independent free charge can be determined from only one vector using the expression

$$\rho_s' = -\nabla \cdot \mathbf{P}_s' , \quad (2.3.1a)$$

and that the density of independent free current is determined from the same vector by

$$\mathbf{J}_s' = \frac{\partial \mathbf{P}_s'}{\partial t} . \quad (2.3.1b)$$

It is easy to prove that the continuity equation is satisfied with such a choice of vector $\mathbf{P}_s'$. Furthermore, any arbitrary distribution of current may be represented in such a way. The source function $\mathbf{P}_s'$ is known as the equivalent density of electric dipole distribution.

The connection between the vector $\mathbf{P}_s'$ and the free charge and currents is the same as the connection between the polarization vector $\mathbf{P}$ and polarized charge and currents according to equations (1.4.9) and (1.4.11). This is only mathematical similarity, and one should realise that ρ_s' and $\mathbf{J}_s'$ represent independent free charge and currents that are external field sources, so $\mathbf{P}_s'$ is actually completely different from polarization $\mathbf{P}$.

The vector $\mathbf{P}_s'$ is measured in *coulombs per square metre* (C/m²). By substituting expressions (2.3.1a,b) into the non-homogeneous wave equations (2.1.19a,b) we obtain equations for electromagnetic potentials

$$\begin{aligned} \Delta \mathbf{A} - \mu\epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} &= -\mu \frac{\partial \mathbf{P}_s'}{\partial t} , \\ \Delta \varphi - \mu\epsilon \frac{\partial^2 \varphi}{\partial t^2} &= \frac{1}{\epsilon} \nabla \cdot \mathbf{P}_s' , \end{aligned} \quad (2.3.2a,b)$$

whose independent sources are given by vector $\mathbf{P}_s'$. If we introduce a new vector $\mathbf{Z}$, in such a way that

$$\begin{aligned} \mathbf{A} &= \mu\epsilon \frac{\partial \mathbf{Z}}{\partial t} , \\ \varphi &= -\nabla \cdot \mathbf{Z} , \end{aligned} \quad (2.3.3a,b)$$

then the Lorentz condition (2.1.5) will be satisfied. If we substitute expressions (2.3.3a,b) into the wave equations (2.3.2a,b) we obtain the same equation for $\mathbf{Z}$, which is

$$\Delta \mathbf{Z} - \mu\epsilon \frac{\partial^2 \mathbf{Z}}{\partial t^2} = -\frac{\mathbf{P}_s'}{\epsilon} . \quad (2.3.4)$$

Therefore, by the introduction of source $\mathbf{P}_s'$ and the vector $\mathbf{Z}$, both equations (2.3.2a,b), and equations (2.1.19a,b) for the potentials $\mathbf{A}$ and φ , are satisfied using one equation (2.3.4). Vector $\mathbf{Z}$ is the *super-potential*, and we often call it either the *polarization potential*, the *Hertz potential* or the *Hertz vector*. It is measured in *volt metres* (Vm).

We can conclude that every solution of vector non-homogeneous wave equation (2.3.4) determines an electromagnetic field, the vectors of which can be calculated by the equations

$$\begin{aligned} \mathbf{B} &= \mu \epsilon \nabla \times \frac{\partial \mathbf{Z}}{\partial t} , \\ \mathbf{E} &= \nabla \nabla \cdot \mathbf{Z} - \mu \epsilon \frac{\partial^2 \mathbf{Z}}{\partial t^2} . \end{aligned} \quad (2.3.5a,b)$$

according to equations (2.1.2), (2.1.3) and (2.3.3a,b). Extending these equations to include homogeneous conductive material is straightforward, and we get

$$\begin{aligned} \mathbf{B} &= \mu \nabla \times \left(\epsilon \frac{\partial \mathbf{Z}}{\partial t} + \kappa \mathbf{Z} \right) , \\ \mathbf{E} &= \nabla \nabla \cdot \mathbf{Z} - \mu \epsilon \frac{\partial^2 \mathbf{Z}}{\partial t^2} - \mu \kappa \frac{\partial \mathbf{Z}}{\partial t} = \nabla \times \nabla \times \mathbf{Z} , \end{aligned} \quad (2.3.6a,b)$$

where

$$\Delta \mathbf{Z} - \mu \epsilon \frac{\partial^2 \mathbf{Z}}{\partial t^2} - \mu \kappa \frac{\partial \mathbf{Z}}{\partial t} = 0 . \quad (2.3.7)$$

We can extend our study further. Due to the antisymmetry of the system of Maxwell's equations, it is possible to introduce a magnetic Hertz vector $\mathbf{Z}^*$ in addition to the electric Hertz vector $\mathbf{Z}$.

Let us analyse a linear homogeneous non-conductive material with a given distribution of independent free charge and currents. It is possible to find the density of an equivalent distribution of electric dipoles $\mathbf{P}_s'$, and the density of an equivalent distribution of magnetic dipoles $\mathbf{M}_s'$, whose electromagnetic fields are the same as the fields produced by the free independent distribution of charge and currents ρ_s' and $\mathbf{J}_s'$. We shall introduce the vectors $\mathbf{P}_s'$ and $\mathbf{M}_s'$ defined by the expressions

$$\begin{aligned} \rho_s' &= -\nabla \cdot \mathbf{P}_s' , \\ \mathbf{J}_s' &= \frac{\partial \mathbf{P}_s'}{\partial t} + \nabla \times \mathbf{M}_s' , \end{aligned} \quad (2.3.8a,b)$$

similarly to equations (2.3.1a,b), so that physical charge and currents satisfy the continuity equation

$$\nabla \cdot \mathbf{J}_s' + \frac{\partial \rho_s'}{\partial t} = 0 . \quad (2.3.9)$$

The density of an equivalent distribution of independent magnetic dipoles $\mathbf{M}_s'$ is measured in *amperes per metre*, while the unit of magnetic Hertz vector $\mathbf{Z}^*$ is *the weber*. If we introduce Hertz vectors $\mathbf{Z}$ and $\mathbf{Z}^*$ in such a way that

$$\begin{aligned} \mathbf{A} &= \mu\epsilon \frac{\partial \mathbf{Z}}{\partial t} + \nabla \times \mathbf{Z}^* , \\ \varphi &= -\nabla \cdot \mathbf{Z} , \end{aligned} \quad (2.3.10a,b)$$

hold, then the Lorentz condition (2.1.5) is automatically satisfied as well as the continuity equation. Substitution of equations (2.3.8a,b) and (2.3.10a,b) into wave equations (2.1.19a,b) results in new wave equations

$$\begin{aligned} \Delta \mathbf{Z} - \mu\epsilon \frac{\partial^2 \mathbf{Z}}{\partial t^2} &= -\frac{\mathbf{P}_s'}{\epsilon} , \\ \Delta \mathbf{Z}^* - \mu\epsilon \frac{\partial^2 \mathbf{Z}^*}{\partial t^2} &= -\mu \mathbf{M}_s' . \end{aligned} \quad (2.3.11a,b)$$

We have to choose Hertz vectors in such a way that they satisfy these equations.

2.4. INTEGRAL REPRESENTATION

In this unit, we shall present an approach to problems in electromagnetism based on integral equations. We shall develop this approach parallel to one based on differential equations. Mathematical models in electromagnetism based on integral equations form the foundation of the method of moments.

We shall study sources and the fundamental problem of determining the electromagnetic field caused by a given distribution of independent charge and currents. A way of obtaining a solution to the problem of linking fields to their sources is to solve the non-homogeneous wave equations for electromagnetic potentials. Let us start the study with partial solutions of the non-homogeneous wave equations described in Section 2.1.3.

The scalar potential φ and Cartesian components of the vector potential $\mathbf{A}$ in equations (2.1.19a,b) for given distributions of charge and current, with Lorentz gauging, satisfy a scalar non-homogeneous wave equation of the form

$$\Delta \psi - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = -g(\mathbf{r}, t) . \quad (2.4.1)$$

Here, the function ψ represents the scalar potential or any Cartesian component of the field vectors or vector potential, and the function $g(\mathbf{r}, t)$ represents the density of sources. We assume that a given distribution of sources is possible, meaning that it is not in contradiction with the field equations and that it is independent, meaning the calculated field does not affect it.

In equation (2.4.1) we set $\mu\epsilon$ to be equal to $1/c^2$, where c possesses the dimensions of velocity (m/s). We assume that the space is filled with a homogeneous isotropic and non-conductive material.

We shall begin with a solution of the field of a point source, and after that we shall derive a general integral of equation (2.4.1) for a continuous source distribution in an

unbounded, as well as a bounded, space. In reality we can solve only a few idealized cases in a relatively simple way. For practical applications, we shall use numerical methods for finding integral solutions to problems with complex geometry.

2.4.1. The scalar potential of a point charge

The simplest distribution of charge that we can analyse is one that can be neglected everywhere except in the vicinity of the origin of a system. This brings us to the hypothetically idealised model of a charge concentrated at an origin. We call such sources point charges. The density of a point charge is equal to zero everywhere except in the actual charge itself where it is infinite. Let us imagine a point charge with a time dependent magnitude $f(t)$ at rest. This is physically impossible, but this assumption will help us to take the first step towards the solution of our problem. We put the charge at the origin of a spherical coordinate system. The homogeneous part of equation (2.4.1)

$$\Delta\varphi - \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} = 0, \quad (2.4.2)$$

should be valid at all points of an unbounded space except at the origin of the system. By φ we have denoted the scalar potential of a point charge placed at the origin. Potential φ obviously depends on the radial distance from the origin r , and is independent of direction. Due to spherical symmetry, φ is a function only of r and t . Therefore, equation (2.4.2) written in terms of spherical coordinates is

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \varphi}{\partial r} \right) - \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} = 0. \quad (2.4.3)$$

The final solution of equation (2.4.2), or (2.4.3), is given by the expression

$$\varphi = \frac{f\left(t - \frac{r}{c}\right)}{r} + \frac{h\left(t + \frac{r}{c}\right)}{r}, \quad (2.4.4)$$

where f and h are arbitrary functions. The first term of φ represents an *outward wave* with an amplitude that decays as the wave propagates due to the factor $1/r$. Similarly, the second term of φ represents an *inward wave* that propagates towards the origin and whose amplitude increases as it is propagating towards the origin.

The exact determination of functions f and h depends on the imposed conditions. If, for example, we suppose we have suddenly set the charge at the origin of an unbounded space, this will result in a wave propagating outwards, and we thus have to set h to be identically equal to zero, as it represents an inward wave. Generally, h represents a disturbance which can be observed before the source has produced it. Since such behaviour is in contradiction with macroscopic experimental observations in electromagnetism, we shall assume that h is equal to zero.

If the charge possesses a constant value Q , then the electrostatic potential is $\varphi = Q/4\pi\epsilon r$. This includes the assumption that the potential vanishes at infinity. Finally, the scalar potential of a point charge of magnitude $f(t)$ placed at the origin of an unbounded space is

$$\varphi = \frac{f\left(t - \frac{r}{c}\right)}{4\pi\epsilon r} . \quad (2.4.5)$$

Although this is only a heuristic analysis that cannot be replaced by rigid analysis as we cannot accomplish the imagined model, the result (2.4.5) is useful. It represents a solution of the wave equation (2.4.2) applied to a point charge of magnitude $f(t)$ placed at the origin of an unbounded space.

2.4.2. Potentials of a time dependent distribution of space charge in an unbounded space

The continuous distribution of independent space charge $\rho'(\mathbf{r}, t)$ in a volume V is given. It satisfies the wave equation

$$\Delta\varphi - \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} = - \frac{\rho'(\mathbf{r}, t)}{\epsilon_0} . \quad (2.4.6)$$

An essential step towards the general solution of wave equation (2.4.6) in an unbounded free space will be made if we observe the charge density $\rho'(\mathbf{r}_i, t_j)$ at a point $\mathbf{r}_i$ at a moment in time t_j , in order to express the sudden occurrence of a point charge with value $\rho'(\mathbf{r}_i, t_j)\Delta V'$ at a point $\mathbf{r}_i$ at a moment in time t_j over a time interval of $\Delta t'$. The potential produced by this point charge at a point $\mathbf{r}$ at a moment in time t will be equal to $[\rho'(\mathbf{r}_i, t_j)\Delta V']$ – a time-dependent potential of a unit point charge that occurs within a time interval of $\Delta t'$ at a point $\mathbf{r}_i$ at a moment in time t . In such a way, we obtain the potential of a point charge of value $\rho'(\mathbf{r}_i, t_j)\Delta V'$, which can be expressed as

$$\varphi(\mathbf{r}, t | \mathbf{r}_i, t_j) = \frac{1}{4\pi\epsilon_0} \frac{\rho'(\mathbf{r}_i, t_j)\Delta V' \delta_t\left(t - t_j - \frac{R_i}{c}\right) \Delta t'}{R_i} , \quad (2.4.7)$$

where $R_i = |\mathbf{r} - \mathbf{r}_i|$. If $\rho' = 0$ outside volume V as shown in Figure 2.4.1, the potential at a point $\mathbf{r}$ at a moment in time t produced by the total charge in V for unbounded free space can be calculated by

$$\varphi(\mathbf{r}, t) = \sum_i \sum_j \varphi(\mathbf{r}, t | \mathbf{r}_i, t_j) = \frac{1}{4\pi\epsilon_0} \sum_i \sum_j \frac{\rho'(\mathbf{r}_i, t_j)\Delta V' \delta_t\left(t - t_j - \frac{R_i}{c}\right) \Delta t'}{R_i} ,$$

or in the limit where $\Delta t' \rightarrow 0$ and $\Delta V' \rightarrow 0$, and with the help of Green's function for an

unbounded space

$$G_0^{(\pm)}(\mathbf{r}, t | \mathbf{r}', t') = \frac{1}{4\pi\epsilon_0} \frac{\delta\left(t - t' \mp \frac{R}{c}\right)}{R},$$

we obtain

$$\varphi(\mathbf{r}, t) = \int \int \rho'(\mathbf{r}', t') G_0^{(+)}(\mathbf{r}, t | \mathbf{r}', t') dV' dt' = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho'\left(\mathbf{r}', t - \frac{R}{c}\right)}{R} dV'. \quad (2.4.8)$$

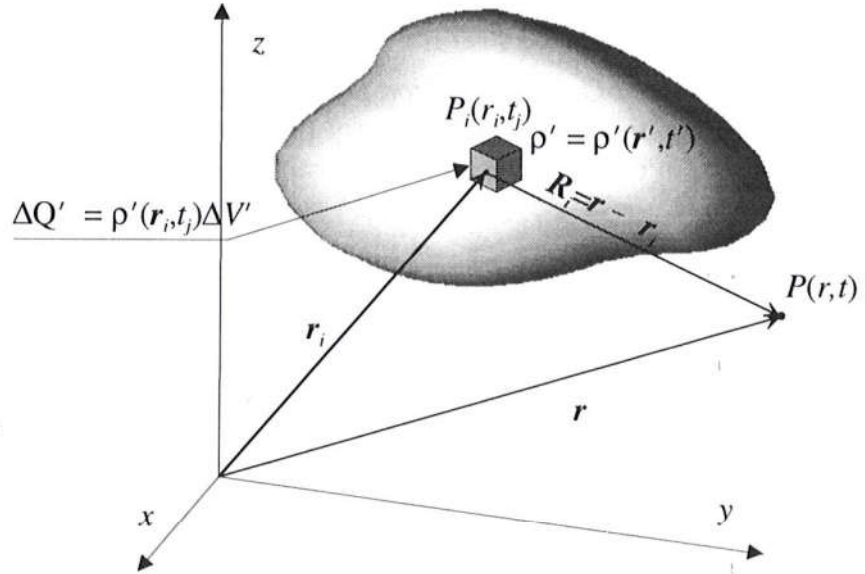


Figure 2.4.1. Determination of the retarded potential

Particular integrals of the non-homogeneous wave equation (2.4.6) for a free unbounded space are

$$\varphi^{(\pm)}(\mathbf{r}, t) = \int \int G_0^{(\pm)}(\mathbf{r}, t | \mathbf{r}', t') \rho'(\mathbf{r}', t') dV' dt', \quad (2.4.9)$$

where ρ' is an ordinary function in addition to φ . It can be proved that φ is a continuous function if ρ' is a continuous function, and that it possesses continuous derivatives if ρ' possesses these continuous derivatives as well. The potentials $\varphi^{(+)}$ and $\varphi^{(-)}$ in equation (2.4.9) are known as *the retarded and advanced potentials*. They are not the only solutions of equation (2.4.6), because we can add them to any solution of the homogeneous equation

$$\Delta\varphi - \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} = 0, \quad (2.4.10)$$

in order to solve a particular physical problem. Before we can say that (2.4.9) is the solution to equation (2.4.6), we have to set up additional conditions.

A given distribution of sources $\rho'(\mathbf{r}', t')$ exists in space and time. It is different from zero only for a finite interval around $t' = 0$. Two extreme situations are of interest. In the first we shall assume that at a moment $t \rightarrow -\infty$, a wave $\varphi_{in}(\mathbf{r}, t)$ exists that satisfies wave

equation (2.4.10). The wave φ_{in} propagates in time and space, and the source activates and produces its own waves. A complete solution in this situation for a complete time span is

$$\varphi(\mathbf{r}, t) = \varphi_{in}(\mathbf{r}, t) + \int \int G_0^{(+)}(\mathbf{r}, t | \mathbf{r}', t') \rho'(\mathbf{r}', t') dV' dt' . \quad (2.4.11)$$

The existence of the *retarded Green's function* $G_0^{(+)}$ ensures that at a distant moment in time t before the activation of the source there is no contribution from the integral. Only the prescribed wave φ_{in} exists. In the second situation we assume that at a moment in time $t \rightarrow +\infty$ the wave is given as $\varphi_{out}(\mathbf{r}, t)$, which is a well-known solution of the homogeneous wave equation (2.4.10). The complete solution for a complete time span is

$$\varphi(\mathbf{r}, t) = \varphi_{out}(\mathbf{r}, t) + \int \int G_0^{(-)}(\mathbf{r}, t | \mathbf{r}', t') \rho'(\mathbf{r}', t') dV' dt' . \quad (2.4.12)$$

Here an *advanced Green's function* $G_0^{(-)}$ ensures that there is no signal from the source after it stops acting. The most frequent physical state is described by equation (2.4.11) with $\varphi_{out} = 0$. In this case we directly substitute Green's function for unbounded space to obtain

$$\varphi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{[\rho'] dV'}{R} , \quad (2.4.13a)$$

where the symbol

$$[\rho'] = \rho'(\mathbf{r}', t') = \rho'\left(\mathbf{r}', t - \frac{R}{c}\right) , \quad (2.4.13b)$$

denotes a function with a retarded phase. This means that we calculate time t' as $t' = t - R/c$. The time interval R/c is the time needed for the signal to travel from the source to an observer. The velocity of information carried by an electromagnetic wave from the source at the point $\mathbf{r}'$ to an observer at the point $\mathbf{r}$ with distance $R = |\mathbf{r} - \mathbf{r}'|$, is in free space equal to $c = c_0 = 1/(\epsilon_0\mu_0)^{1/2} = 2.9979245 \cdot 10^8$ m/s and equals the speed of light.

2.4.3. Retarded potentials in an unbounded space

If we separate equation (2.1.19a) for the vector potential into Cartesian coordinates, we obtain three equations of the form (2.4.1). Accordingly, a particular integral of the *retarded vector potential* in unbounded free space is

$$\begin{aligned} \mathbf{A}(\mathbf{r}, t) &= \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}'(\mathbf{r}', t') \delta\left(t - t' - \frac{R}{c}\right)}{R} dV' dt' = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}'\left(\mathbf{r}', t - \frac{R}{c}\right)}{R} dV' = \\ &= \frac{\mu_0}{4\pi} \int_V \frac{[\mathbf{J}'] dV'}{R} , \end{aligned} \quad (2.4.14)$$

for a given distribution of independent currents $\mathbf{J}'(\mathbf{r}', t')$ in a finite volume V , together with the assumption that $\mathbf{J}' = 0$ outside V . Equations (2.4.8), (2.4.13a) and (2.4.14) enable the calculation of fields and potentials from a given distribution of free charge and current in a free unbounded space. We have to keep in mind that the non-homogeneous equations (2.1.19a,b) are valid only if the Lorentz condition

$$\nabla \cdot \mathbf{A} + \mu_0 \epsilon_0 \frac{\partial \phi}{\partial t} = 0, \quad (2.4.15)$$

is satisfied. The solutions (2.4.8) and (2.4.14) satisfy this condition, and ensure that the densities of independent charge and currents satisfy the continuity equation.

Finally, we can write the retarded solutions for Hertz vectors described in Unit 2.3. that satisfy wave equations (2.3.11a,b) as

$$\begin{aligned} \mathbf{Z} &= \frac{1}{4\pi\epsilon} \int_V \frac{[\mathbf{P}'_s]}{|\mathbf{r} - \mathbf{r}'|} dV', \\ \mathbf{Z}^* &= \frac{\mu}{4\pi} \int_V \frac{[\mathbf{M}'_s]}{|\mathbf{r} - \mathbf{r}'|} dV'. \end{aligned} \quad (2.4.16a,b)$$

These solutions are valid for an unbounded space filled with a non-conductive homogeneous linear isotropic material with constitutive parameters ϵ and μ .

2.5. WAVE EQUATIONS AND INTEGRALS OF POTENTIALS

We have showed that in a non-conductive material ($\epsilon, \mu, \kappa = 0$) with given independent sources ($\rho'_s, \mathbf{J}'_s, \mathbf{P}'_s, \mathbf{M}'_s$) we can determine the scalar electric potential ϕ and Cartesian components of the vector potential $\mathbf{A}$ and Hertz electric and magnetic vectors $\mathbf{Z}, \mathbf{Z}^*$ with the help of the wave equation (2.4.1)

$$\begin{aligned} \Delta \phi - \mu \epsilon \frac{\partial^2 \phi}{\partial t^2} &= -\frac{\rho'_s}{\epsilon}, \\ \Delta \mathbf{A} - \mu \epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} &= -\mu \mathbf{J}'_s, \\ \Delta \mathbf{Z} - \mu \epsilon \frac{\partial^2 \mathbf{Z}}{\partial t^2} &= -\frac{\mathbf{P}'_s}{\epsilon}, \\ \Delta \mathbf{Z}^* - \mu \epsilon \frac{\partial^2 \mathbf{Z}^*}{\partial t^2} &= -\mu \mathbf{M}'_s. \end{aligned} \quad (2.5.1a - d)$$

While determining the general integrals of these equations, we have to distinguish two main cases: the first when the region of interest is *unbounded*, and the second when the

region of interest is *bounded*. In both cases, we can find the retarded solutions with the help of Green's functions.

A *retarded Green's function* for an unbounded space is determined by the expression

$$G_0^{(+)} = \frac{\delta\left(t - t' - \frac{R}{c}\right)}{4\pi\epsilon R}, \quad (2.5.2)$$

where $c^2 = 1/\mu\epsilon$ denotes the velocity of propagation of the disturbance, while $R = |\mathbf{r} - \mathbf{r}'|$ is the distance between the field point and the source point. When Green's function is determined, we can determine the integral of the wave equation of the scalar potential in an unbounded space according to (2.4.8) as being

$$\varphi = \frac{1}{4\pi\epsilon} \int \frac{\rho_s' \left(\mathbf{r}, t - \frac{R}{c} \right)}{R} dV'. \quad (2.5.3)$$

If we apply the notion $[\rho_s'] = \rho_s'(\mathbf{r}', t - R/c)$ to all other sources, we can write solutions of the wave equations (2.5.1a - d) for potentials and super-potentials as

$$\begin{aligned} \varphi &= \frac{1}{4\pi\epsilon} \int \frac{[\rho_s'] dV'}{R}, \\ A &= \frac{\mu}{4\pi} \int \frac{[\mathbf{J}_s'] dV'}{R}, \\ Z &= \frac{1}{4\pi\epsilon} \int \frac{[\mathbf{P}_s'] dV'}{R}, \\ Z^* &= \frac{\mu}{4\pi} \int \frac{[\mathbf{M}_s'] dV'}{R}. \end{aligned} \quad (2.5.4a - d)$$

In the case of a bounded space, integration of the wave equation is a very complex problem. We need to determine the corresponding Green's function of a bounded space for every problem with a specific geometry from the wave equation

$$\Delta G^{(+)} - \mu\epsilon \frac{\partial^2 G^{(+)}}{\partial t^2} = -\frac{\delta(\mathbf{r} - \mathbf{r}')\delta(t - t')}{\epsilon}. \quad (2.5.5)$$

Determination of the special Green's function corresponding to the given geometry of a bounded space is the most complex part of the problem. In most cases, it is impossible to obtain the solution analytically, except in some special configurations. An essential improvement in the area of solving boundary and initial value problems was achieved recently by a powerful breakthrough in numerical procedures and computers. Therefore, in determining $G^{(+)}$ from equation (2.5.5), we use the principle of superposition. It is composed of two parts

$$G^{(+)} = \frac{\delta\left(t - t' - \frac{R}{c}\right)}{4\pi\epsilon R} + F^{(+)} , \quad (2.5.6)$$

where the first term represents the response of an unbounded space. We denote it as $G_0^{(+)}$ according to expression (2.5.2), and we obtain it as a particular solution of the non-homogeneous part of wave equation (2.5.5). The second term $F^{(+)}(\mathbf{r}, t | \mathbf{r}', t')$ in the solution of Green's function $G^{(+)}(\mathbf{r}, t | \mathbf{r}', t')$ for a bounded space is determined as a particular solution of the homogeneous part of the wave equation (2.5.6)

$$\Delta F^{(+)} - \mu\epsilon \frac{\partial^2 F^{(+)}}{\partial t'^2} = 0 , \quad (2.5.7)$$

particular to the specific geometry of a given problem. In spite of such a comprehensive and complex procedure, solutions can be obtained for a relatively small number of problems with regular geometries. Therefore, this description should be judged from a practical point of view as only a desire to determine *solutions of the wave equation* in an unbounded space as a formality. Finally, we can write the solution of the wave equation (2.5.1a) for the scalar potential in a bounded space as being

$$\begin{aligned} \varphi^{(+)} = & \int_{t_0}^{t'} dt' \int_V G^{(+)} \rho_s' dV' + \epsilon \int_{t_0}^{t'} dt' \oint_S \left\{ G^{(+)} \frac{\partial \varphi_s}{\partial n'} - \varphi_s \frac{\partial G^{(+)}}{\partial n'} \right\} dS' - \\ & - \frac{\epsilon}{c^2} \int_V \left\{ \left(\frac{\partial G^{(+)}}{\partial t'} \right)_{t'=t_0} \varphi_0 - (G^{(+)})_{t'=t_0} \left(\frac{\partial \varphi}{\partial t'} \right)_{t'=t_0} \right\} dV' . \end{aligned} \quad (2.5.8)$$

It is obvious from this expression that all required boundary conditions on the boundary surfaces S of a given geometry and initial conditions within volume V at a moment in time t_0 are to be prescribed in order to obtain a unique solution of the scalar wave equation (2.5.1a).

PROBLEMS

- 2.1. Prove that solution of the wave equation

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \varphi}{\partial r} \right) - \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} = 0 ,$$

is

$$\varphi = \frac{f\left(t - \frac{r}{c}\right)}{r} + \frac{h\left(t + \frac{r}{c}\right)}{r} ,$$

where f and h are arbitrary functions!

- 2.2. Prove that solutions for retarded potentials in an unbounded space

$$\varphi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho'\left(\mathbf{r}', t - \frac{R}{c}\right)}{R} dV' ,$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}'\left(\mathbf{r}', t - \frac{R}{c}\right)}{R} dV' ,$$

satisfy the continuity equation!

- 2.3. Derive wave equations for the pair of field vectors $(\mathbf{E}, \mathbf{B})$ in a non-conductive, material ($\epsilon, \mu, \kappa = 0$)!
(The independent sources ρ_s' and $\mathbf{J}_s'$ are prescribed.)

- 2.4. Write down the solutions of the wave equations for fields $\mathbf{E}$ and $\mathbf{B}$ in an unbounded space filled with a non-conductive material derived in the previous problem!

Hint: use the analogy to the solutions (2.5.4a – d)!

3. ENERGY AND FORCES IN AN ELECTROMAGNETIC FIELD

The subject of this chapter is the introduction of an energetic point of view into the study of electromagnetic fields. So far we have defined an electromagnetic field as a space in which electromagnetic forces act on charge and currents. As a next step, we shall associate energy with an electromagnetic field.

It is obvious from the definitions of the field vectors $\mathbf{E}$ and $\mathbf{B}$, which are based on the expression of the Lorentz force on a charge in motion, that the space in which an electromagnetic field exists is in a specific state. We may connect a specific energy with each element of this space. Regarding this, one can question the existence of an electromagnetic field as an independent physical entity. It is well known that Maxwell dared to attribute a great degree of physical reality to an electromagnetic field in a direct mechanical way. We emphasise Maxwell's understanding of an electromagnetic field as a real physical essence in which energy is stored, and not as a mathematical quantity that is used to describe forces acting on unit charge.

The space in which a field is acting is changed. The field carries an energy per se, regardless of the existence of charge in the space. In many books on electromagnetism, one frequently gets the impression that a problem is solved when the fields are calculated. Since many practical devices and machines are founded on the transmission or conversion of electromagnetic energy into other forms, an understanding of the characteristics of storing and transmitting energy is essential for electrical engineers.

Our approach to electromagnetic energy in this chapter is founded on the power the electromagnetic field delivers to charge moving in free space. Afterwards, we shall include the charge on which the macroscopic model of a material has been built. The resulting equation of energetic balance, known as Poynting's theorem, will be our guide in establishing the different forms of energy stored and transmitted by means of an electromagnetic field in free space and in a material.

3.1. THE INFLUENCE OF AN ELECTROMAGNETIC FIELD ON MOVING CHARGES

From mechanics we know that the acting of a force on a material particle causes a change of the velocity of motion, which causes a change in the kinetic energy of the particle or in its energetic state. In other words, when a charged particle is moving in an electromagnetic field in a vacuum, the field will act with a Lorentz force on the charge and thus do work that will be used to change the trajectory and velocity of the particle.

3.1.1. Forces on charges in motion

Let us focus our attention on determining the interactive forces with which the electromagnetic field affects different distributions of electric charge and currents as well as equivalent magnetic charge and currents. We assume that all charges are moving and that all currents flow in free space.

We shall start with expression (1.1.1) for the Lorentz force $\mathbf{F}_e = q(\mathbf{E} + \mathbf{v}_e \times \mathbf{B})$ on a discrete electrical point charge q , which is moving with velocity $\mathbf{v}_e$ in a vacuum. Here, $\mathbf{E}$ and $\mathbf{B}$ are given vectors of an externally applied electromagnetic field. $\mathbf{F}_e$ is the force with which the field acts on the electric charge. The unit of force is the *newton* (N).

Analogously, due to the harmony and mathematical similarity between the electric and magnetic charge, we can calculate the force with which the electromagnetic field acts on a discrete magnetic point charge q_m

$$\mathbf{F}_m = q_m(\mathbf{H} - \mathbf{v}_m \times \mathbf{D}) . \quad (3.1.1)$$

Here, q_m is the quantity of magnetic point charge measured in *volt seconds* (Vs), or *webers* (Wb), which is moving with a velocity $\mathbf{v}_m$ given in *metres per second* (m/s).

Namely, if we use the magnetic charge model to describe magnetic dipoles, it can be easily shown that external fields $\mathbf{E}$ and $\mathbf{H}$ act with the same moment on both electric and magnetic dipoles (which are physically real) when a magnetic charge q_m , which is moving with velocity $\mathbf{v}_m$, is subject to a Lorentz force of the form (3.1.1).

It is easy to go from an expression for forces on individual charges to an expression for forces on charges in an elemental volume dV . In the first place, it is necessary to replace the notion of a singular distribution of charge with a continuous distribution of charge in a vacuum, which is described by continuous functions of spatial coordinates $\rho = \rho(\mathbf{r}, t)$ and $\rho_m = \rho_m(\mathbf{r}, t)$. The force on an infinitesimally small volume dV , which contains the electrical charge $dQ = \rho dV$ is

$$d\mathbf{F}_e = dQ(\mathbf{E} + \mathbf{v}_e \times \mathbf{B}) = (\rho \mathbf{E} + \rho \mathbf{v}_e \times \mathbf{B}) dV , \quad (3.1.2)$$

and for magnetic charge $dQ_m = \rho_m dV$

$$d\mathbf{F}_m = dQ_m(\mathbf{H} - \mathbf{v}_m \times \mathbf{D}) = (\rho_m \mathbf{H} - \rho_m \mathbf{v}_m \times \mathbf{D}) dV . \quad (3.1.3)$$

The total force with which the electromagnetic field acts on a system of continuously distributed electric and magnetic charge in motion in a vacuum within a finite region of volume V is determined by integration over volume V

$$\begin{aligned} \mathbf{F}_e &= \int_V (\rho \mathbf{E} + \rho \mathbf{v}_e \times \mathbf{B}) dV , \\ \mathbf{F}_m &= \int_V (\rho_m \mathbf{H} - \rho_m \mathbf{v}_m \times \mathbf{D}) dV . \end{aligned} \quad (3.1.4a,b)$$

Finally, we need to determine the volume density of forces with which the field acts on a

continuous distribution of electric and magnetic charge in unit volume around the point of interest

$$\begin{aligned} \mathbf{f}_e &= \rho \mathbf{E} + \rho \mathbf{v}_e \times \mathbf{B} , \\ \mathbf{f}_m &= \rho_m \mathbf{H} - \rho_m \mathbf{v}_m \times \mathbf{D} . \end{aligned} \quad (3.1.5,6)$$

These forces are often called “volume forces” and their unit is *the newton per cubic metre* (N/m³). If we introduce the electric current density in a vacuum $\mathbf{J} = \rho \mathbf{v}$ according to the expression for convection currents (1.2.5), we obtain the volume force on electric charge ρ and electric currents $\mathbf{J}$ in free space with the help of expression (3.1.5) as

$$\mathbf{f}_e = \rho \mathbf{E} + \mathbf{J} \times \mathbf{B} . \quad (3.1.7)$$

Analogously, the magnetic current density is $\mathbf{J}_m = \rho_m \mathbf{v}_m$, and the volume force on magnetic charge and currents in an electromagnetic field is

$$\mathbf{f}_m = \rho_m \mathbf{H} - \mathbf{J}_m \times \mathbf{D} . \quad (3.1.8)$$

3.1.2. Energy and power delivered to charges in motion

Our approach to electromagnetic energy will be based on the determination of the power delivered from an electromagnetic field to charge in motion in free space. We grasp that power is the velocity of energy transfer from the field to the charge in motion. The instantaneous value of power is determined by the expression

$$P = \frac{dW}{dt} . \quad (3.1.9)$$

It is measured in *volt amperes*, i.e. *watts* (W). By W we have denoted energy measured in *watt seconds* (Ws), which means *joules* (J), and with t time measured in *seconds* (s).

We shall begin with a determination of the energy delivered in unit time, which means the power delivered from an electromagnetic field ($\mathbf{E}, \mathbf{B}$) to an electric point charge q that is moving with velocity $\mathbf{v}_e$ in a vacuum. It is equal to the work done in unit time when the field acts with force $\mathbf{F}_e$ on a charge in motion. We shall calculate the delivered power, or work done by a force $\mathbf{F}_e$ on a path ds in unit time, with the help of the definition

$$P_e = \mathbf{F}_e \cdot \frac{d\mathbf{s}}{dt} = \mathbf{F}_e \cdot \mathbf{v}_e = q(\mathbf{E} + \mathbf{v}_e \times \mathbf{B}) \cdot \mathbf{v}_e = q \mathbf{v}_e \cdot \mathbf{E} . \quad (3.1.10)$$

Similarly, the power delivered to a magnetic point charge q_m that is moving with velocity $\mathbf{v}_m$ in an electromagnetic field ($\mathbf{D}, \mathbf{H}$) in a vacuum can be calculated as

$$P_m = \mathbf{F}_m \cdot \frac{d\mathbf{s}}{dt} = \mathbf{F}_m \cdot \mathbf{v}_m = q_m(\mathbf{H} - \mathbf{v}_m \times \mathbf{D}) \cdot \mathbf{v}_m = q_m \mathbf{v}_m \cdot \mathbf{H} . \quad (3.1.11)$$

Let us focus our attention on electric and magnetic charge with a continuous distribution in free space given by functions $\rho(\mathbf{r}, t)$ and $\rho_m(\mathbf{r}, t)$ moving with velocities $\mathbf{v}_e(\mathbf{r}, t)$ and $\mathbf{v}_m(\mathbf{r}, t)$.

The forces with which the electromagnetic field acts on the charge in an elemental volume dV are determined with the help of (3.1.5) and (3.1.6). These forces deliver a total quantity of energy in unit time, or a total instantaneous power, to the electric and magnetic charge in motion within the elemental volume defined by the expression

$$dP = (p_e + p_m)dV = d\mathbf{F}_e \cdot \mathbf{v}_e + d\mathbf{F}_m \cdot \mathbf{v}_m = (\mathbf{f}_e \cdot \mathbf{v}_e + \mathbf{f}_m \cdot \mathbf{v}_m)dV. \quad (3.1.12)$$

The quantity $p = p_e + p_m$, which is measured in *watts per cubic metre* (W/m^3), is called *the volume density of power* or simply *the volume power*. It can be determined with the help of afore mentioned expressions as

$$p = \rho \mathbf{v}_e \cdot \mathbf{E} + \rho_m \mathbf{v}_m \cdot \mathbf{H}. \quad (3.1.13)$$

If we introduce the density of electric current in a vacuum $\mathbf{J} = \rho \mathbf{v}_e$ according to (1.2.5) and analogously the density of magnetic current $\mathbf{J}_m = \rho_m \mathbf{v}_m$, we can reduce expression (3.1.13) for total instantaneous volume power that an electromagnetic field delivers to a continuously distributed electric and magnetic charge in motion to

$$p = \mathbf{J} \cdot \mathbf{E} + \mathbf{J}_m \cdot \mathbf{H}. \quad (3.1.14)$$

Here $\mathbf{J}$ and $\mathbf{J}_m$ are the macroscopic densities of the electric and magnetic currents at a point. The total energy delivered from the electromagnetic field in unit time (instantaneous delivered power) to all charges within volume V is determined by the integration

$$P = \int_V p dV = \int_V (\mathbf{J} \cdot \mathbf{E} + \mathbf{J}_m \cdot \mathbf{H}) dV. \quad (3.1.15)$$

In expression (3.1.14) $\mathbf{J}$ represents contributions from the movement of free electric charge and of bound electric charge that result from the macroscopic model of a polarized material to the current and $\mathbf{J}_m$ represents the density of the magnetic current that results from the macroscopic model of a magnetized material.

Therefore, we can conclude that, although the volume energy delivered in unit time is related to the distributions $\mathbf{J}$ and $\mathbf{J}_m$ in a vacuum, the expression (3.1.14) may be useful and general enough for application in equivalent macroscopic models in materials, which we represent by the distributions of electric and magnetic charge in free space.

3.2. ELECTROMAGNETIC ENERGY

The total instantaneous power delivered from an electromagnetic field to both free (conductive and convective) and bound (electric and magnetic) charge, which is always in a state of movement according to experience from mechanics, is partially converted into some other *reversible form of energy*. For example, acceleration and deceleration of charge results in an increase or decrease in kinetic energy, or a change in mutual positions of charge results in an increase or decrease in the potential energy of a system of charge.

It is reasonable to assume that another part of the energy delivered to charge is converted into an *irreversible form of energy* (heat). The heat generated by displacement, and generally by movement of charge, is irretrievably lost electromagnetic energy. Therefore

we call it energy loss or simply loss (of power).

On the other hand, the universal natural *principle of conservation of energy* refers to the assumption that the power (3.1.15) delivered by an electromagnetic field to charge in motion must be supplied and expressed as the decrease in field energy and as the flow of power transmitted by the field from a source.

Contrary to the principle of action at a distance, the field principle introduces a field of forces that are transmitting their effects from one point to another through space by means of local interaction. With this idea of a field, we are speaking about *energy that is stored* in the field itself and that is *transmitted by the field* through space. *This enables us to call energy that remains in a field or that is transmitted by an electromagnetic field “electromagnetic energy”.*

3.2.1. The conservation of electromagnetic energy. Poynting’s theorem

3.2.1.1. Poynting’s theorem in differential form

We shall derive Poynting’s theorem, which is a *fundamental expression of the conservation of energy in an electromagnetic system*, starting from expression (3.1.14) for the total instantaneous volume power that an electromagnetic field delivers to both electric and magnetic charge in motion $p = \mathbf{J} \cdot \mathbf{E} + \mathbf{J}_m \cdot \mathbf{H}$. As we have mentioned before, all external and internal, free and bound charge generally contribute to currents $\mathbf{J}$ and $\mathbf{J}_m$. This analysis as well as conclusions related to the motion of charge in a vacuum, will be equally valid for energetic relationships in free space and within a macroscopic model of a material. We shall obtain this if we substitute $\mathbf{J}$ and $\mathbf{J}_m$ with the expressions

$$\begin{aligned} \mathbf{J} &= \mathbf{J}_s + \frac{\partial \mathbf{P}}{\partial t} , \\ \mathbf{J}_m &= \mu_0 \frac{\partial \mathbf{M}}{\partial t} , \end{aligned} \tag{3.2.1a,b}$$

that are valid for macroscopic laws in a material as described in Unit 1.5. Starting from Maxwell’s equations (1.5.9a, b) for a model with magnetic charge, we can express the total electric and magnetic currents $\mathbf{J}$ and $\mathbf{J}_m$ as

$$\begin{aligned} \mathbf{J} &= \mathbf{J}_s + \frac{\partial \mathbf{P}}{\partial t} = \nabla \times \mathbf{H} - \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} , \\ \mathbf{J}_m &= \mu_0 \frac{\partial \mathbf{M}}{\partial t} = -\nabla \times \mathbf{E} - \mu_0 \frac{\partial \mathbf{H}}{\partial t} . \end{aligned} \tag{3.2.2a,b}$$

If we substitute these equations for currents $\mathbf{J}$ and $\mathbf{J}_m$ into the expression $p = \mathbf{J} \cdot \mathbf{E} + \mathbf{J}_m \cdot \mathbf{H}$, we shall transform the expression for power delivered to charge into a form that contains only the field quantities $\mathbf{E}$ and $\mathbf{H}$. After rearrangement we can write the expression for power

$$p = \mathbf{E} \cdot \mathbf{J} + \mathbf{H} \cdot \mathbf{J}_m = -\nabla \cdot (\mathbf{E} \times \mathbf{H}) - \frac{\partial}{\partial t} \left(\frac{\epsilon_0}{2} |\mathbf{E}|^2 + \frac{\mu_0}{2} |\mathbf{H}|^2 \right), \quad (3.2.3)$$

that the electromagnetic field delivers to all electric and magnetic charge in motion within unit volume of free space or a material. *The equation (3.2.4) is Poynting's theorem in differential form.* It represents a fundamental expression of the conservation of electromagnetic energy in a general case.

3.2.1.2. Poynting's theorem in an integral form

By integration of both sides of expression (3.2.4) over the total volume V enclosed by surface S , as shown in Figure 3.2.1, we obtain

$$\int_V p dV = \int_V (\mathbf{E} \cdot \mathbf{J} + \mathbf{H} \cdot \mathbf{J}_m) dV = - \int_V \nabla \cdot (\mathbf{E} \times \mathbf{H}) dV - \frac{d}{dt} \int_V \left(\frac{\epsilon_0}{2} |\mathbf{E}|^2 + \frac{\mu_0}{2} |\mathbf{H}|^2 \right) dV.$$

The total derivative has substituted for the partial derivative because volume V is independent of time.

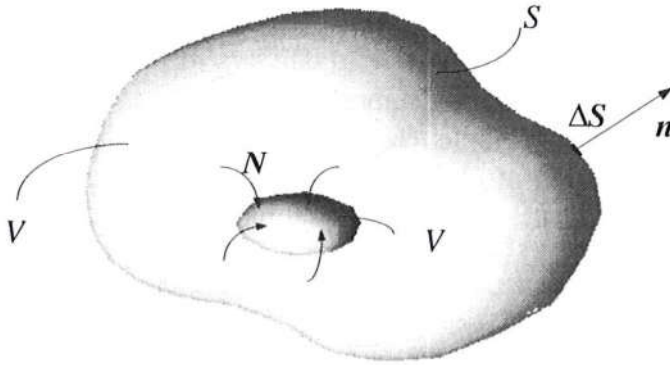


Figure 3.2.1. Flow of energy in a volume V enclosed by a surface S .

By application of Gauss' divergence theorem, we obtain an *expression of Poynting's energetic theorem in integral form*

$$\int_V (\mathbf{E} \cdot \mathbf{J} + \mathbf{H} \cdot \mathbf{J}_m) dV = - \oint_S (\mathbf{E} \times \mathbf{H}) \cdot \mathbf{n} dS - \frac{d}{dt} \int_V \left(\frac{\epsilon_0}{2} |\mathbf{E}|^2 + \frac{\mu_0}{2} |\mathbf{H}|^2 \right) dV. \quad (3.2.4)$$

Expressions (3.2.3) and (3.2.4) represent the differential and integral form of Poynting's theorem. They are valid in general in both free space and in space filled with a material.

3.2.1.3. Conservation of electromagnetic energy in a vacuum

We shall begin the explanation of Poynting's theorem about the conservation of energy in a time dependent electromagnetic field in general by interpreting individual terms in expressions (3.2.3) and (3.2.4). The meaning of the right-hand side of both expressions will become more clear if we introduce two new quantities. First of all we shall introduce a scalar quantity

$$w = \frac{1}{2} \epsilon_0 |\mathbf{E}|^2 + \frac{1}{2} \mu_0 |\mathbf{H}|^2 , \quad (3.2.5)$$

which we can describe as *the volume density of energy stored in an electromagnetic field* ($\mathbf{E}$, $\mathbf{H}$) in free space.

The volume density of stored energy w for time dependent electromagnetic fields distributed throughout free space is a function of time too. Therefore, the notion of time dependent energy density at different points in space should be associated with a hypothesis about the flow of energy in the field. This idea is implicit if we want to retain the principle of conservation of energy. In any case, the change in field strengths ($\mathbf{E}$, $\mathbf{H}$) and in energy density w should be associated with the idea of energy flow in an electromagnetic field. Therefore, we introduce a new vector quantity

$$\mathbf{N} = \mathbf{E} \times \mathbf{H} , \quad (3.2.6)$$

known as *Poynting's vector*.

First of all, we shall examine the differential form of Poynting's theorem (3.2.3), which we can now write in a simpler form as

$$-\nabla \cdot \mathbf{N} - \frac{\partial w}{\partial t} = p . \quad (3.2.7)$$

The right-hand side is derived from the forces acting on all kinds of electric and magnetic charge and it represents the energy an electromagnetic field delivers in unit time to all charge in motion inside unit volume of free space. In other words, it is the power an electromagnetic field supplies to charge in motion in unit volume. This power shall be extracted from the electromagnetic field either from its locally stored energy in the vicinity in which the charge is placed, or from somewhere else, including from the transmission of energy by the field through space. The left-hand side of this equation is reduced to only two terms that contain only fields $\mathbf{E}$ and $\mathbf{H}$ and their space and time derivatives. This suggests that these terms represent inherent characteristics of an electromagnetic field associated with the storage and flow of energy within it. From the mathematical form of these expressions, we shall identify one as the divergence of energy flow, and the other as the value of time change of electromagnetic energy stored in a field.

The first term in equation (3.2.7) is the density with which the vector field $\mathbf{N}$ sinks at any point in space in which an electromagnetic field exists. We can explain it as being the density of the power flow from the electromagnetic field outside ΔV into ΔV (according to Figure 3.2.1). *This means that $\mathbf{N}$ can be interpreted as the flow of electromagnetic power per unit of surface that is transmitted through a space occupied by the electromagnetic field.*

The second term on the left-hand side of equation (3.2.7) is a negative time derivative of the scalar quantity w , which possesses the dimensions of energy per unit volume. This leads us to the explanation of the second term as being the density of which the power is delivered locally to charge by means of the electromagnetic field. In this case, our assumption that w in expression (3.2.5) is the density of energy stored in an electromagnetic field is valid.

Accordingly, expression (3.2.7) representing Poynting's theorem in a differential form can be explained as follows: if the power density is delivered to the charge in unit volume ΔV , then this power is partially taken from the energy w stored locally with the help of fields $\mathbf{E}$ and $\mathbf{H}$ within ΔV ($-\partial w/\partial t$ is the amount of decrease in energy in ΔV) and partially transmitted from a surrounding space wherever the electromagnetic field ($\mathbf{E}$, $\mathbf{H}$) exists ($-\nabla \cdot \mathbf{N}$ is the amount of energy that flows from the whole outer space into a unit volume ΔV in unit time). An interesting case arises in a charge-free space. Then the power delivered to charge is $p = 0$. The equation (3.2.7) in this case takes the form

$$\nabla \cdot \mathbf{N} + \frac{\partial w}{\partial t} = 0, \quad (3.2.8)$$

which is similar to the continuity equation for charge conservation. Analogously, the continuity equation (3.2.8) is a statement of the *conservation of electromagnetic energy in a charge-free region*, where w is a function of energy distribution and $\mathbf{N}$ is a vector of energy flow.

An integral form of Poynting's theorem given by expression (3.2.4) is used more frequently. Its simpler form is

$$-\oint_S \mathbf{N} \cdot \mathbf{n} dS - \frac{d}{dt} \int_V w dV = P, \quad (3.2.9)$$

where P is the total (instantaneous) electromagnetic power delivered to the charge in motion inside a closed volume V . The negative integral of $\mathbf{N}$ over the closed surface S (according to Figure 3.2.1) on the left-hand side must be the total electromagnetic power that flows into volume V from the surrounding space through the closed surface S . The second term represents a time change (or a decrease if we denote it with a negative sign) of the total energy stored in an electromagnetic field inside closed volume V .

On the basis of equation (3.2.9) and the principle of energy conservation, we can conclude that the total electromagnetic power delivered to all electric and magnetic charge inside volume V must have come from a time change resulting in a decrease of the total electromagnetic energy W stored in the electromagnetic field inside V , as well as from the "pure" flow of electromagnetic power P_N into V from the surrounding space (the negative sign denotes a flow into V through closed surface S).

3.2.2. Stored electromagnetic energy in a vacuum

So far we have restricted our analysis to the study of the energy of an electromagnetic field in a vacuum. We have posed no restrictions in regard to the nature of time changes of the field quantities. In considering these facts, we shall begin with the expression (3.2.5) for a scalar quantity

$$w = \frac{1}{2} \epsilon_0 |\mathbf{E}|^2 + \frac{1}{2} \mu_0 |\mathbf{H}|^2, \quad (3.2.10)$$

which we introduced as the volume density of energy stored in an electromagnetic field in

a vacuum. Therefore, it is obvious that electromagnetic energy is stored in any part of space where the electromagnetic field ($\mathbf{E}, \mathbf{H}$) exists. In this way, we associate energy with an electromagnetic field. As can be seen from (3.2.10), *the stored energy* w is a function of the field quantities $\mathbf{E}$ and $\mathbf{H}$, which means that it is independent of both free and bound sources in a material. This fact enables the separation of electromagnetic energy from other forms of energy that can simultaneously exist in a space. Furthermore, we can separate the density of stored energy w in an electromagnetic field into one part,

$$w_E = \frac{1}{2} \epsilon_0 |\mathbf{E}|^2, \quad (3.2.11)$$

associated with the electric field, thus representing the density of energy stored in the electric field in a vacuum. and into another part,

$$w_H = \frac{1}{2} \mu_0 |\mathbf{H}|^2, \quad (3.2.12)$$

associated with the magnetic field, thus representing the density of energy stored in the magnetic field in a vacuum.

On the other hand, Poynting's vector, i.e. the density of flux of electromagnetic power, results from the interaction of the electric and magnetic fields, and therefore it cannot be resolved into components associated with $\mathbf{E}$ and $\mathbf{H}$ separately. From a known density of electromagnetic energy w , it is possible to determine the total stored electromagnetic energy W in a space with a given volume V with the help of

$$W = \int_V w dV = \int_V \left(\frac{\epsilon_0}{2} |\mathbf{E}|^2 + \frac{\mu_0}{2} |\mathbf{H}|^2 \right) dV. \quad (3.2.13)$$

In the same way we can separate the total stored electromagnetic energy into the total energy stored in the electric field in a vacuum inside V

$$W_E = \int_V w_E dV = \int_V \frac{\epsilon_0}{2} |\mathbf{E}|^2 dV, \quad (3.2.14)$$

and into the total energy stored in the magnetic field in a vacuum inside V

$$W_H = \int_V w_H dV = \int_V \frac{\mu_0}{2} |\mathbf{H}|^2 dV. \quad (3.2.15)$$

Finally, we get $W = W_E + W_H$, which is the total electromagnetic energy stored in the system in a vacuum. This is only correct for free space. If a material is in the field, then $W = W_E + W_H$ represents only a part of the total energy stored in fields $\mathbf{E}$ and $\mathbf{H}$ in a given system. In this case, additional energy will be stored as a result of the polarization and magnetization of charge within the system that contains the material.

We can notice a certain non-uniqueness in the derivation of terms for stored electric and magnetic energy in an electromagnetic field. The energy densities introduced by

expressions (3.2.5), (3.2.11) and (3.2.12) are derived from the term with a time derivative in Poynting's theorem (3.2.3). It is obvious that one can always add any time independent term to both w_E and w_H without affecting their time derivative, meaning without affecting the balance of power in Poynting's theorem. This ambiguity can be easily avoided if we simply choose an arbitrary zero level of density of stored electromagnetic energy in a system when it is electromagnetically "dead", which means that all electromagnetic quantities are equal to zero.

3.2.3. The flow of electromagnetic energy. Poynting's vector

In Section 3.2.1. we derived Poynting's theorem in differential form, resulting in expression (3.2.4) for the conservation of electromagnetic energy in an elemental volume in a vacuum. In this way, we introduced the vector quantity given by expression (3.2.7)

$$\mathbf{N} = \mathbf{E} \times \mathbf{H} , \quad (3.2.16)$$

known as *Poynting's vector*. It represents the density of flow of electromagnetic power, which means the energy that is flowing in unit time through a unit surface in the vicinity of a point in free space in which the electromagnetic field exists. The unit for $\mathbf{N}$ in SI system is the *watt per square metre* (W/m²). The integral of $\mathbf{N}$,

$$\oint_S (\mathbf{E} \times \mathbf{H}) \cdot \mathbf{n} dS = \oint_S \mathbf{N} \cdot \mathbf{n} dS = P_N , \quad (3.2.17)$$

which appears in expressions (3.2.4) and (3.2.9) uniquely determines the flow of electromagnetic energy through a closed surface S . It is important to warn that our explanation of Poynting's vector $\mathbf{N} = \mathbf{E} \times \mathbf{H}$ as being the density of electromagnetic power at every point in a space is not strictly justified for both forms of Poynting's theorem. A difficulty arises from the fact that, although the divergence of $\mathbf{E} \times \mathbf{H}$ (or an integral of $\mathbf{E} \times \mathbf{H}$ over a closed surface) always gives an exact value of the power transmitted by an electromagnetic field, we can always add any source-free vector $\mathbf{C}$ (whose divergence is equal to zero: $\nabla \cdot \mathbf{C} = 0$) to the vector $\mathbf{N} = \mathbf{E} \times \mathbf{H}$ to obtain a new vector $\mathbf{N}' = \mathbf{N} + \mathbf{C} = \mathbf{E} \times \mathbf{H} + \mathbf{C}$. Both vectors $\mathbf{N}$ and $\mathbf{N}'$ satisfy both forms of Poynting's theorem. In other words, an extrapolation of the true value of vector $\mathbf{N}$ from its divergence in a given region is not a unique procedure. Acceptance of the definition of Poynting's vector $\mathbf{N} = \mathbf{E} \times \mathbf{H}$ itself, without an additional term, as the density of flux of electromagnetic power at a point has a few advantages:

- 1) it always gives exact values for the total power that flows from or into a closed region;
- 2) in an electrostatic system where $\mathbf{H} = 0$ and in a magnetostatic system where $\mathbf{E} = 0$, the resulting value for $\mathbf{N} = \mathbf{E} \times \mathbf{H} = 0$, which is in accordance with the non-existence of any experimentally observed flux of power.
- 3) it is very simple.

Accordingly, we postulate that the flux of electromagnetic power through a unit surface at every point in space is defined by Poynting's vector $\mathbf{N} = \mathbf{E} \times \mathbf{H}$.

3.3. A MATERIAL IN AN ELECTROMAGNETIC FIELD

In this unit we shall analyse the transmission and storage of electromagnetic energy in a material that has been put into an electromagnetic field. We shall establish that expressions derived for energetic relationships for the motion of charge in an electromagnetic field in a vacuum can be used for a material added to the field.

3.3.1. Electromagnetic energy absorbed in a material

If we put a material into an electromagnetic field, a mutual interaction will start. Due to free and bound charge in the material, various effects will occur. The field will be changed too, and the material's state will be different to what it was before it was put into the field.

We are interested in energetic relationships. In order to establish them, we shall use Poynting's theorem in differential form given by expressions (3.2.3) and (3.2.7). The electromagnetic power delivered from the field to the material (absorbed at a point in a material) is determined by

$$p = \mathbf{J} \cdot \mathbf{E} + \mathbf{J}_m \cdot \mathbf{H} , \quad (3.3.1)$$

where $\mathbf{J}$ and $\mathbf{J}_m$ are densities of electric and magnetic current in the material. It was shown in Unit 1.4, Sections 1.4.2, 1.4.5 and 1.4.8 that $\mathbf{J}$ and $\mathbf{J}_m$ are in direct relationship with the conductive, dielectric and magnetic properties of the material.

Therefore, we are now able to transform expression (3.3.1) into a form that expresses the absorbed electromagnetic power consumed during the polarization and magnetization of the material and during the process of conduction (of charge) in the material. In the model of a material with magnetic charge and current, the density of magnetic currents $\mathbf{J}_m$ in a magnetic material at rest results from a time variation in magnetization $\mathbf{M}$ is

$$\mathbf{J}_m = \mu_0 \frac{\partial \mathbf{M}}{\partial t} . \quad (3.3.2)$$

It is obvious that in expression (3.3.1), the density of power delivered by the electromagnetic field to a material when its magnetization varies with time is represented by the term

$$p_M = \mathbf{J}_m \cdot \mathbf{H} = \mathbf{H} \cdot \mu_0 \frac{\partial \mathbf{M}}{\partial t} . \quad (3.3.3)$$

In other words, the time dependent magnetization of a material causes an exchange of energy p_M (in unit time and in unit volume) between the electromagnetic field and the material.

In addition, the density of the electric current $\mathbf{J}$ is generally constituted of a free (conductive) current $\mathbf{J}_s$ and of a polarization component $\mathbf{J}_p$, and we can express it as

$$\mathbf{J} = \mathbf{J}_s + \frac{\partial \mathbf{P}}{\partial t} . \quad (3.3.4)$$

The polarization component of current $\mathbf{J}_p$ represents the power density that the macroscopic electromagnetic field delivers to a material when its polarization is changing with time and contributes to the component of power density

$$p_p = \mathbf{E} \cdot \frac{\partial \mathbf{P}}{\partial t} . \quad (3.3.5)$$

In other words, the term p_p represents the exchange of energy between an electromagnetic field and a material caused by a time dependent polarization. The component of power density

$$p_g = \mathbf{J}_s \cdot \mathbf{E} , \quad (3.3.6)$$

which results from free currents $\mathbf{J}_s$ can be associated with various physical phenomena. If free currents result only from conduction (of charge) in a material, the forces produced by the electric field are in balance with the forces of friction that depend on the velocity of displacement of charge. In this case the power delivered by the electromagnetic field is lost as heat. This process leads to *Joule's heating of a resistor*. The density of conductive current is

$$\mathbf{J}_s = \kappa \mathbf{E} , \quad (3.3.7)$$

and the density of the power lost in a linear conductor is given by

$$p_g = \kappa |\mathbf{E}|^2 . \quad (3.3.8)$$

Also, $\mathbf{J}_s$ can represent other densities of free currents as well. For example, $\mathbf{J}_s$ can be the density of independent currents $\mathbf{J}_i$ flowing from a voltage or current source. In this case, $\mathbf{E} \cdot \mathbf{J}_i$ represents the power delivered from the field to a source (if it is positive), and from a source to the field (if it is negative). Therefore,

$$p_i = -\mathbf{J}_i \cdot \mathbf{E} , \quad (3.3.9)$$

(with included negative sign) represents the density of power delivered from a source to the field. In general, we can say that $\mathbf{J}_s \cdot \mathbf{E}$ represents the electromagnetic power converted into some other form of energy (if it is positive) or vice versa (if it is negative). Finally, the right-hand side of expression (3.3.1) becomes

$$p = \mathbf{E} \cdot \mathbf{J}_i + \kappa |\mathbf{E}|^2 + \mathbf{E} \cdot \frac{\partial \mathbf{P}}{\partial t} + \mathbf{H} \cdot \mu_0 \frac{\partial \mathbf{M}}{\partial t} = -p_i + p_g + p_p + p_M , \quad (3.3.10)$$

if we combine the different forms of density of electromagnetic power delivered from the field to a material (strictly speaking to charge in a material). It is obvious that the power delivered in unit volume to charge can be absorbed in different ways within the material. It can be converted to another form of energy inside the independent source ($-p_i$), converted to heat and lost in a conductor (p_g), transformed into an increase in polarization of

a dielectric material (p_p), or transformed into an increase in magnetization of a magnetic material (p_M). Any one, all or none of these processes can happen at a given point of space, depending on the values of $\mathbf{E}$ and $\mathbf{H}$ and on the type of material at the point of interest.

3.3.2. Poynting's theorem in a material

When we put a material in a field, we can apply Poynting's theorem in differential form too. According to (3.2.3) and (3.2.7), Poynting's theorem in this case is

$$-\nabla \cdot \mathbf{N} - \frac{\partial w}{\partial t} = \mathbf{J} \cdot \mathbf{E} + \mathbf{J}_m \cdot \mathbf{H} = p. \quad (3.3.11)$$

In order to obtain a final differential form of Poynting's theorem in a material, we shall substitute expression (3.3.10) for the density of electromagnetic power p delivered from the field to a material and obtain

$$\begin{aligned} -\nabla \cdot (\mathbf{E} \times \mathbf{H}) - \mathbf{E} \cdot \mathbf{J}_i = & \kappa |\mathbf{E}|^2 + \mathbf{E} \cdot \frac{\partial \mathbf{P}}{\partial t} + \mathbf{H} \cdot \mu_0 \frac{\partial \mathbf{M}}{\partial t} + \\ & + \frac{\partial}{\partial t} \left(\frac{\epsilon_0}{2} |\mathbf{E}|^2 + \frac{\mu_0}{2} |\mathbf{H}|^2 \right), \end{aligned} \quad (3.3.12)$$

or in symbolic form

$$[-\nabla \cdot \mathbf{N} + p_i] = [p_g + p_P + p_M] + \left[\frac{\partial}{\partial t} (w_E + w_H) \right]. \quad (3.3.13)$$

The left-hand side of Poynting's theorem (3.3.13) represents the total power per unit volume delivered to a material at the point of interest. Part of that power is delivered from outside the unit volume (which is given by $-\nabla \cdot \mathbf{N}$), while the rest is delivered from an independent source within the unit volume (denoted as p_i).

The right-hand side of this expression of energetic balance must give all the forms in which the delivered power density can be converted at a given point in space. We can see from the right-hand side of equation (3.3.12) or (3.3.13) that the density of the delivered power can be converted into heat loss due to conduction (p_g), transformed into power needed for an increase in polarization (p_P) or magnetization (p_M) in a material, or it can supply an increase in stored electric and magnetic energy in unit volume around the point of interest ($\partial w_E / \partial t$ and $\partial w_H / \partial t$).

An integral form of Poynting's theorem in a material is obtained by the integration of expression (3.3.12) in a volume V enclosed by a surface S , as is shown in Figure 3.2.1. If we use Gauss' theorem for the transformation of the first term on the left-hand side into a surface integral, and if we replace the symbol of the total differential with a partial derivative as volume V is time independent, we obtain

$$\begin{aligned}
-\oint_S \mathbf{N} \cdot \mathbf{n} dS - \int_V \mathbf{E} \cdot \mathbf{J}_i dV = \int_V \left(\kappa |\mathbf{E}|^2 + \mathbf{E} \cdot \frac{\partial \mathbf{P}}{\partial t} + \mathbf{H} \cdot \mu_0 \frac{\partial \mathbf{M}}{\partial t} \right) dV + \\
+ \frac{d}{dt} \int_V \left(\frac{\epsilon_0}{2} |\mathbf{E}|^2 + \frac{\mu_0}{2} |\mathbf{H}|^2 \right) dV .
\end{aligned} \tag{3.3.14}$$

Poynting's theorem can be expressed as follows: the two terms on the left-hand side of equation (3.3.14) represent the total power that is delivered in volume V . A part of that delivered power enters into V from outside in the form of flux of electromagnetic power $\mathbf{N}$ passing through any surface S that encloses V . The rest of the delivered power is defined by the volume integral of p_i over V and originates from independent sources inside V .

The terms on the right-hand side of equation (3.3.14) represent the different forms in which the delivered power is absorbed in the material within volume V . The second integral on the right-hand side represents the total power needed for an increase in stored energy in the electric and magnetic field inside V .

3.3.3. Energy of polarization and electric energy

The answer to the question about what is actually happening to the electromagnetic power that drives the processes of polarization and magnetization in our macroscopic model of a material is not yet given. It is our intention to exactly determine that part of the electromagnetic power which is stored as energy in a form which can be reversed by polarization charge in a material, and how large the part lost in overcoming the internal friction forces that counteract the process of polarization is.

On the basis of analysis in Section 3.3.1, we know that the density of power delivered to bound charge that takes part in the process of polarization in a material (dielectric, insulator) is, according to equation (3.3.5), given as

$$p_p = \mathbf{E} \cdot \frac{\partial \mathbf{P}}{\partial t} . \tag{3.3.15}$$

The energy delivered per unit of the polarized material when the material is slowly polarized from an initial non-polarized state at a moment in time $t' = 0$ where $\mathbf{P}' = 0$, to a final state at a moment in time t'' where polarization $\mathbf{P}''$ is determined by the integral

$$W_p = \int_{t'=0}^{t''} \mathbf{E} \cdot \frac{\partial \mathbf{P}}{\partial t} dt = \int_{\mathbf{P}'=0}^{\mathbf{P}''} \mathbf{E} \cdot d\mathbf{P} . \tag{3.3.16}$$

In order to explain this integral, we shall use constitutive relationships for dielectric materials and insulators. To begin with, we shall analyse the non-linear, but single-valued (unique) constitutive expression that shows the relationship between $\mathbf{E}$ and $\mathbf{P}$ according to Figure 3.3.1a. The integral in (3.3.16) represents the area above the polarization curve $\mathbf{P} = \mathbf{P}(\mathbf{E})$.

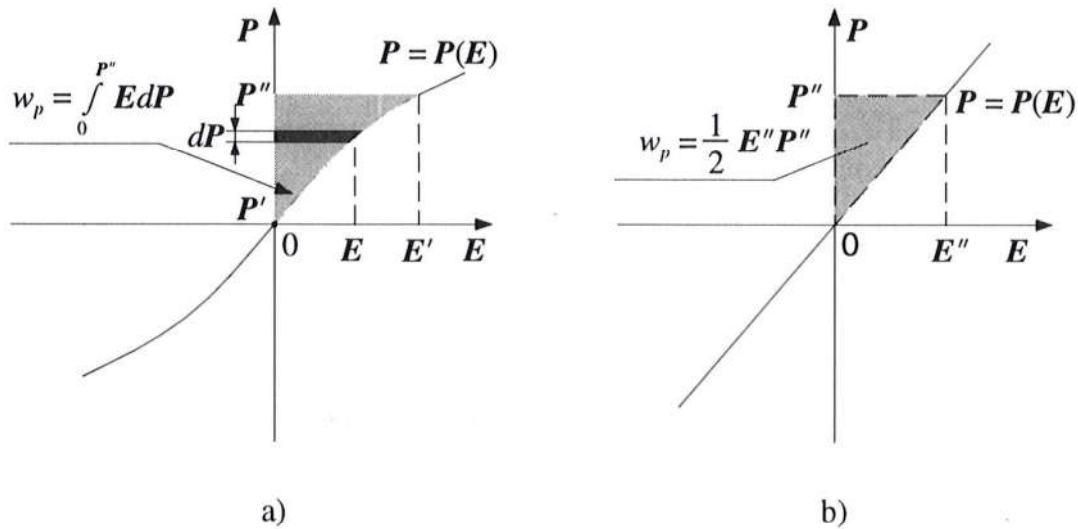


Figure 3.3.1. Graphical presentation of the density of polarization energy in a lossless dielectric material;
a) non-linear and b) linear

As long as the curve $P = P(E)$ is single-valued the amount of energy that will be retrieved when the fields E and P become equal to zero in the process of polarization will be equal to the accumulated energy and the density of polarization energy, given by equation (3.3.16), is stored in a dielectric in a reversible form.

There is a special case of a unique constitutive relationship for a linear dielectric material. The polarization curve $P = P(E)$ is then represented by a straight line through the origin as shown in Figure 3.3.1.b). The energy of polarization is reduced to the surface of the triangle W_p .

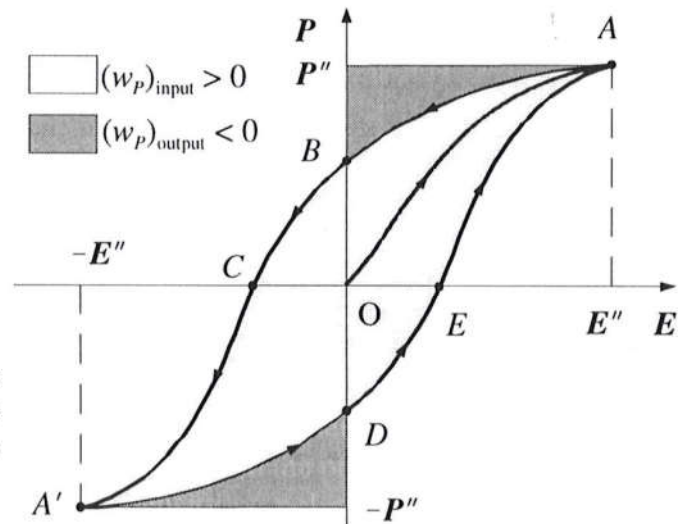


Figure 3.3.2. Graphical presentation of the delivering and retrieving of electromagnetic energy in the process of polarization in a lossy dielectric

On the other hand, a dielectric material can have a non-unique polarization curve $P = P(E)$, which has the form of a hysteresis loop. Let us analyse a ferroelectric material which has a hysteresis loop as shown in Figure 3.3.2. If we neglect the initial part of the curve from 0 to A, we see that moving along a branch of the hysteresis loop from A over B and C to A' results in the delivering of energy to the material, and that moving along a branch from A' over D and E to A results in a partial retrieving of the energy from the

material to the field in the form of electromagnetic energy. Starting from the point A , over B , C , A' , D , E , and again back to A , meaning one cycle has been completed, we discover that the amount of energy density equals the area enclosed by the hysteresis loop, as shown in Figure 3.3.2, and which is lost from the system when each cycle around the hysteresis loop is completed. This energy per unit volume is converted into the work needed to overcome the frictional forces that cause hysteresis. Finally, it usually leaves the material as heat. Therefore, the density of polarization energy that is calculated in one cycle as

$$w_p = \oint \mathbf{E} \cdot d\mathbf{P} , \quad (3.3.17)$$

is generally known as the hysteresis loss.

It is obvious that in an electromagnetic field that is alternating with *frequency* f (*period per second* or *hertz*) a material whose polarization in one cycle is represented by the hysteresis loop shown in Figure 3.3.2. will have a hysteresis loss given by the expression

$$p_p = f \oint \mathbf{E} \cdot d\mathbf{P} . \quad (3.3.18)$$

These losses are measured in *watts per cubic metre* (W/m^3).

It is often convenient to link the density of energy stored in an electric field to that which is delivered to a material in the process of polarization. We call the sum of these two energies the density of electric energy w_e delivered to a material

$$w_e = w_E + w_p = \frac{\epsilon_0}{2} |\mathbf{E}|^2 + \int_0^{\mathbf{P}} \mathbf{E} \cdot d\mathbf{P} . \quad (3.3.19)$$

We can rearrange this equation as

$$w_e = \epsilon_0 \int_0^{\mathbf{E}} \mathbf{E} \cdot d\mathbf{E} + \int_0^{\mathbf{P}} \mathbf{E} \cdot d\mathbf{P} = \int_0^{(\epsilon_0 \mathbf{E} + \mathbf{P})} \mathbf{E} \cdot d(\epsilon_0 \mathbf{E} + \mathbf{P}) = \int_0^{\mathbf{D}} \mathbf{E} \cdot d\mathbf{D} , \quad (3.3.20)$$

since the flux density is $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$. Equation (3.3.20) for macroscopic electric energy stored in unit volume of a dielectric material with a single-valued curve $\mathbf{D} = \mathbf{D}(\mathbf{E})$ represents the area between this curve and the $\mathbf{D}$ axis. For the special case of a linear dielectric, i.e. a dielectric whose permittivity ϵ in the expression $\mathbf{D} = \epsilon \mathbf{E}$ is independent of $\mathbf{E}$, equation (3.3.20) reduces to

$$w_e = \epsilon \int_0^{\mathbf{E}} \mathbf{E} \cdot d\mathbf{E} = \frac{\epsilon}{2} |\mathbf{E}|^2 = \frac{1}{2} \mathbf{E} \cdot \mathbf{D} . \quad (3.3.21)$$

We can say that in a dielectric material with a single-valued polarization curve the total electric energy, which is equal to the sum of the stored energy in the electric field and the energy of polarization, is stored in the system in a reversible form.

The total electric energy W_e delivered to a system is simply the volume integral of w_e over the whole of space V

$$W_e = \int_V w_e dV = \int_V \left(\int_0^D \mathbf{E} \cdot d\mathbf{D} \right) dV . \quad (3.3.22)$$

In the special case of a linear material with permittivity ϵ , this integral is

$$W_e = \int_V \frac{1}{2} \epsilon |\mathbf{E}|^2 dV . \quad (3.3.23)$$

We can derive an expression for electromagnetic energy in a thermodynamic way by equating the energy and work needed for forming a system of electric charge, insulated conductors and dielectric bodies in static balance.

3.3.4. Energy of magnetization and magnetic energy

The total energy delivered in unit time to a material that is slowly magnetized from an initial unexcited state at a moment in time $t' = 0$, where $\mathbf{M}' = 0$ everywhere, to a final state at a moment in time t'' , where the density of magnetization is $\mathbf{M}''$ is determined by

$$w_M = \int_{t'=0}^{t''} \mathbf{H} \cdot \frac{\partial}{\partial t} (\mu_0 \mathbf{M}) dt = \mu_0 \int_{\mathbf{M}'=0}^{\mathbf{M}''} \mathbf{H} \cdot d\mathbf{M} . \quad (3.3.24)$$

The meaning of this integral for density of magnetization energy is completely analogous to expression (3.3.16) for the density of polarization energy derived in the previous section. For magnetic materials with a single-valued magnetization curve, the density of magnetization energy can be represented by the area enclosed by curve $\mathbf{M} = \mathbf{M}(\mathbf{H})$ and the $\mathbf{M}$ axis. This energy is stored in a magnetic material in a reversible form. If the material is linear, the expression for density of magnetization energy is simplified to the form

$$w_M = \frac{1}{2} \mathbf{H} \cdot \mu_0 \mathbf{M} . \quad (3.3.25)$$

In a magnetic material with a hysteresis loop, an amount of energy equivalent to the area enclosed by the hysteresis loop is lost over every magnetization cycle

$$w_M = \mu_0 \oint \mathbf{H} \cdot d\mathbf{M} . \quad (3.3.26a)$$

This energy is lost from the electromagnetic system and is usually converted into heat in a magnetic material. The corresponding hysteresis power loss in a material which is in an alternating electromagnetic field with frequency f is

$$P_M = f \mu_0 \oint \mathbf{H} \cdot d\mathbf{M} . \quad (3.3.26b)$$

We can prove that hysteresis loss can be expressed as

$$p_M = f \oint \mathbf{H} \cdot d\mathbf{B} , \quad (3.3.26c)$$

since magnetic induction is $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$.

As in the previous case, we shall link the density of magnetization energy w_M to the density of energy stored in the magnetic field w_H defined by equation (3.2.12) in order to obtain the total density of magnetic energy w_m delivered to a material

$$w_m = w_H + w_M = \frac{\mu_0}{2} |\mathbf{H}|^2 + \mu_0 \int_0^{\mathbf{M}} \mathbf{H} \cdot d\mathbf{M} = \mu_0 \int_0^{\mathbf{H}} \mathbf{H} \cdot d\mathbf{H} + \mu_0 \int_0^{\mathbf{M}} \mathbf{H} \cdot d\mathbf{M} = \int_0^{\mathbf{B}} \mathbf{H} \cdot d\mathbf{B} , \quad (3.3.27)$$

since the induction is $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$ and the expression $d\mathbf{B} = \mu_0(d\mathbf{H} + d\mathbf{M})$ is valid.

For a magnetic material with a single-valued constitutive relationship $\mathbf{B} = \mathbf{B}(\mathbf{H})$, w_m represents the total magnetic energy stored in unit volume around a given point inside the material. In the special case of a linear material, equation (3.3.27) reduces to

$$w_m = \mu \int_0^{\mathbf{H}} \mathbf{H} \cdot d\mathbf{H} = \frac{\mu}{2} |\mathbf{H}|^2 = \frac{1}{2} \mathbf{H} \cdot \mathbf{B} . \quad (3.3.28)$$

The total magnetic energy w_m delivered to a system over a whole given space V is

$$W_m = \int_V w_m dV = \int_V \left(\int_0^{\mathbf{B}} \mathbf{H} \cdot d\mathbf{B} \right) dV , \quad (3.3.29)$$

for a system in its final state defined by $\mathbf{H}$ and $\mathbf{B}$. For a linear material with permeability μ this amounts to

$$W_m = \int_V \frac{1}{2} \mu |\mathbf{H}|^2 dV .$$

3.3.5. Poynting's theorem and electric and magnetic energy in materials

For linear materials with single-valued polarization and magnetization curves, the differential and integral forms of Poynting's theorem can be simply written as

$$\begin{aligned} -\nabla \cdot (\mathbf{E} \times \mathbf{H}) - \mathbf{E} \cdot \mathbf{J}_i &= \kappa |\mathbf{E}|^2 + \frac{\partial}{\partial t} \left(\frac{\epsilon}{2} |\mathbf{E}|^2 + \frac{\mu}{2} |\mathbf{H}|^2 \right) , \\ -\oint_S (\mathbf{E} \times \mathbf{H}) \cdot \mathbf{n} dS - \int_V \mathbf{E} \cdot \mathbf{J}_i dV &= \int_V \kappa |\mathbf{E}|^2 dV + \frac{d}{dt} \int_V \left(\frac{\epsilon}{2} |\mathbf{E}|^2 + \frac{\mu}{2} |\mathbf{H}|^2 \right) dV \end{aligned} \quad (3.3.30a,b)$$

Finally, we can say that integration of the expression for macroscopic density of total stored energy, and of the expression for lost specific power over the whole volume of a

passive material inside a given system must always result in expressions that are dependent on the amounts of energy and power delivered to the material from outside by means of physically measurable electromagnetic fields in neighbouring regions of free space.

3.4. VOLUME FORCES AND SURFACE STRESSES

Historically, the first research in electricity and magnetism in the eighteenth century was carried out in the area of the effects of forces acting on charged particles at rest and in motion. According to this, the force between charges at rest is calculated by Coulomb's law, which is a typical way of representing the sense of electrical effects with a so called "acting at a distance" approach.

From the middle of the nineteenth century, greater significance was attributed to the electric and magnetic field around a charge. We denote this other modern approach as "near acting". A direct consequence of the near acting concept was Maxwell's theory of an electromagnetic field as expressed by Maxwell's equations.

One of the significant successes of Maxwell's electromagnetic field theory was its ability to express volume force using the surface stress transmitted through the surface enclosing a given volume. This enabled a description of mechanical effects in electromagnetic fields with the help of the action of this force transmitted through space over an infinitesimally small distance.

3.4.1. Surface stresses in free space

In order to express the volume force on charge and currents with the help of the surface stress transmitted locally from point to point, we shall imagine a bounded region of free space that contains a given distribution of free charge and currents. Let the field have been produced partially by charge and currents inside the region and partially by charge and currents outside the given region. Then Maxwell's equations are valid at any point of free space within volume V bounded by closed surface S

$$\begin{aligned}\nabla \times \mathbf{B} &= \mu_0 \mathbf{J}_s + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \quad \nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t}, \\ \nabla \cdot \mathbf{E} &= \frac{\rho_s}{\epsilon_0}, \quad \nabla \cdot \mathbf{B} = 0.\end{aligned}\tag{3.4.1a - d}$$

The fields $\mathbf{E}$ and $\mathbf{B}$ are acting with volume forces on the distribution of free charge ρ_s and currents $\mathbf{J}_s$

$$\mathbf{F}_{MECH} = \mathbf{F}_e + \mathbf{F}_m = \int_V \rho_s \mathbf{E} dV + \int_V (\mathbf{J}_s \times \mathbf{B}) dV.\tag{3.4.2}$$

In order to obtain an expression for surface forces, we shall rearrange equations (3.4.1a - d) and (3.4.2) and introduce an electric tensor of second order $\tilde{S}^e$ with its nine components S_{jk}^e given in Table 3.4.1,

Table 3.4.1. Components S_{jk}^e of the electric tensor

	$k = x$	$k = y$	$k = z$
$j = x$	$\epsilon_0 E_x^2 - \frac{\epsilon_0}{2} E ^2$	$\epsilon_0 E_x E_y$	$\epsilon_0 E_x E_z$
$j = y$	$\epsilon_0 E_y E_x$	$\epsilon_0 E_y^2 - \frac{\epsilon_0}{2} E ^2$	$\epsilon_0 E_y E_z$
$j = z$	$\epsilon_0 E_z E_x$	$\epsilon_0 E_z E_y$	$\epsilon_0 E_z^2 - \frac{\epsilon_0}{2} E ^2$

as well as a magnetic tensor of second order $\tilde{S}^m$, with components shown in Table 3.4.2.

Table 3.4.2. Components S_{jk}^m of the magnetic tensor

	$k = x$	$k = y$	$k = z$
$j = x$	$B_x^2 \frac{1}{\mu_0} - B ^2 \frac{1}{2\mu_0}$	$B_x B_y \frac{1}{\mu_0}$	$B_x B_z \frac{1}{\mu_0}$
$j = y$	$B_y B_x \frac{1}{\mu_0}$	$B_y^2 \frac{1}{\mu_0} - B ^2 \frac{1}{2\mu_0}$	$B_y B_z \frac{1}{\mu_0}$
$j = z$	$B_z B_x \frac{1}{\mu_0}$	$B_z B_y \frac{1}{\mu_0}$	$B_z^2 \frac{1}{\mu_0} - B ^2 \frac{1}{2\mu_0}$

If we introduce a new tensor $\tilde{S} = \tilde{S}^e + \tilde{S}^m$ with components

$$S_{jk} = S_{jk}^e + S_{jk}^m, \quad (3.4.3)$$

we obtain

$$\mathbf{f}_{MECH} = \mathbf{f}_e + \mathbf{f}_m = \rho_s \mathbf{E} + \mathbf{J}_s \times \mathbf{B} = \text{div} \tilde{S} - \epsilon_0 \frac{\partial}{\partial t} (\mathbf{E} \times \mathbf{B}). \quad (3.4.4)$$

Equation (3.4.4) is a very important expression by which one can express and calculate the volume force that acts on elements of charge and currents in unit volume at any point of free space using only field vectors $\mathbf{E}$ and $\mathbf{B}$. After integration of equation (3.4.4) over volume V and the application of the divergence theorem on the tensors, we obtain

$$\mathbf{F}_T = \oint_S \tilde{S} \cdot \mathbf{n} dS = \mathbf{F}_e + \mathbf{F}_m + \epsilon_0 \frac{d}{dt} \int_V \mathbf{E} \times \mathbf{B} dV. \quad (3.4.5)$$

The surface integral on the left-hand side of this equation must be the *total force* $\mathbf{F}_T$ that acts on a given volume of free space. Let us introduce a new vector

$$\mathbf{t} = \tilde{S} \cdot \mathbf{n}, \quad (3.4.6)$$

which is known from mechanics as the *vector of (electromagnetic) stress*. We shall call it

“the electromagnetic stress” and it generally represents the total force per unit surface in an electromagnetic field, or simply the density of surface force in an electromagnetic field. Obviously, we can calculate the total electromagnetic force acting on volume V of free space in which free charge ρ_s and currents $\mathbf{J}_s$ exist

$$\mathbf{F}_T = \oint_S \mathbf{t} dS = \mathbf{F}_e + \mathbf{F}_m + \epsilon_0 \frac{d}{dt} \int_V \mathbf{E} \times \mathbf{B} dV = \mathbf{F}_{MECH} + \frac{1}{c_0^2} \frac{d}{dt} \int_V \mathbf{N} dV . \quad (3.4.7)$$

This expression states that electromagnetic stress exists over an imagined surface that encloses a defined volume in a free space, if an electromagnetic field exists in the volume, even in the case where there is no free charge and currents within the volume. As a consequence, a total non-zero force $\mathbf{F}_T$ may act within this free space. Only in static fields is this force always equal to zero.

In the case of a static distribution of free charge and currents, the fields are independent of time. The last term on the right-hand side of equation (3.4.7) vanishes and the force $\mathbf{F}_T$ with which the static fields act on a static distribution of charge and currents is

$$\mathbf{F}_T = \oint_S \mathbf{t} dS = \mathbf{F}_e + \mathbf{F}_m = \mathbf{F}_{MECH} , \quad (3.4.8)$$

which is an integral of the stress vector $\mathbf{t}$ over any closed surface that encloses all of these charges and currents within a free space.

3.4.1.1. Maxwell's stress tensor

We have introduced the electric and magnetic tensors $\tilde{S}^e$ and $\tilde{S}^m$ that together form the so-called Maxwell's stress tensor $\tilde{S} = \tilde{S}^e + \tilde{S}^m$ as an arranged form of nine functions of time and space S_{jk}

$$S_{jk}(\mathbf{r}, t) = \epsilon_0 (E_j E_k - \frac{1}{2} |\mathbf{E}|^2 \delta_{jk}) + \frac{1}{\mu_0} (B_j B_k - \frac{1}{2} |\mathbf{B}|^2 \delta_{jk}) , \quad (3.4.9)$$

from which we can calculate the density of electromagnetic forces and total forces. In this equation, the *Kronecker delta* δ_{jk} is defined as $\delta_{jk} = 0$ for $j \neq k$ and $\delta_{jk} = 1$ for $j = k$. It is suitable to write Maxwell's stress tensor in matrix form

$$\tilde{S} = [S_{jk}] = \begin{bmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{yx} & S_{yy} & S_{yz} \\ S_{zx} & S_{zy} & S_{zz} \end{bmatrix} . \quad (3.4.10)$$

It is obvious from equation (3.4.9) and from Tables 3.4.1,2 that Maxwell's stress tensor is diagonally symmetric, $S_{jk} = S_{kj}$, and we have only to calculate using six components.

Maxwell and Faraday introduced the stress tensor in order to avoid calculations with forces acting at a distance. They postulated the existence of aether that occupies the whole of space and that is in a state of stress determined by $\tilde{S}$. This state of stress was responsi-

ble for the transmission of force from one charge to another, similar to the transmission of forces in an elastic medium. Research that followed showed that the idea of aether was not viable, and that the components of stress $\tilde{S}$ possess no physical reality. What we can mostly say is that forces can be calculated on the basis of the fictional existence of a stress state outlined by $\tilde{S}$. Therefore, the stress tensor $\tilde{S}$ is a notion that is independent of the actual presence of a material.

3.4.1.2. The vector of electromagnetic stress

We shall determine the vector of electromagnetic stress that was introduced with the help of (3.4.6) as $\mathbf{t} = \tilde{S} \cdot \mathbf{n}$. Using the equation

$$t_j = \sum_k S_{jk} n_k \quad ; \quad (j, k = x, y, z) , \quad (3.4.11)$$

we calculate its three orthogonal components that form the surface stress vector $\mathbf{t} = t_x \mathbf{a}_x + t_y \mathbf{a}_y + t_z \mathbf{a}_z$.

If we substitute the components of Maxwell's stress tensor given by the expression (3.4.10) into equation (3.4.11), we obtain components of the stress vector. We can arrange these components into a stress vector that acts on an element of the surface with a positive normal $\mathbf{n}$

$$\mathbf{t} = \left[(\mathbf{n} \cdot \epsilon_0 \mathbf{E}) \mathbf{E} - \mathbf{n} \frac{1}{2} \epsilon_0 |\mathbf{E}|^2 \right] + \left[\left(\mathbf{n} \cdot \frac{\mathbf{B}}{\mu_0} \right) \mathbf{B} - \mathbf{n} \frac{1}{2} \frac{|\mathbf{B}|^2}{\mu_0} \right] . \quad (3.4.12)$$

From equation (3.4.7) it follows that the surface density of total force given by expression (3.4.12) can be referred to as the electromagnetic stress acting on a surface that encloses the volume of interest.

It is suitable to divide the electromagnetic stress in equation (3.4.12) into two parts, so we write $\mathbf{t} = \mathbf{t}^e + \mathbf{t}^m$. By $\mathbf{t}^e$ we denote the stress caused only by the electric field $\mathbf{E}$, and by $\mathbf{t}^m$ we denote the stress caused only by the magnetic field $\mathbf{B}$, and we get

$$\mathbf{t}^e = \tilde{S}^e \cdot \mathbf{n} = (\mathbf{n} \cdot \epsilon_0 \mathbf{E}) \mathbf{E} - \mathbf{n} \frac{1}{2} \epsilon_0 |\mathbf{E}|^2, \quad (3.4.13a,b)$$

$$\mathbf{t}^m = \tilde{S}^m \cdot \mathbf{n} = \left(\mathbf{n} \cdot \frac{\mathbf{B}}{\mu_0} \right) \mathbf{B} - \mathbf{n} \frac{1}{2} \frac{|\mathbf{B}|^2}{\mu_0} .$$

Since electric and magnetic terms of stress possess the same form, in further analysis we shall focus our attention on the stress $\mathbf{t}^e$ caused only by the electric field $\mathbf{E}$, which is generally time dependent. The modulus of the electric stress vector is

$$|\mathbf{t}^e| = \frac{1}{2} \epsilon_0 |\mathbf{E}|^2 = w_E . \quad (3.4.14)$$

The direction of the stress vector $\mathbf{t}^e$ is determined by equation (3.4.13a). As can be seen from this equation, the stress vector consists of two components. One component of the vector is in the direction of the field $\mathbf{E}$, and another is directed perpendicular to an element

of the surface S , as shown in Figure 3.4.1. Vector $\mathbf{t}^e$ obviously lies in the plane of $\mathbf{E}$ and $\mathbf{n}$.

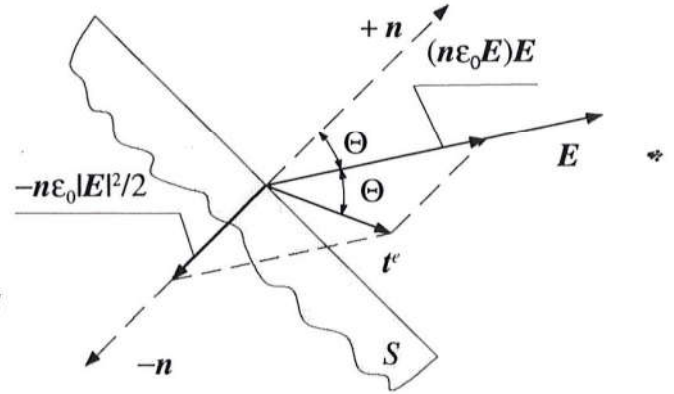


Figure 3.4.1. Relationship between vectors $\mathbf{n}$, $\mathbf{E}$ and $\mathbf{t}^e$

Let us assume that the angle between $\mathbf{E}$ and $\mathbf{n}$ is Θ . Then we get $\mathbf{E} \cdot \mathbf{n} = E \cos \Theta$, and the normal component of $\mathbf{t}^e$ is $|\mathbf{t}^e| \cos 2\Theta$. It follows that the angle between $\mathbf{t}^e$ and $\mathbf{n}$ is 2Θ . Therefore, *the vector of electric field strength bisects the angle between $\mathbf{n}$ and $\mathbf{t}^e$ into two halves*, as is shown in Figure 3.4.1. Additionally, the modulus $|\mathbf{t}^e|$ depends only on the modulus of field $|\mathbf{E}|$ and is independent of the direction of the field.

We can analyse the surface stress density $\mathbf{t}^m$ caused only by the magnetic field $\mathbf{B}$ in a similar way, and obtain similar results. The modulus of stress is

$$|\mathbf{t}^m| = \frac{1}{2} \frac{|\mathbf{B}|^2}{\mu_0} = \frac{1}{2} \mu_0 |\mathbf{H}|^2 = w_H . \quad (3.4.15)$$

Furthermore, we can say that vector $\mathbf{t}^m$ lies in the plane of $\mathbf{B}$ and $\mathbf{n}$ in free space in such a way that the vector $\mathbf{B}$ bisects the angle between $\mathbf{n}$ and $\mathbf{t}^m$ into two halves. The modulus of $\mathbf{t}^m$ depends only on the modulus of $\mathbf{B}$, and is independent of its direction.

It is convenient to describe the perpendicular stress that is directed inside a closed boundary surface S (in the direction of $\mathbf{n}$), or the normal component of such a stress, in terms of the *pressure* p on the surface S , and we get

$$p = -\mathbf{n} \cdot \mathbf{t} . \quad (3.4.16)$$

Therefore, we can say that *an electromagnetic field acts with a pressure on a boundary surface*.

In static fields it is possible to express the total force with which the electric field $\mathbf{E}$ acts on a static distribution of free charge ρ_s (i.e. on charged conductors) as

$$\mathbf{F}_e = \int_V \rho_s \mathbf{E} dV = \oint_S \mathbf{t}^e dS = \oint_S [(\mathbf{n} \epsilon_0 \mathbf{E}) \mathbf{E} - \mathbf{n} \frac{1}{2} \epsilon_0 |\mathbf{E}|^2] dS . \quad (3.4.17)$$

In the same way we can express the total force

$$\mathbf{F}_m = \int_V \mathbf{J}_s \times \mathbf{B} dV = \oint_S \mathbf{t}^m dS = \oint_S \left[\left(\mathbf{n} \cdot \frac{\mathbf{B}}{\mu_0} \right) \mathbf{B} - \mathbf{n} \frac{1}{2} \frac{|\mathbf{B}|^2}{\mu_0} \right] dS , \quad (3.4.18)$$

by which the magnetic field $\mathbf{B}$ acts on a static distribution of free currents $\mathbf{J}_s$ (i.e. on current-carrying conductors).

3.4.2. Forces and surface stresses in materials

So far we have been analysing the volume forces and surface stresses in an electromagnetic field in a vacuum. The expressions are different when we put a material into a field. The material is polarized and magnetized, and additional so-called polarization and magnetization forces arise. In the previous Section 3.4.1, we showed that the total force which is transmitted by the electromagnetic field through any closed surface S in free space can be expressed by the integral

$$\begin{aligned} \mathbf{F}_T &= \oint_S \left[(\mathbf{n} \cdot \epsilon_0 \mathbf{E}) \mathbf{E} - \mathbf{n} \frac{1}{2} \epsilon_0 |\mathbf{E}|^2 + \left(\mathbf{n} \cdot \frac{\mathbf{B}}{\mu_0} \right) \mathbf{B} - \mathbf{n} \frac{1}{2} \frac{|\mathbf{B}|^2}{\mu_0} \right] dS = \\ &= \mathbf{F}_{MECH} + \frac{1}{c_0^2} \frac{d}{dt} \int_V \mathbf{N} dV . \end{aligned} \quad (3.4.19)$$

With $\mathbf{F}_{MECH} = \mathbf{F}_e + \mathbf{F}_m$ we have denoted the sum of forces with which the electromagnetic field acts on free charge and currents in free space.

If we put a material in an electromagnetic field, we can assume that the corresponding expression for total force transmitted by the electromagnetic field through a closed surface S that encloses free charge, currents and the material, will have the same form as expression (3.4.19). We have to replace the constitutive parameters of free space ϵ_0 and μ_0 with the corresponding values for a material ϵ and μ . Thus, the expression for the total force on a material in a volume V enclosed by a surface S that contains free charge and currents, is

$$\mathbf{F}_T = \oint_S \left[(\mathbf{n} \cdot \epsilon \mathbf{E}) \mathbf{E} - \mathbf{n} \frac{1}{2} \epsilon |\mathbf{E}|^2 + \left(\mathbf{n} \cdot \frac{\mathbf{B}}{\mu} \right) \mathbf{B} - \mathbf{n} \frac{1}{2} \frac{|\mathbf{B}|^2}{\mu} \right] dS . \quad (3.4.20)$$

This equation is valid if the material is linear, which means that the relationship between $\mathbf{E}$ and $\mathbf{D}$ as well as between $\mathbf{B}$ and $\mathbf{H}$ are linear and isotropic.

In the same way we can determine vectors of surface stresses in a material in an electromagnetic field as

$$\mathbf{t} = \left[(\mathbf{n} \cdot \epsilon \mathbf{E}) \mathbf{E} - \mathbf{n} \frac{1}{2} \epsilon |\mathbf{E}|^2 \right] + \left[\left(\mathbf{n} \cdot \frac{\mathbf{B}}{\mu} \right) \mathbf{B} - \mathbf{n} \frac{1}{2} \frac{|\mathbf{B}|^2}{\mu} \right] , \quad (3.4.21)$$

and in a static electric and magnetic field as

$$\begin{aligned} \mathbf{t}^e &= (\mathbf{n} \cdot \epsilon \mathbf{E}) \mathbf{E} - \mathbf{n} \frac{1}{2} \epsilon |\mathbf{E}|^2 , \\ \mathbf{t}^m &= \left(\mathbf{n} \cdot \frac{\mathbf{B}}{\mu} \right) \mathbf{B} - \mathbf{n} \frac{1}{2} \frac{|\mathbf{B}|^2}{\mu} . \end{aligned} \quad (3.4.22a,b)$$

An electromagnetic field acts on free charge and currents in a volume V enclosed by a

closed surface S with a volume force $\mathbf{F}_{MECH}$ that can be calculated by

$$\mathbf{F}_{MECH} = \int_V [\rho_s \mathbf{E} + \mathbf{J}_s \times \mathbf{B}] dV . \quad (3.4.23)$$

In accordance with equations (3.4.20) and (3.4.23), we may determine the resultant mechanical volume force with which the electromagnetic field acts on free charge, currents and polarized and magnetized material in a volume V enclosed by a surface S using

$$\begin{aligned} \mathbf{F}_{MECH} = \oint_S \left[(\mathbf{n} \cdot \epsilon \mathbf{E}) \mathbf{E} - \mathbf{n} \frac{1}{2} \epsilon |\mathbf{E}|^2 + \left(\mathbf{n} \cdot \frac{\mathbf{B}}{\mu} \right) \mathbf{B} - \mathbf{n} \frac{1}{2} \frac{|\mathbf{B}|^2}{\mu} \right] dS - \\ - \frac{1}{c^2} \frac{d}{dt} \int_V \mathbf{N} dV . \end{aligned} \quad (3.4.24)$$

The forces $\mathbf{F}_T$ and $\mathbf{F}_{MECH}$ do not differ much because the contribution of the second term on the right-hand side of equation (3.4.24) is very small due to the factor $1/c^2$. In static fields these forces are exactly equal. We use the expression

$$\begin{aligned} S_{jk} = \epsilon \left(E_j E_k - \frac{1}{2} |\mathbf{E}|^2 \delta_{jk} \right) + \frac{1}{\mu} \left(B_j B_k - \frac{1}{2} |\mathbf{B}|^2 \delta_{jk} \right) , \\ \mathbf{t} = \tilde{\mathbf{S}} \cdot \mathbf{n} = \left[(\mathbf{n} \cdot \epsilon \mathbf{E}) \mathbf{E} - \mathbf{n} \frac{1}{2} \epsilon |\mathbf{E}|^2 \right] + \left[\left(\mathbf{n} \cdot \frac{\mathbf{B}}{\mu} \right) \mathbf{B} - \mathbf{n} \frac{1}{2} \frac{|\mathbf{B}|^2}{\mu} \right] . \end{aligned} \quad (3.4.25a,b)$$

for the calculation of components of stress tensor that form the overall stress vector in a material.

3.4.3. Surface stresses in static fields at the boundary between two materials

Forces and surface stresses at a boundary between two materials in static fields can be explained by fundamental analysis of the fields at the boundary, (the conclusions of which are summarized in the boundary conditions equations (1.5.16a – c)) as well as from analysis of surface stresses in materials according to equations (3.4.22a,b).

We shall assume that the dielectric and magnetic materials are linear and isotropic. The boundary between the two materials is sharp. Consequently we shall substitute the discontinuity of dielectric and magnetic constitutive parameters with a thin layer with a very steep, but continuous change of material constitutive parameters. The gradient of change of constitutive parameters ϵ and μ becomes infinite as the thickness of the boundary layer tends to zero, but the volume of the layer vanishes too. We have to seek the limit of transformation of the volume force density $\mathbf{f}$ into surface force density $\mathbf{t}$ in order to find the resulting force that acts on the surface of the discontinuity S . At first we shall determine the forces on the boundary in a static electric field.

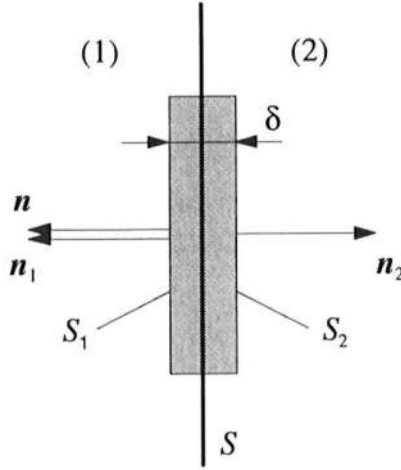


Figure 3.4.2. Boundary layer between two dielectrics

The dark surface in Figure 3.4.2. represents the transitional layer between dielectric materials (1) and (2). The force with which the electric field acts on a material and free charge in this layer can be determined by integration of (3.4.22a) over two surfaces S_1 and S_2 . The resultant force on unit of surface S in direction $\mathbf{n}$, meaning from dielectric (2) to (1), becomes equal to

$$\mathbf{t}^e = [(\mathbf{n} \cdot \epsilon_1 \mathbf{E}_1) \mathbf{E}_1 - (\mathbf{n} \cdot \epsilon_2 \mathbf{E}_2) \mathbf{E}_2] - \frac{1}{2} \mathbf{n} [\epsilon_1 |\mathbf{E}_1|^2 - \epsilon_2 |\mathbf{E}_2|^2], \quad (3.4.26)$$

as the thickness of the boundary layer limits to zero. In a case where there is no free surface charge ($\sigma_s = 0$) on the boundary surface between the two dielectrics, from equation (1.5.16c) it follows that $D_{n1} = D_{n2} = D_n$, which means $\mathbf{n} \cdot \epsilon_1 \mathbf{E}_1 = \mathbf{n} \cdot \epsilon_2 \mathbf{E}_2$. Also, from equation (1.5.16a) it follows that $E_{t1} = E_{t2} = E_t$. The tangential field components on both sides of the boundary are always equal. The difference between them that appears only within the first square brackets of equation (3.4.26) is thus equal to zero. The first term involves only the normal component on the boundary, as does the second term. Therefore, in the case where $\sigma_s = 0$, only the normal component of the resultant surface stress on the boundary between the two dielectrics exists. According to equation (3.4.26), this component is calculated as

$$t_n^e = \mathbf{t}^e \cdot \mathbf{n} = [\epsilon_1 (\mathbf{n} \cdot \mathbf{E}_1)^2 - \epsilon_2 (\mathbf{n} \cdot \mathbf{E}_2)^2] - \frac{1}{2} [\epsilon_1 |\mathbf{E}_1|^2 - \epsilon_2 |\mathbf{E}_2|^2]. \quad (3.4.27)$$

The resultant stress is perpendicular to the boundary surface S and positive (which means in the direction of the normal $+\mathbf{n}$) if $\epsilon_1 < \epsilon_2$. It acts in a direction from dielectric (2) to dielectric (1) as pressure on the boundary surface S . Therefore we say that the *material with greater permittivity tends to penetrate and fill space occupied by the material with lower permittivity*. This is a general rule according to which one can easily determine the direction of forces and stresses in a static electric field.

If a solid body with permittivity ϵ_2 is totally enclosed by a dielectric ϵ_1 , as shown in Figure 3.4.3, an electrostatic field will act on the body (on the dielectric and free charge inside it) with a resultant mechanical force

$$\mathbf{F}_{TOT} = \oint_{S_1} [(\mathbf{n} \cdot \epsilon_1 \mathbf{E}_1) \mathbf{E}_1 - \frac{1}{2} \mathbf{n} \epsilon_1 |\mathbf{E}_1|^2] dS_1. \quad (3.4.28)$$

This force replaces the action of the outer space on a solid body.

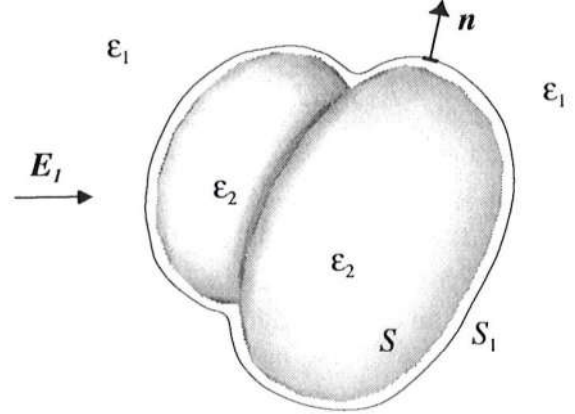


Figure 3.4.3. Solid body enclosed by dielectric ϵ_1

When the solid body is a conductor the field $\mathbf{E}_1$ is normal to the boundary surface and $\mathbf{E}_2 = 0$, and the resultant force on the insulated conductor is

$$\mathbf{F}_{TOT} = \frac{1}{2} \oint_{S_1} \epsilon_1 |\mathbf{E}_1|^2 \mathbf{n} dS_1 = \frac{1}{2} \oint_{S_1} \sigma_s \mathbf{E}_1 dS_1, \quad (3.4.29)$$

where $\sigma_s = \epsilon_1 \mathbf{n} \cdot \mathbf{E}_1$ is the surface charge density on the surface of the conductor.

By using a similar procedure, we can determine the stresses on the boundary surface between two linear magnetic materials in a static magnetic field. Analogous to equation (3.4.27), the resulting surface stress on the boundary is

$$\mathbf{t}^m = \left[\left(\mathbf{n} \cdot \frac{\mathbf{B}_1}{\mu_1} \right) \mathbf{B}_1 - \left(\mathbf{n} \cdot \frac{\mathbf{B}_2}{\mu_2} \right) \mathbf{B}_2 \right] - \frac{1}{2} \mathbf{n} \left[\frac{|\mathbf{B}_1|^2}{\mu_1} - \frac{|\mathbf{B}_2|^2}{\mu_2} \right]. \quad (3.4.30)$$

In the case where there are no free surface current $\mathbf{K}_s = 0$ on the boundary surface between two magnetic materials, from equation (1.5.16b) it follows that $H_{t1} = H_{t2} = H_t$, which means that the tangential components are equal on both sides of the boundary. The normal components of induction are always equal, $B_{n1} = B_{n2} = B_n$, according to equation (1.5.16d). Therefore, the resulting pressure perpendicular to the boundary surface S acts in a direction from material (2) to material (1) if $\mu_1 < \mu_2$, and its value is

$$t_n^m = \mathbf{t}^m \cdot \mathbf{n} = \left[\frac{1}{\mu_1} (\mathbf{n} \cdot \mathbf{B}_1)^2 - \frac{1}{\mu_2} (\mathbf{n} \cdot \mathbf{B}_2)^2 \right] - \frac{1}{2} \left[\frac{|\mathbf{B}_1|^2}{\mu_1} - \frac{|\mathbf{B}_2|^2}{\mu_2} \right]. \quad (3.4.31)$$

In a magnetic field, the material with a greater permeability tends to push out the material with a lower permeability.

PROBLEMS

- 3.1. Prove the expression for power that the electromagnetic field delivers to all electric and magnetic charge in motion within unit volume of free space or a material

$$p = \mathbf{E} \cdot \mathbf{J} + \mathbf{H} \cdot \mathbf{J}_m = -\nabla \cdot (\mathbf{E} \times \mathbf{H}) - \frac{\partial}{\partial t} \left(\frac{\epsilon_0}{2} |\mathbf{E}|^2 + \frac{\mu_0}{2} |\mathbf{H}|^2 \right) !$$

Hint: apply the vector identity $-\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{E} \cdot (\nabla \times \mathbf{H}) - \mathbf{H} \cdot (\nabla \times \mathbf{E})$!

- 3.2. Prove Poynting's theorem directly by usage of the principle of energy conservation in an electromagnetic system!
- 3.3. Prove that the modulus of the electric stress vector $\mathbf{t}^e$ is $0.5\epsilon_0 |\mathbf{E}|^2$!
- 3.4. Prove that the normal component of the electric stress vector $\mathbf{t}^e$ defined in Figure 3.4.1. is $|\mathbf{t}^e| \cos 2\Theta$!
- 3.5. Determine the directions of the electric and magnetic stress vectors in special cases where the element of the surface S lies perpendicular or parallel to the field vectors $\mathbf{E}$ and $\mathbf{B}$!

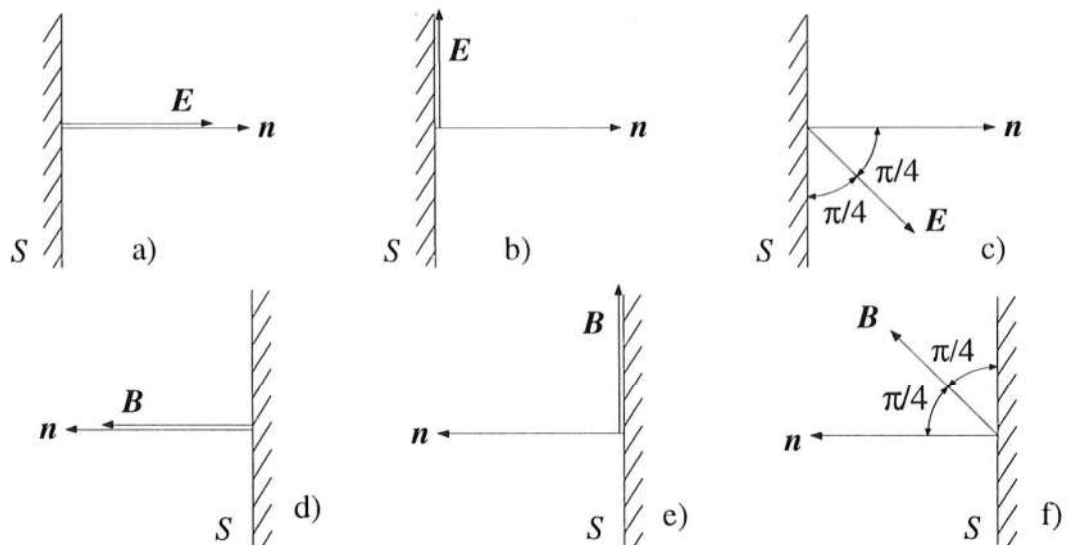


Figure P3.5. Special cases for the determination of the directions of electric and magnetic stress vectors

4. A STATIC ELECTRIC FIELD

In previous chapters, we derived a system of Maxwell's equations and an electromagnetic theory that is founded upon them. In continuation, we shall develop the *application* of this theory to solutions of problems in electromagnetism. We can classify such problems according to their complexity. The simplest problems in electromagnetism involve static fields, where field quantities are independent of time. So we shall begin with the analysis of problems in static electric fields caused by charge at rest.

First of all, we shall present the fundamental equations of a static electric field representing a special case of Maxwell's equations and study energy in such a field. Afterwards we shall define capacitance as a fundamental integral characteristic of simple geometrical arrangements, and define coefficients of capacitance for more complex arrangements. Special attention will be paid to analytical procedures used for solving static electric fields in three basic coordinate systems: Cartesian, cylindrical and spherical.

4.1. EQUATIONS OF A STATIC ELECTRIC FIELD

A static electric field is an electromagnetic field in which there is *no charge motion*. In this case there are no currents ($\mathbf{J} = 0$ is valid), all derivatives with respect to time are equal to zero, and from equations (1.5.15a – d) and (1.5.17a – c) we can derive that $\mathbf{H} = \mathbf{B} = 0$. The system of Maxwell's equations reduces to

$$\begin{aligned}\nabla \times \mathbf{E} &= 0, \\ \nabla \cdot \mathbf{D} &= \rho,\end{aligned}\tag{4.1.1,2}$$

with the constitutive relationship

$$\mathbf{D} = \epsilon \mathbf{E},\tag{4.1.3}$$

where ϵ is the permittivity of a material. This is a scalar constant for linear, isotropic and homogeneous materials. The integral forms of equations (4.1.1) and (4.1.2) are

$$\begin{aligned}\oint_l \mathbf{E} \cdot d\mathbf{l} &= 0, \\ \oint_S \mathbf{D} \cdot \mathbf{n} dS &= \int_V \rho dV.\end{aligned}\tag{4.1.4,5}$$

Here l is a closed curve, and S is a surface that encloses a volume V . From the initial assumption that there is no charge in motion in a static electric field, it follows that in such

a field we can have only two types of materials: *conductors and dielectrics (insulators)*. The mobility of charge in conductors is infinite and every possible rearrangement of charge happens over an infinitesimally small period. Charge in dielectrics is immovable and is subjected only to the mechanism of polarization.

We derived the conditions that the field vectors $\mathbf{E}$ and $\mathbf{D}$ have to satisfy on the boundary between two materials with different permittivities in Unit 1.5. and they are defined by equations (1.5.16a and 1.5.16c). In short, a tangential component of electric field strength continuously passes through the boundary

$$\mathbf{n} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0 \quad .$$

If the boundary surface separates a dielectric and conductor, then $\mathbf{E}$ must equal zero inside the conductor, and the tangential component of $\mathbf{E}$ on the boundary must be equal to zero in both materials. Accordingly, *the electric field strength is always perpendicular to the surface of a conductor in a static electric field*. The normal component of electric induction has a discontinuity that is equal to the value of the surface charge density of free charge on the boundary

$$\mathbf{n} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \sigma \quad .$$

In the conductor $\mathbf{D} = 0$, and the normal component of the vector of electric induction $\mathbf{D}$ on its surface must be equal to *the surface density of free charge σ on the surface of the conductor*.

4.1.1. Scalar electric potential

Introduction of the scalar electric potential ϕ into equations (4.1.1) and (4.1.2) greatly simplifies the solution of problems in static electric fields. In homogeneous isotropic materials it satisfies *the scalar differential equation*

$$\nabla\phi \nabla\epsilon + \epsilon \Delta\phi = -\rho \quad . \quad (4.1.6)$$

We need to prescribe the function of distribution of permittivity in order to solve this equation. In practical usage of this equation, we most frequently come across two cases:

1. The field in a linear homogeneous dielectric with density of free volume charge ρ_s . In this case equation (4.1.6) is reduced to a *Poisson's differential equation*

$$\Delta\phi = -\frac{\rho_s}{\epsilon} \quad . \quad (4.1.7)$$

2. The field in a linear homogeneous dielectric with no free charge ($\rho_s = 0$). In this case, equation (4.1.6) is reduced to *Laplace's differential equation*, known from many areas of application of theoretical physics

$$\Delta\phi = 0 \quad . \quad (4.1.8)$$

An expression for the relationship between scalar potential and electric field strength in a static electric field follows from expression (2.1.3)

$$\mathbf{E} = -\nabla\varphi \quad \text{or} \quad \varphi(\mathbf{r}) - \varphi(\mathbf{r}_0) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l} , \quad (4.1.9,10)$$

where $\mathbf{r}_0$ is the position vector of the point with the reference (zero) potential. With the help of this equation, we can define scalar electric potential as the work that should be done in order to bring unit charge moving in the opposite direction to the field from the reference point to a point defined by vector $\mathbf{r}$. Namely, if we need to bring unit charge in a direction opposite to that of the field, we need to overcome the force $-\mathbf{E}$ by a mechanical force, and the work that should be done is determined by equation (4.1.10).

From equation (4.1.4) it follows that this integral is independent of the path taken between points $\mathbf{r}_0$ and $\mathbf{r}$. It depends only on the position (potential) of the points. We should emphasise that only the potential difference between two points, ie. voltage, is a really measurable quantity. From equation (4.1.10), we can derive an expression for determining the potential difference (voltage) between arbitrarily chosen points A and B

$$U_{AB} = \varphi_A - \varphi_B = -\int_B^A \mathbf{E} \cdot d\mathbf{l} . \quad (4.1.11)$$

The choice of point with the reference potential is arbitrary. Its influence disappears in the calculation of the potential difference (voltage).

The scalar electric potential is continuous and finite at all points of the field in which there are no point or line charges, and in a static electric field it must limit to zero at infinity if the distribution of charge is finite and it must be constant in every conductor because all charge is at rest. Additionally, the scalar electric potential must continuously pass through the boundary between the two different materials.

4.2. VISUAL PRESENTATION OF A STATIC ELECTRIC FIELD

Presentation of a static electric field in the form of field plots greatly aids the understanding of a field. A field plot in a three-dimensional space is composed of field lines and equipotential surfaces. Field lines may be *lines of force* ($\mathbf{E}$ -lines) or *flux lines* ($\mathbf{D}$ -lines). Vectors $\mathbf{E}$ and $\mathbf{D}$ must be tangential to these lines at every point on the lines.

In a linear isotropic and homogeneous material, these lines are identical. Lines of force and flux lines differ only at the boundary between two materials or in a non-homogeneous anisotropic dielectric. We determine these field lines with the help of the differential equations

$$\frac{dx}{E_x} = \frac{dy}{E_y} = \frac{dz}{E_z} . \quad (4.2.1)$$

We define an *equipotential surface* as the set of all points that possess the same scalar electric potential and we determine them by the equation

$$\varphi(x,y,z) = \text{const.} \quad (4.2.2)$$

Since the difference in potential between two near points of an equipotential surface is equal to zero, with the help of equation (4.1.9) we can conclude that vector $\mathbf{E}$ (field line) is always perpendicular to an equipotential surface.

The presentation of three-dimensional images that consist of equipotential surfaces and field lines is very complicated, and we most frequently use two-dimensional presentations in a plane. Such presentations consist of field lines and equipotential lines that form a mesh of orthogonal trajectories and we call them field plots. When we plot these lines, we usually obey the following convention:

- 1) the difference in potential between neighbouring equipotential lines is equal for all equipotential lines in the drawing,
- 2) the change of electric flux between two neighbouring field lines is equal for all field lines in the drawing.

If we plot a field obeying this convention, we obtain a drawing with a qualitative and quantitative distribution of the field in space. For example, the static electric field of a point charge in two and three dimensions is shown in Figure 4.2.1.

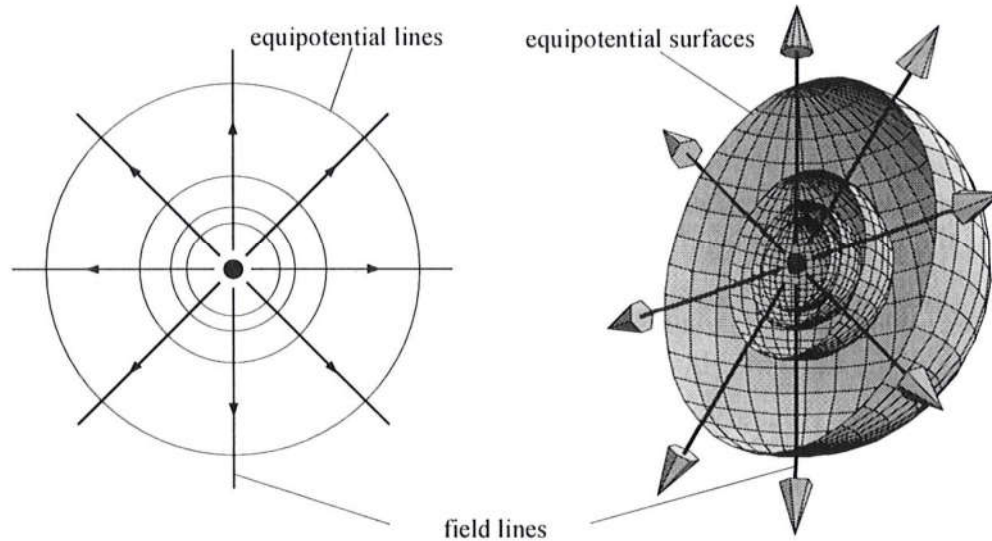


Figure 4.2.1. Static electric field of a point charge

4.3. ENERGY STORED IN A STATIC ELECTRIC FIELD

In order to assemble any static distribution of charge in a system, a transient dynamic process of bringing the charge to conductors and dielectrics of the system is necessary. During this process, currents flow and a magnetic field exists, but these currents and magnetic field disappear with the establishing of a static state.

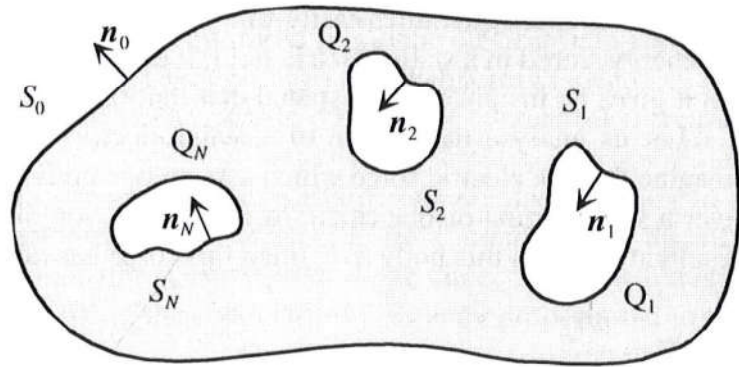
In Section 3.3.3. we showed that the energy consumed in establishing an electric field in a linear material is equal to

$$W_e = \frac{1}{2} \int_V \mathbf{E} \cdot \mathbf{D} dV . \quad (4.3.1)$$

We can transform this expression for the energy stored in a static electric field and express it as a function of charge and potential. Application of the identity regarding divergence of the product of both the scalar and vector function to the product $\varphi \mathbf{D}$ results in $\nabla \cdot (\varphi \mathbf{D}) = \varphi \nabla \cdot \mathbf{D} + \mathbf{D} \cdot \nabla \varphi$. Taking into consideration that $\nabla \cdot \mathbf{D} = \rho$ and $\nabla \varphi = -\mathbf{E}$, we obtain the expression $\mathbf{E} \cdot \mathbf{D} = \rho \varphi - \nabla \cdot (\varphi \mathbf{D})$.

Let us assume that the static electric field is composed of N charged conductive bodies with surfaces $S_1, \dots, S_N$, and that surface S_0 encloses the whole space of the field (Figure 4.3.1). Free charge with density ρ may be located between conductive bodies.

Figure 4.3.1. Static electric field of N charged conductive bodies



Substitution of expression for $\mathbf{E} \cdot \mathbf{D}$ into (4.3.1) and application of Gauss' divergence theorem results in

$$W_e = \frac{1}{2} \int_V \rho \varphi dV - \frac{1}{2} \left\{ \oint_{S_0} \varphi \mathbf{D} \cdot \mathbf{n} dS + \sum_{k=1}^N \oint_{S_k} \varphi \mathbf{D} \cdot \mathbf{n} dS \right\} . \quad (4.3.2)$$

The integral over surface S_0 is equal to zero since, according to our initial assumption, it is outside the field ($\mathbf{D} = 0$). We showed that vector $\mathbf{E}$ is perpendicular to the conductor at every point of its surface. Therefore, from equation (4.1.9) it follows that between any near points of a conductor, there is no potential difference. Thus, the *surface of a conductor in a static electric field is an equipotential surface*. Accordingly,

$$W_e = \frac{1}{2} \int_V \rho \varphi dV - \frac{1}{2} \sum_{k=1}^N \left\{ \varphi_k \oint_{S_k} \mathbf{D} \cdot \mathbf{n} dS \right\} , \quad (4.3.3)$$

where φ_k is the potential of the k -th conductive body. If we take into consideration the direction of the normal to surface S_k directed into the conductive body k , we obtain

$$W_e = \frac{1}{2} \int_V \rho \varphi dV + \frac{1}{2} \sum_{k=1}^N Q_k \varphi_k , \quad (4.3.4)$$

where Q_k is the total charge on the conductive body k . In most cases there is no free

charge between conductive bodies ($\rho = 0$), and therefore the total energy stored in a static electric field composed of N charged conductive bodies may be calculated as being

$$W_e = \frac{1}{2} \sum_{k=1}^N Q_k \varphi_k . \quad (4.3.5)$$

4.3.1. Energy-based determination of forces in a static electric field

In Chapter 3. we explained in detail the determination of the forces in a static electric field in terms of known field quantities and distribution of charge. It is often not possible to determine these quantities easily, and in these cases we need to determine the forces via the energy stored in a static electric field. It enables a determination of the resultant force, but it gives no insight into the spatial distribution of the forces.

Let us analyse the system of a static electric field shown in Figure 4.3.1. Let us imagine that the electric force which acts on one body of the system has moved this body over a small virtual displacement δs in the direction of the s -axis. If the s -component of resultant force on this body is F_s , then this force has done work $\delta A_e = F_s \delta s$, and we get

$$F_s = \frac{\delta A_e}{\delta s} . \quad (4.3.6)$$

We can express the work done by an electric force δA_e in terms of the energy stored in the static electric field in two distinct cases:

1. *The amount of charge within the system remains constant (insulated system).* This means that we have connected conductors of the system to the external source generating the field and after establishing a static state we have disconnected the source. All displacement in the system (work) in this case must be compensated for by a change in the energy stored in the system, and we get $\delta A_e = -\delta W_e$, which means

$$F_s = -\frac{\delta W_e}{\delta s} . \quad (4.3.7)$$

2. *The potentials within the system remain constant (uninsulated system).* This means that conductors of the system remain connected to a source that keeps their potential fixed regardless of their displacement. In this case, a small displacement δs of one of the components of the system requires an amount of additional charge $\delta Q_1 \dots \delta Q_N$ to be supplied by the source. According to the definition of potential, the work done is

$$\delta A_i = \sum_{k=1}^N \varphi_k \delta Q_k . \quad (4.3.8)$$

According to the law of energy conservation, this work represents the work done by the electric forces δA_e and the increase of energy stored in the field $\delta A_i = \delta A_e + \delta W_e$. From expression (4.3.5) it follows that the increase of energy stored in the field is

$$\delta W_e = \frac{1}{2} \sum_{k=1}^N \varphi_k \delta Q_k, \quad (4.3.9)$$

and we get $\delta A_e = \delta W_e$. Accordingly, the expression

$$F_s = \frac{\delta W_e}{\delta s}, \quad (4.3.10)$$

gives the component of the resultant force in the direction of the s -axis in this case.

4.4. CAPACITANCE

In the previous unit, we analysed the energy stored in a static electric field and concluded that in order to charge conductors, i.e. in order to establish a static electric field, we have to supply energy. Therefore, the scalar electric potential of conductors increases with supply of charge given to them.

The simplest system of a static electric field consists of one conductive body in a linear dielectric. Due to this linearity the vector of electric field strength $\mathbf{E}$ is directly proportional to the charge on the conductor at every point in the dielectric. From expression (4.1.10) it follows that the potential of the conductor φ is directly proportional to the charge on the conductor Q

$$Q = C\varphi. \quad (4.4.1)$$

We call the constant $C = Q/\varphi$ the *capacitance* (or *capacity*) of the observed body. The unit of capacitance is the *farad* (F), ($1\text{F} = 1\text{C}/1\text{V}$). In linear dielectrics, *capacitance is a geometrical characteristic of an electric field and permittivity* and is independent of the charge and potential of the body.

4.4.1. Capacitance of a condenser

We can utilise the capability of storing energy in an electric field using devices that we call condensers. An *ideal condenser* is a system of two conductive bodies charged with an equal amount of charge Q of opposite sign placed in an infinite space filled with a material with permittivity ϵ (Figure 4.4.1). These conductive bodies are called electrodes.

The electric field $\mathbf{E}$ in this system is linked to the charge on the conductors and the voltage (potential difference) between the conductors is linked to the field $\mathbf{E}$ by equation (4.1.11). If the material between the electrodes of the condenser is linear, the vector $\mathbf{E}$ is directly proportional to charge Q (or $-Q$) at every point. This means that the voltage between the electrodes is proportional to the charge.

$$Q = C(\varphi_Q - \varphi_{-Q}) = CU. \quad (4.4.2)$$

The constant C is called the *capacitance of the condenser*. It depends on the shape, dimensions and distance between the electrodes as well as the permittivity of the material.

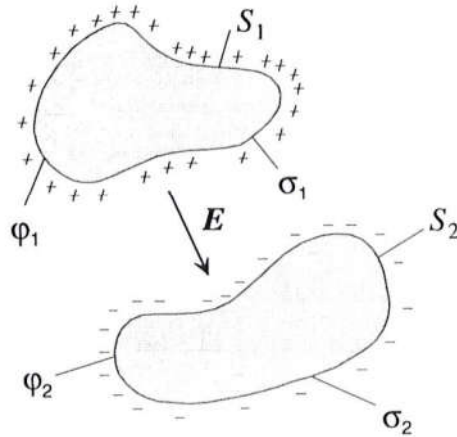


Figure 4.4.1. A condenser as a system of two solitary electrodes

According to (4.3.5), the energy stored in this system is

$$W_e = \frac{1}{2} [Q\varphi_Q + (-Q)\varphi_{-Q}] = \frac{1}{2} QU, \quad (4.4.3)$$

and using equation (4.4.2), we can express the amount of energy stored in the electric field of the condenser as

$$W_e = \frac{1}{2} CU^2 = \frac{1}{2} \frac{Q^2}{C}. \quad (4.4.4)$$

We use these equations in calculating the forces in the condenser with the help of the stored energy. From expressions (4.3.7) and (4.3.10) it follows that the force that acts in a direction s in a condenser of capacity C can be determined as

$$F_s = - \left\{ \frac{\partial W_e}{\partial s} \right\}_{Q=const.} = - \frac{1}{2} Q^2 \frac{\partial}{\partial s} \left(\frac{1}{C} \right), \quad (4.4.5)$$

for an insulated condenser, and as

$$F_s = \left\{ \frac{\partial W_e}{\partial s} \right\}_{U=const.} = \frac{1}{2} U^2 \frac{\partial C}{\partial s}, \quad (4.4.6)$$

for an uninsulated condenser.

4.4.2. A system of conductors and partial capacitances

Now we are going to study the various relationships in a static electric field assembled from N separated conductive bodies. All of these bodies are electrically connected. The charge on each body has arisen under the influence of charge on all the other bodies, and the static electric field has arisen under the influence of the charge on all the conductors.

In such a system, we choose a reference equipotential surface to which we associate a zero potential. Most frequently, we choose the ground for this surface since the ground is assumed to be an ideal conductor in usual static electric problems (Figure 4.4.2).

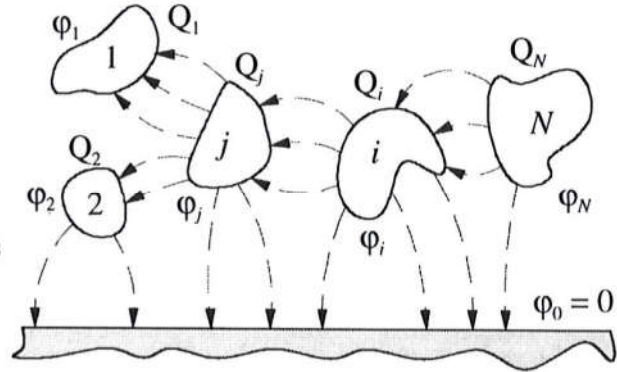


Figure 4.4.2. Electric field of N charged conductors and reference surface

We divide each conductive body into N parts from which D -lines impart to the other $N - 1$ bodies as well as to the reference surface with index 0, and we may write the total charge Q_i on the i -th body as $Q_i = Q_{i0} + \dots + Q_{i,i-1} + Q_{i,i+1} + \dots + Q_{iN}$, ($i = 0, \dots, N$). We have shown that the charge in linear dielectrics is proportional to the potential difference, and now we can introduce the notion of *partial capacitances* as

$$C_{ij} = \begin{cases} \frac{Q_{ij}}{\varphi_i - \varphi_j} & ; \quad i = 0, \dots, N \quad ; \quad j = 0, \dots, i-1, i+1, \dots, N , \\ 0 & ; \quad i = j . \end{cases} \quad (4.4.7)$$

Because of a symmetry in influence, $C_{ij} = C_{ji}$ is valid. Substitution of the *partial capacitances* into the expression for the total charge Q_i on the i -th body results in

$$Q_i = \sum_{j=0}^N C_{ij} (\varphi_i - \varphi_j) \quad ; \quad i = 0, \dots, N . \quad (4.4.8)$$

In numerous applications, it is convenient to rearrange this expression into a linear dependence between charges and potentials of individual conductors

$$Q_i = \left\{ \sum_{j=0}^N C_{ij} \right\} \varphi_i + \sum_{j=0}^N (-C_{ij}) \varphi_j = \sum_{j=0}^N c_{ij} \varphi_j \quad ; \quad i = 0, \dots, N , \quad (4.4.9)$$

where c_{ij} are *coefficients of induction*, since c_{ij} expresses the charge induced on the i -th conductive body if the j -th body is of unit potential, and all other bodies are of zero potential. The expression

$$c_{ij} = \begin{cases} \sum_{k=0}^N C_{ik} = c_{ii} & ; \quad i = 0, \dots, N \quad ; \quad j = i , \\ -C_{ij} & ; \quad i = 0, \dots, N \quad ; \quad j = 0, \dots, i-1, i+1, \dots, N . \end{cases} \quad (4.4.10)$$

is valid. Coefficients c_{ii} do not represent the charge that has been self-induced on a conductor. We call them *coefficients of capacitance*.

In the static electric field that we are analysing here, the sum of all the charge in the system is equal to zero (such a system is called a complete system) and the total charge on the reference surface is given by

$$Q_0 = - \sum_{i=1}^N Q_i . \quad (4.4.11)$$

Taking this equation into consideration, we obtain a system of N unknown charges with $N + 1$ unknown potentials. It is usual to introduce $\varphi_0 = 0$ and redefine all other potentials as potential differences with respect to the reference surface. This eliminates the first row and the first column in the system (4.4.9) and we thus reduce it to the system

$$Q_i = \sum_{j=1}^N c_{ij} \varphi_j \quad ; \quad i = 1, \dots, N , \quad (4.4.12)$$

with $N(N + 1)/2$ different coefficients ($c_{ij} = c_{ji}$). With the help of these coefficients, we can express the energy stored in the system by substitution of (4.4.12) into (4.3.5)

$$W_e = \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N c_{ij} \varphi_i \varphi_j . \quad (4.4.13)$$

This expression is widely applied in engineering practice because it includes easily measurable quantities.

From equations (4.4.12,13) it follows that we can represent such a system of conductors with an equivalent circuit of partial capacitances. The potential of the i -th node of the circuit corresponds to the potential of the i -th conductor in the field, and we connect the partial capacitance C_{ik} between the nodes i and k . The energy stored in the field is equal to the energy stored in the capacitances of the equivalent network. We can write the system (4.4.12) in matrix form as

$$[Q] = [c][\varphi] , \quad (4.4.14)$$

where $[Q]$ is the vector of charges on the conductive bodies, $[c]$ is the matrix of coefficients of induction with elements c_{ij} defined by expression (4.4.10), and $[\varphi]$ is the vector of potentials of the conductive bodies. If we multiply this system by matrix $[c]^{-1}$ from the left-hand side we obtain a linear relationship between potentials and charges $[\varphi] = [P][Q]$ where $[P] = [c]^{-1}$. Elements of matrix $[P]$ are called *the coefficients of potential* and they are related to the coefficients of induction by the operation of matrix inversion.

4.5. THE SOLUTION OF STATIC ELECTRIC FIELDS

The solution of static electric fields implies the determination of field quantities most frequently performed in the following steps:

1. The determination of the static electric potential φ .
2. The determination of the electric field strength by usage of equation (4.1.9).
3. The determination of other required quantities (charge densities, electric forces, capacitances etc.) with the help of solutions of the first two steps.

This approach reduces the solving of a static electric field to that of solving Poisson's (or Laplace's) equation. Therefore, we shall analyse some characteristics of this equation.

4.5.1. The superposition of solutions of Poisson's equation

Let us assume the density of free charge in the form of a linear combination to be

$$\rho = \sum_{i=1}^N a_i \rho_i, \quad (4.5.1)$$

where a_i , ($i = 1 \dots N$) are arbitrary constants. Let electric charges with densities ρ_{si} , ($i = 1 \dots N$) act in the system of a static electric field. If the solutions that correspond to these charges are potentials φ_i ($i = 1 \dots N$), then the expression

$$\Delta \varphi_i = -\frac{\rho_{si}}{\epsilon} \quad ; \quad i = 1, \dots, N, \quad (4.5.2)$$

is valid. If we multiply the i -th equation of this system by coefficient a_i , and then sum all equations, we obtain

$$\sum_{i=1}^N \{a_i \Delta \varphi_i\} = \Delta \left\{ \sum_{i=1}^N a_i \varphi_i \right\} = -\frac{1}{\epsilon} \left\{ \sum_{i=1}^N a_i \rho_{si} \right\}. \quad (4.5.3)$$

From equations (4.5.1 – 3) it follows: if φ_i ($i = 1 \dots N$) are solutions of Poisson's equation caused by free charge ρ_i , ($i = 1 \dots N$), then the solution

$$\varphi = \sum_{i=1}^N a_i \varphi_i. \quad (4.5.4)$$

corresponds to a linear combination given by equation (4.5.1). We call this property of Poisson's equation the principle of superposition. *This principle is valid for all phenomena that can be described by linear differential equations.*

4.5.2. The uniqueness of solutions of Poisson's equation

Let us analyse the system of a static electric field shown in Figure 4.3.1. If there is a linear homogeneous dielectric material with permittivity ϵ inside a volume V enclosed by a surface S that consists of surfaces $S_0, S_1, \dots, S_N$, and if there is free charge of density ρ inside V , then let us assume that there are two different solutions φ_1 and φ_2 of Poisson's equation in V . Therefore

$$\Delta \varphi_1 = -\frac{\rho}{\epsilon} \quad ; \quad \Delta \varphi_2 = -\frac{\rho}{\epsilon}, \quad (4.5.5)$$

and subtraction of these equations results in $\Delta \varphi_1 - \Delta \varphi_2 = \Delta(\varphi_1 - \varphi_2) = 0$. The difference between the solutions must satisfy Laplace's equation. Therefore, solution of the static electric field is unique if $\varphi_1 - \varphi_2 = 0$ or $\varphi_1 - \varphi_2 = \text{const.}$ and we shall see under which circumstances we can fulfil these conditions. If we introduce the first Green's identity

$$\oint_S u \frac{\partial v}{\partial n} dS = \int_V [u \Delta v + \nabla u \nabla v] dV, \quad (4.5.6)$$

and substitute $u = v = \varphi_1 - \varphi_2$ into it, we obtain

$$\oint_S (\varphi_1 - \varphi_2) \frac{\partial (\varphi_1 - \varphi_2)}{\partial n} dS = \int_V [\nabla (\varphi_1 - \varphi_2)]^2 dV. \quad (4.5.7)$$

The right-hand side of this equation is equal to zero when $\nabla(\varphi_1 - \varphi_2) = 0$. We can satisfy this condition only when the difference between solutions $\varphi_1 - \varphi_2$ is independent of any coordinate, meaning that the difference $\varphi_1 - \varphi_2$ is constant within the whole volume V . This is exactly the case we are interested in. It is easy to prove that the left-hand side of equation (4.5.7) is equal to zero in the following characteristic cases:

1. $\varphi_1 - \varphi_2 = 0$, or $\varphi_1 = \varphi_2$ at all points of surface S . Then $\varphi_1 = \varphi_2$ is valid within the complete domain V .
2. $\partial\varphi_1/\partial n - \partial\varphi_2/\partial n = 0$, or $\partial\varphi_1/\partial n = \partial\varphi_2/\partial n$ at all points of surface S . Then $\varphi_1 = \varphi_2 + \varphi_0$ is valid within the complete domain V .
3. $\varphi_1 - \varphi_2 = 0$ on one part of the surface S , and $\partial\varphi_1/\partial n - \partial\varphi_2/\partial n = 0$ on the rest of the surface S . Then $\varphi_1 = \varphi_2$ is valid within the complete domain V .

Accordingly, *the solution of Poisson's (Laplace's) equation is unique if the boundary conditions defined in these three characteristic cases are given at the boundaries of the domain*. On the basis of this, we can differentiate between three formulations of the boundary conditions in the solution of Poisson's equation:

1. *Dirichlet conditions:* The solution of Poisson's equation is unique within the domain of calculation V if the potential is given on the closed surface that encloses the domain of calculation.
2. *Neumann conditions:* The solution of Poisson's equation is unique to a constant within the domain of calculation V if $\partial\varphi/\partial n$ is given on a closed surface that encloses the domain of calculation.
3. *Mixed conditions:* The solution of Poisson's equation is unique within the domain of calculation V if the potential on part of the closed surface S that encloses the domain of calculation is given, and $\partial\varphi/\partial n$ is given on the rest of S .

In each of these formulations, the vectors $\mathbf{E}$ and $\mathbf{D}$ will be unique in the domain of calculation V and on the closed surface S that encloses V .

A direct consequence of uniqueness of the solution of Poisson's equation is that *a static electric field remains unaffected if we put an infinitesimally thin metal layer on any part of (or whole of) the equipotential surface*. Namely, if we put a metal layer on a surface that is already equipotential, the potential on the surface is not going to change, as metals in static electric fields are always equipotential on their surface. Due to the uniqueness of the solution of Poisson's equation, the potential is not going to change at any point of the field. Analogously, if we put a metal body in a static electric field, and if the boundary surface of the body coincides with an equipotential surface of the field in the absence of the body, the potential at all points of the boundary surface and outside the body will remain the same as if it were in the field in the absence of the body.

4.5.3. Integral of Poisson's equation and Green's functions

We have determined the general solution of the scalar wave equation in Unit 2.5. Poisson's equation can be regarded as a special case of the scalar wave equation in which there are no time derivatives, and we can derive its general solution from equation (2.5.8)

$$\varphi(\mathbf{r}) = \int_V \rho(\mathbf{r}') G(\mathbf{r}, \mathbf{r}') dV' + \varepsilon \oint_S \left[G(\mathbf{r}, \mathbf{r}') \frac{\partial \varphi}{\partial n'} - \varphi(\mathbf{r}') \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n'} \right] dS' . \quad (4.5.8)$$

This means that we can calculate the potential of a given charge distribution in a bounded space as the sum of the contribution of the given charge distribution in an unbounded space (the first term of the solution) and of the contribution on the boundary surface of the space S that replaces the influence of all external field sources.

Liberty in defining Green's function G enables the simplification of calculation of the surface integral in (4.5.8) when we are solving a Dirichlet's or Neumann's boundary value problem. For a *Dirichlet's boundary value problem* we prescribe $G_D(\mathbf{r}, \mathbf{r}') = 0$ if $\mathbf{r}' \in S$, and the potential in this case can be determined by

$$\varphi(\mathbf{r}) = \int_V \rho(\mathbf{r}') G_D(\mathbf{r}, \mathbf{r}') dV' - \varepsilon \oint_S \varphi(\mathbf{r}') \frac{\partial G_D(\mathbf{r}, \mathbf{r}')}{\partial n'} dS' . \quad (4.5.9)$$

We have to be careful while prescribing the boundary condition for G in solving a *Neumann's boundary value problem* (derivatives of the potential on a boundary are given). The simplest choice

$$\frac{\partial G_N(\mathbf{r}, \mathbf{r}')}{\partial n'} = 0 \quad ; \quad \mathbf{r}' \in S , \quad (4.5.10)$$

does not equate to the fundamental equation of Green's function of a static electric field, and we accept that

$$\Delta G(\mathbf{r}, \mathbf{r}') = - \frac{\delta(\mathbf{r}, \mathbf{r}')}{\varepsilon} , \quad (4.5.11)$$

since integration of this equation over a closed surface S results in

$$\oint_S \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n'} dS' = - \frac{1}{\varepsilon} . \quad (4.5.12)$$

Accordingly, the simplest boundary condition for G that we can prescribe while solving a Neumann's boundary value problem is

$$\frac{\partial G_N(\mathbf{r}, \mathbf{r}')}{\partial n'} = - \frac{1}{\varepsilon S} \quad ; \quad \mathbf{r}' \in S . \quad (4.5.13)$$

In this case, the solution for potential is

$$\varphi(\mathbf{r}) = \int_V \rho(\mathbf{r}') G_N(\mathbf{r}, \mathbf{r}') dV' + \varepsilon \oint_S \frac{\partial \varphi(\mathbf{r}')}{\partial n'} G_N(\mathbf{r}, \mathbf{r}') dS' + \frac{1}{S} \oint_S \varphi(\mathbf{r}') dS' . \quad (4.5.14)$$

The last term of this solution represents the mean value of potential on the whole boundary surface. In the case of an infinite boundary surface, this contribution is equal to zero.

Green's functions for solving Dirichlet and Neumann's boundary value problems that are chosen in such a way are independent of the value and distribution of boundary conditions. They depend only on the shape of the boundary surface S .

In practical applications, the analytical determination of Green's functions is possible only in a relatively small number of geometries, and we most frequently determine them by the *method of images*.

4.6. THE SOLUTION OF LAPLACE'S EQUATION BY A SEPARATION OF VARIABLES

In some cases it is possible to solve boundary value problems in a static electric field by the direct solution of Laplace's equation with a separation of variables. This procedure consists of reducing the partial differential equation to a system of ordinary differential equations by way of the assumption that

$$\varphi(u_1, u_2, u_3) = F(u_1)G(u_2)H(u_3) . \quad (4.6.1)$$

We most frequently use this assumption in cases when the boundaries of the domain of calculation coincide with coordinate surfaces ($u_1 = \text{const}$, $u_2 = \text{const}$. and $u_3 = \text{const}$.). Partial derivatives in this case are reduced to ordinary derivatives as every scalar function F, G, H depends on only one coordinate. We shall present an application of this method in three different coordinate systems: Cartesian, cylindrical and spherical.

4.6.1. Cartesian coordinate system

Laplace's equation (4.1.8) in this coordinate system transforms to

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0 . \quad (4.6.2)$$

We assume the solution of this equation takes the form $\varphi(x, y, z) = F(x)G(y)H(z)$. Substitution of this assumption into equation (4.6.2) results in

$$\frac{1}{F} \frac{d^2 F}{dx^2} + \frac{1}{G} \frac{d^2 G}{dy^2} + \frac{1}{H} \frac{d^2 H}{dz^2} = 0 . \quad (4.6.3)$$

We can separate the variables x, y and z in this equation by differentiation with respect to x, y and z . Differentiation with respect to x results in an ordinary differential equation with one variable

$$\frac{d}{dx} \left(\frac{1}{F} \frac{d^2 F}{dx^2} \right) = 0 . \quad (4.6.4)$$

By integration with respect to x , this equation transforms to

$$\frac{d^2 F}{dx^2} + k_x^2 F = 0 , \quad (4.6.5)$$

where k_x is a constant of integration. In a similar way, we obtain the equations

$$\frac{d^2 G}{dy^2} + k_y^2 G = 0 \quad \text{and} \quad \frac{d^2 H}{dz^2} + k_z^2 H = 0 . \quad (4.6.6,7)$$

Quantities k_x^2 , k_y^2 and k_z^2 are real constants that can be both positive and negative. They are mutually dependent due to the fact that substitution of (4.6.5 – 7) into (4.6.3) results in

$$k_x^2 + k_y^2 + k_z^2 = 0 . \quad (4.6.8)$$

The solution of equations (4.6.5 – 7) is

$$F(x) = \begin{cases} A_x e^{k_x x} + B_x e^{-k_x x} & ; \quad k_x^2 < 0 , \\ A_x x + B_x & ; \quad k_x = 0 , \\ A_x \sin(k_x x) + B_x \cos(k_x x) & ; \quad k_x^2 > 0 . \end{cases} \quad (4.6.9)$$

If we are able to satisfy the boundary conditions by some linear combination of these solutions, this combination is therefore the solution of the boundary value problem due to the fact that the solution of Laplace's equation is unique.

Example 4.6.1.1. The field in a rectangular parallelepiped

Let us analyse a rectangular parallelepiped of dimensions $(a \times b \times c)$ as shown in Figure 4.6.1.

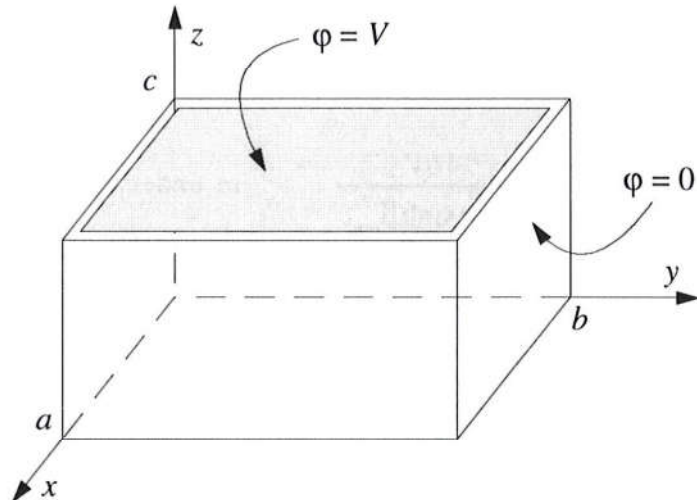


Figure 4.6.1. Rectangular parallelepiped

All surfaces of the rectangular parallelepiped are at zero potential except the surface in the plane $z = c$, which is at a given potential V . This distribution of potential is possible only if a small negligible gap exists between the upper and side surfaces. From the given boundary conditions, it follows that $F(0) = F(a) = G(0) = G(b) = H(0) = 0$ and $H(c) = V$. Solutions of equations (4.6.5 – 7) satisfying these conditions are eigenfunctions of the problem

$$\begin{aligned} F(x) &= A_x \sin \frac{\pi m x}{a} \quad ; \quad m = 1, 2, \dots \\ G(y) &= A_y \sin \frac{\pi n y}{b} \quad ; \quad n = 1, 2, \dots \\ H(z) &= \sinh \beta_{mn} z . \end{aligned} \quad (4.6.10 - 12)$$

We determine the coefficient β_{mn} using equation (4.6.8)

$$k_z = j\beta_{mn} = \sqrt{-(k_x^2 + k_y^2)} = j\sqrt{\left(\frac{\pi m}{a}\right)^2 + \left(\frac{\pi n}{b}\right)^2} \quad ; \quad j = \sqrt{-1} . \quad (4.6.13)$$

Therefore, the solution of Laplace's equation satisfying the given boundary conditions is

$$\varphi_{mn}(x, y, z) = K_{mn} \sin \frac{\pi m x}{a} \sin \frac{\pi n y}{b} \sinh \beta_{mn} z , \quad (4.6.14)$$

where K_{mn} is an arbitrary constant. We shall determine its value from the condition $\varphi(x, y, z = c) = V$. From the principle of superposition it follows that the linear combination

$$\varphi(x, y, z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \varphi_{mn}(x, y, z) , \quad (4.6.15)$$

is also a solution of Laplace's equation. Equation (4.6.15) represents a *Fourier series* of variables x and y . By multiplying expression (4.6.14) by

$$\sin \frac{\pi m x}{a} \sin \frac{\pi n y}{b} , \quad (4.6.16)$$

and afterwards by integration over an area $0 \leq x \leq a$ and $0 \leq y \leq b$ we can determine the coefficients K_{mn} as being

$$K_{mn} = \begin{cases} \frac{16V}{\pi^2 mn \sinh \beta_{mn} c} & ; \quad m \text{ and } n \text{ odd} , \\ 0 & ; \quad m \text{ and } n \text{ are even} , \end{cases} \quad (4.6.17)$$

and the final solution is

$$\varphi(x, y, z) = \frac{16V}{\pi^2} \sum_{m=1,3,\dots}^{\infty} \sum_{n=1,3,\dots}^{\infty} \frac{\sinh \beta_{mn} z}{mn \sinh \beta_{mn} c} \sin \frac{\pi m x}{a} \sin \frac{\pi n y}{b} . \quad (4.6.18)$$

The distribution of potential over characteristic cross-sections of a rectangular parallelepiped with dimensions $a = 1\text{ m}$, $b = 2\text{ m}$, $c = 3\text{ m}$ and potential $V = 100\text{ V}$ is shown in Figures 4.6.2.a,b.

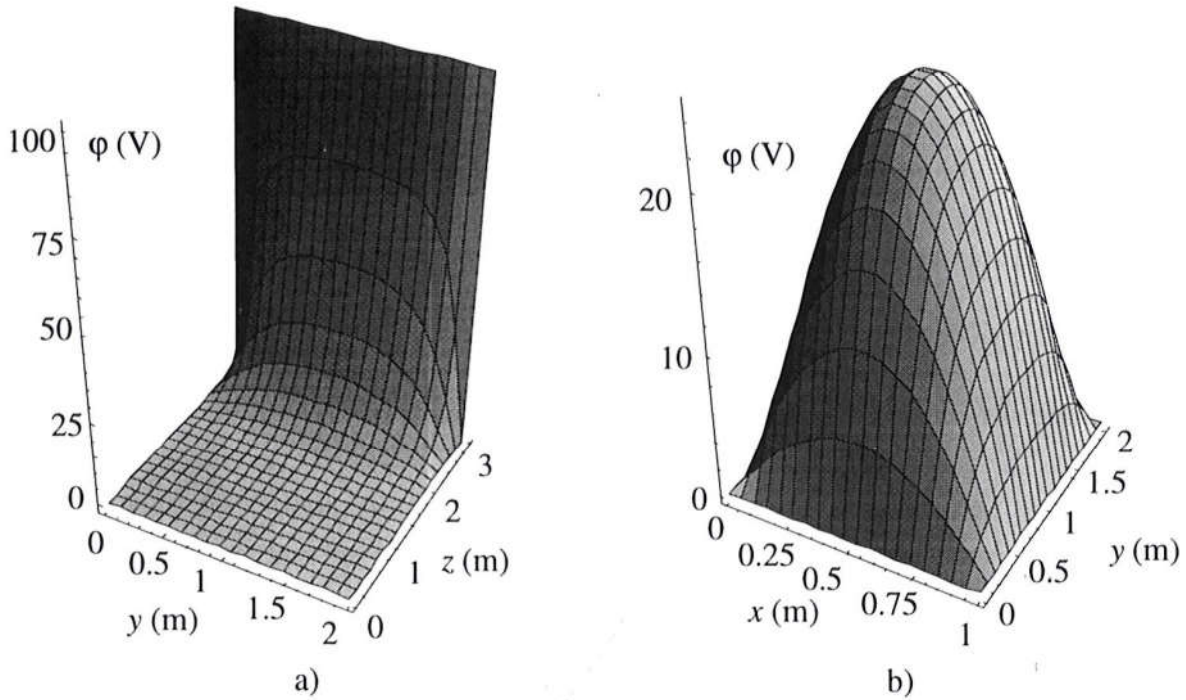


Figure 4.6.2. The distribution of potential in a rectangular parallelepiped a) in the plane $x = 0.5\text{ m}$
b) in the plane $z = 2.5\text{ m}$

4.6.2. Cylindrical coordinate system

Laplace's equation (4.1.8) in this coordinate system transforms to

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \varphi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \varphi}{\partial \alpha^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0. \quad (4.6.19)$$

If we assume a solution of the form $\varphi(r, \alpha, z) = F(r)G(\alpha)H(z)$ we can separate equation (4.6.19) into three ordinary differential equations

$$\begin{aligned} \frac{d^2 H}{dz^2} - k^2 H &= 0, \\ \frac{d^2 G}{d\alpha^2} &= -n^2 G, \\ r \frac{d}{dr} \left(r \frac{dF}{dr} \right) + (k^2 r^2 - n^2) F &= 0. \end{aligned} \quad (4.6.20 - 22)$$

Solutions of equations (4.6.20,21) are defined by expressions (4.6.9). It is worth mentioning that n can only be an integer since the solution of equation (4.6.21) must be equal for

angles α and $\alpha + 2n\pi$.

Equation (4.6.22) possesses the form of *Bessel's differential equation* and its solutions are thus *Bessel functions* $J_n(kr)$ and *Neumann functions* $N_n(kr)$ when k is a real number, and hyperbolic Bessel functions K_m and I_m when k is an imaginary number. Within this unit, we shall analyse cases where $k = 0$ is valid. In these problems, $Z(z)$ is constant, which means that these are two-dimensional problems in a cylindrical coordinate system. In this case, the functions

$$F(r) = \begin{cases} r^n & ; \quad n \neq 0, \\ C_r \ln r + D_r & ; \quad n = 0. \end{cases} \quad (4.6.23)$$

are solutions of equation (4.6.22).

Example 4.6.2.1. A cylindrical conductor enclosed by a dielectric layer in an external field

We suppose that the conductor is of infinite length and that the external field is homogeneous. This problem is illustrated in Figure 4.6.3. and we shall solve it using the principle of superposition.

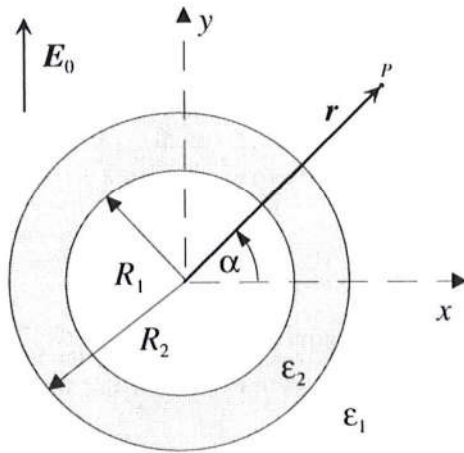


Figure 4.6.3. A cylindrical conductor enclosed by a dielectric layer in a homogeneous external field.

The potential of the homogeneous field E_0 is $\varphi_0 = -E_0 r \sin \alpha$, and we shall assume that the total potential in this problem can be expressed as a linear combination of the solutions (4.6.9) and (4.6.23) of Laplace's equation in a cylindrical coordinate system

$$\begin{aligned} \varphi_I &= -E_0 r \sin \alpha + \sum_n (a_{1n} r^n + b_{1n} r^{-n}) [c_{1n} \sin(n\alpha) + d_{1n} \cos(n\alpha)] \quad ; \quad r \geq R_2, \\ \varphi_{II} &= \sum_n (a_{2n} r^n + b_{2n} r^{-n}) [c_{2n} \sin(n\alpha) + d_{2n} \cos(n\alpha)] \quad ; \quad r \leq R_2. \end{aligned} \quad (4.6.24)$$

The coefficients a_{1n} must be equal to zero because a disturbance of the field caused by the insulated cylinder becomes negligible with increase of distance from the cylinder.

The boundary conditions that the potentials in the two regions of this problem have to satisfy are

$$\begin{aligned}
\varphi_I(R_2) &= \varphi_{II}(R_2) , \\
\epsilon_1 \left(\frac{\partial \varphi_I}{\partial r} \right)_{r=R_2} &= \epsilon_2 \left(\frac{\partial \varphi_{II}}{\partial r} \right)_{r=R_2} , \\
\varphi_{II}(R_1) &= 0 .
\end{aligned} \tag{4.6.25a - c}$$

The assumed solution (4.6.24) will satisfy the continuity condition (4.6.25a) for every value of α only if $n = 1$ and $d_{21} = 0$ are valid. The assumed solution now takes the form

$$\begin{aligned}
\varphi_I(r, \alpha) &= \left(-E_0 r + b_1 \frac{1}{r} \right) \sin \alpha , \\
\varphi_{II}(r, \alpha) &= \left(a_2 r + b_2 \frac{1}{r} \right) \sin \alpha .
\end{aligned} \tag{4.6.26a,b}$$

Substitution of this solution into (4.6.25a - c) results in a system for unknown coefficients b_1 , a_2 and b_2 , and the final form of the expression for the potential is

$$\varphi = \begin{cases} -E_0 \left[r - \frac{R_2^2 - \epsilon_1(R_2^2 - R_1^2) + \epsilon_2(R_2^2 + R_1^2)}{\epsilon_1(R_2^2 - R_1^2) + \epsilon_2(R_2^2 + R_1^2)} \right] \sin \alpha & ; \quad r \geq R_2 , \\ -\frac{2E_0 \epsilon_1 R_2^2}{\epsilon_1(R_2^2 - R_1^2) + \epsilon_2(R_2^2 + R_1^2)} \left(r - \frac{R_1^2}{r} \right) \sin \alpha & ; \quad R_1 \leq r \leq R_2 . \end{cases} \tag{4.6.27}$$

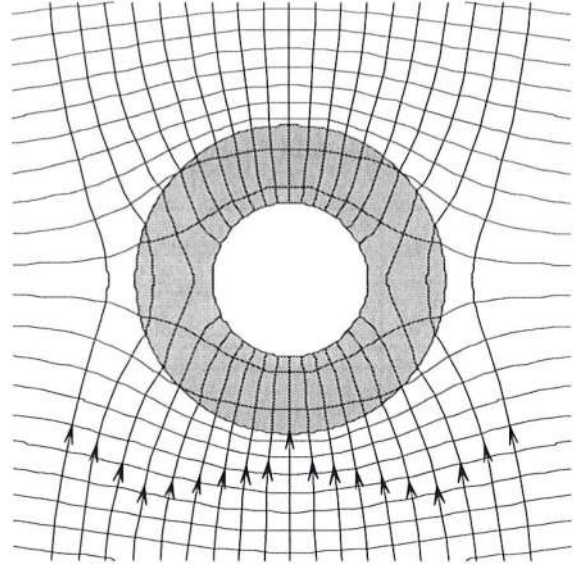


Figure 4.6.4. Plot of an electric field of a cylindrical insulated conductor in an external homogeneous field

The electric field strength is calculated using equation (4.1.9). The static electric field for this example is shown in Figure 4.6.4.

4.6.3. A spherical coordinate system

Laplace's equation (4.1.8) in this coordinate system transforms to

$$\frac{\partial}{\partial r} \left(r^2 \frac{\partial \varphi}{\partial r} \right) + \frac{1}{\sin \vartheta} \frac{\partial}{\partial \vartheta} \left(\sin \vartheta \frac{\partial \varphi}{\partial \vartheta} \right) + \frac{1}{\sin^2 \alpha} \frac{\partial^2 \varphi}{\partial \alpha^2} = 0 . \quad (4.6.28)$$

If we assume that the solution is of the form $\varphi(r, \vartheta, \alpha) = F(r)G(\vartheta)H(\alpha)$, we may apply a procedure similar to those described in previous units and can thus separate equation (4.6.28) into three ordinary differential equations

$$\begin{aligned} \frac{1}{H} \frac{d^2 H}{d\alpha^2} + m^2 &= 0 , \\ \frac{1}{\sin \vartheta} \frac{d}{d\vartheta} \left(\sin \vartheta \frac{dG}{d\vartheta} \right) + \left[n(n+1) - \frac{m^2}{\sin^2 \vartheta} \right] G &= 0 , \\ \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dF}{dr} \right) - \frac{n(n+1)}{r^2} F &= 0 , \end{aligned} \quad (4.6.29 - 31)$$

where we have denoted constants by m^2 and $n(n+1)$. Consequently, the solution of Laplace's equation in a spherical coordinate system may be written as

$$\varphi(r, \vartheta, \alpha) = (A r^n + B r^{-n-1}) [\cos(m\alpha) + \sin(m\alpha)] P_n^m(\cos \vartheta) . \quad (4.6.32)$$

Such functions are called *spherical harmonics*. We shall study only axially symmetrical problems in this unit. In these cases $m = 0$, and solutions of Laplace's equation are Legendre polynomials $P_n(\cos \vartheta)$ of $\cos \vartheta$. If this is the case, solutions of Laplace's equation in this case are called *zonal harmonics*.

Table 4.6.1. Solutions of Laplace's equation in the spherical coordinate system

n	$P_n(\cos \vartheta)$	<i>Zonal harmonics</i>	
0	1	1	$\frac{1}{r}$
1	$\cos \vartheta$	$r \cos \vartheta$	$\frac{1}{r^2} \cos \vartheta$
2	$\frac{1}{2}(3 \cos^2 \vartheta - 1)$	$\frac{r^2}{2}(3 \cos^2 \vartheta - 1)$	$\frac{1}{2r^3}(3 \cos^2 \vartheta - 1)$

Example 4.6.3.1. A sphere in an external homogeneous field

We are going to construct the solution of a problem of a conductive sphere with radius R in an external homogeneous field (Figure 4.6.5).

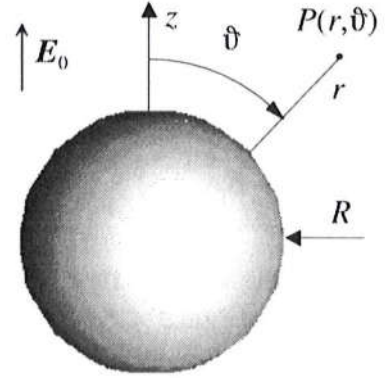


Figure 4.6.5. Sphere in an external homogeneous field

The conditions that the solution for potential must satisfy are $\varphi = V$ for $r = R$ and $\varphi = -E_0 z = -E_0 r \cos \vartheta$ for $r \gg R$, because the influence of the sphere is local and the field far away from the sphere is homogeneous.

We shall assume that the solution is of the form of a linear combination of zonal harmonics of both zero and first order

$$\varphi(r) = A + \frac{B}{r} + C r \cos \vartheta + \frac{D}{r^2} \cos \vartheta . \quad (4.6.33)$$

From the condition for $r \gg R$ it follows that $A = 0$ and $C = -E_0$. From the condition for $r = R$ it follows that $B = VR$ and $D = E_0 R^3$. The final expression for the potential is

$$\varphi(r, \vartheta) = \frac{RV}{r} + E_0 \left[-1 + \left(\frac{R}{r} \right)^3 \right] r \cos \vartheta . \quad (4.6.34)$$

The electric field strength is

$$\mathbf{E} = -\nabla \varphi = \left\{ \frac{RV}{r^2} + E_0 \left[1 + 2 \left(\frac{R}{r} \right)^3 \right] \right\} \cos \vartheta \mathbf{a}_r + E_0 \left[-1 + \left(\frac{R}{r} \right)^3 \right] \sin \vartheta \mathbf{a}_\vartheta . \quad (4.6.35)$$

The static electric field of a metal sphere of potential equal to zero ($V = 0$) in a homogeneous external field is plotted in Figure 4.6.6a.

If we put a dielectric sphere of permittivity ϵ_1 in place of the metal sphere into the homogeneous electric field, the potential must satisfy the conditions

$$\begin{aligned} r = R & \quad ; \quad \epsilon_1 \frac{\partial \varphi_1}{\partial r} = \epsilon_0 \frac{\partial \varphi_0}{\partial r} , \\ r \gg R & \quad ; \quad \varphi_0 = E_0 z = E_0 r \cos \vartheta . \end{aligned} \quad (4.6.36)$$

The conditions in this case are more complex because we need to determine the potentials φ_i and φ_o in *two regions*: the region inside the sphere and the region outside the sphere.

In each region, the potential satisfies Laplace's equation and we assume that the solutions are linear combinations of the zonal harmonics for $n = 0$ and $n = 1$. We determine unknown constants by substitution into the boundary conditions. The final solution for the potential inside and outside the sphere is

$$\varphi_i = -\frac{3E_0\varepsilon_0}{2\varepsilon_0+\varepsilon_1}r\cos\vartheta, \quad (4.6.37,38)$$

$$\varphi_o = E_0\left[-1 + \left(\frac{R}{r}\right)^3 \frac{\varepsilon_1-\varepsilon_0}{2\varepsilon_0+\varepsilon_1}\right]r\cos\vartheta.$$

The static electric field of a dielectric sphere in an external homogeneous field is presented in Figure 4.6.6b.

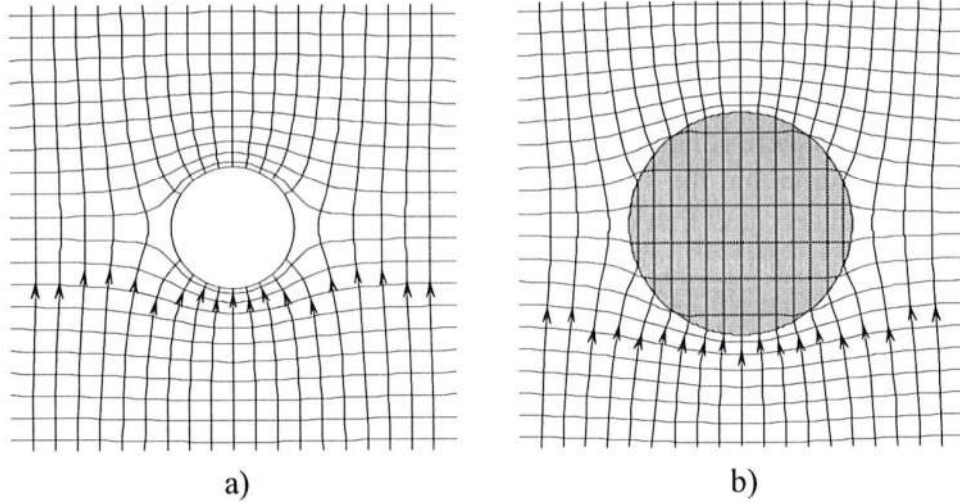


Figure 4.6.6. Static electric field of a sphere in an external homogeneous field;
a) metal sphere of zero potential, b) dielectric sphere, $\varepsilon_r = 3$

4.7. A GIVEN DISTRIBUTION OF CHARGE IN AN UNBOUNDED SPACE

We derive the solution of such problems from the general integral solution of Poisson's equation as defined by expression (4.5.8). The space is unbounded and therefore the contribution of the term representing an integral over the boundary surface S of the space in (4.5.8) is equal to zero. The general integral form of the solution of such a problem is

$$\varphi(\mathbf{r}, \mathbf{r}') = \int_V \rho(\mathbf{r}) G(\mathbf{r}, \mathbf{r}') dV', \quad (4.7.1)$$

where G is Green's function of an unbounded space, which represents the potential of a unit point charge in an unbounded space.

4.7.1. A group of point charges

In Section 2.4.2. we proved that the solution of Poisson's equation for one point charge Q placed at a point $\mathbf{r}'$ of an unbounded space is

$$\varphi = \frac{Q}{4\pi\epsilon |\mathbf{r} - \mathbf{r}'|} , \quad (4.7.2)$$

and we can apply the principle of superposition to determine the potential and electric field strength at a point $\mathbf{r}$ of a group of N point charges placed at points $\mathbf{r}_i'$

$$\varphi = \sum_{i=1}^N \frac{Q_i}{4\pi\epsilon |\mathbf{r} - \mathbf{r}_i'|} , \quad (4.7.3,4)$$

$$\mathbf{E} = -\nabla\varphi = \sum_{i=1}^N \frac{Q_i}{4\pi\epsilon |\mathbf{r} - \mathbf{r}_i'|^2} \frac{\mathbf{r} - \mathbf{r}_i'}{|\mathbf{r} - \mathbf{r}_i'|} .$$

Example 4.7.1.1. The field of two point charges

Two point charges Q_1 and Q_2 are placed in an unbounded space as shown in Figure 4.7.1.

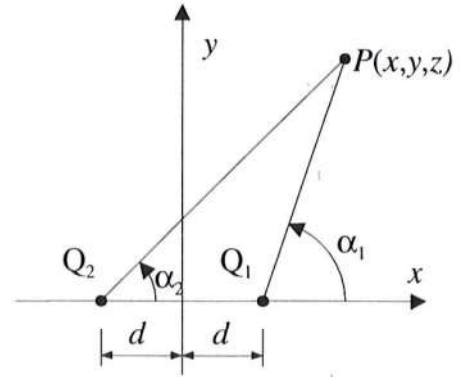


Figure 4.7.1 Two point charges in an unbounded space

Their resultant potential at point $P(x, y, z)$ according to (4.7.3) is

$$\varphi(x, y, z) = \frac{1}{4\pi\epsilon} \left[\frac{Q_1}{\sqrt{(x-d)^2 + y^2 + z^2}} + \frac{Q_2}{\sqrt{(x+d)^2 + y^2 + z^2}} \right] . \quad (4.7.5)$$

Equipotential lines in the $x - y$ plane ($z = 0$) are implicitly defined by the expression $\varphi(x, y, z = 0) = C$. The electric field strength of given point charges at point $P(x, y, z)$ is

$$\begin{aligned} \mathbf{E}(x, y, z) = & \frac{Q_1}{4\pi\epsilon [(x-d)^2 + y^2 + z^2]^{3/2}} [(x-d)\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z] + \\ & + \frac{Q_2}{4\pi\epsilon [(x+d)^2 + y^2 + z^2]^{3/2}} [(x+d)\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z] . \end{aligned} \quad (4.7.6)$$

Field lines in the $x - y$ plane are defined by equation (4.2.1) and it follows that

$$Q_1 \frac{x+d}{\sqrt{(x+d)^2 + y^2}} + Q_2 \frac{x-d}{\sqrt{(x-d)^2 + y^2}} = k , \quad (4.7.7)$$

is the equation for the field lines. Field plots for cases $\pm Q_1 = Q_2$ are shown in Figure 4.7.2.

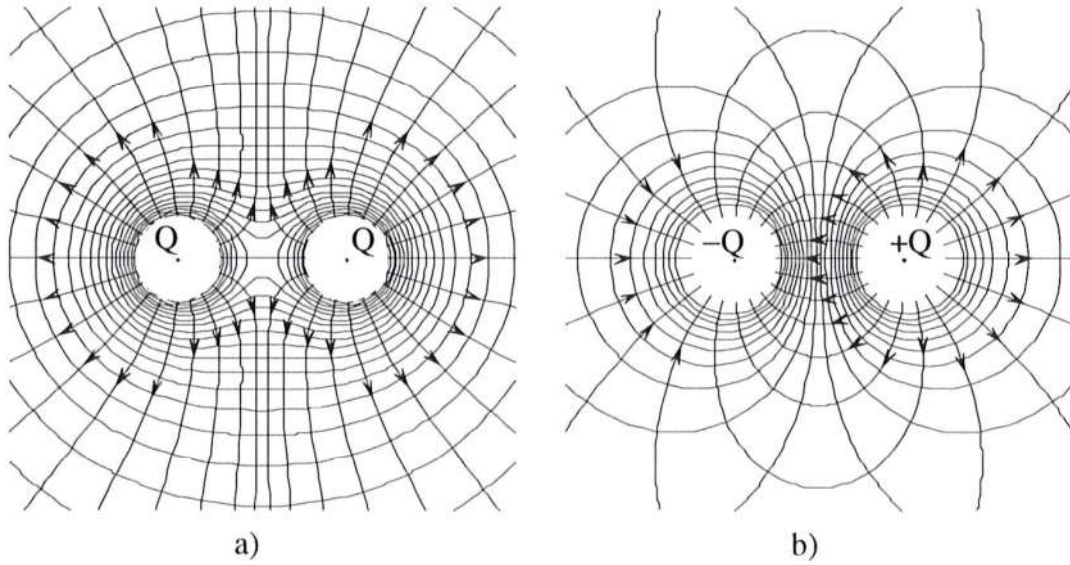


Figure 4.7.2. Plot of the static electric field of two point charges with a) equal size, b) equal size with opposite signs

Example 4.7.1.2. The field of a dipole

The notion of a dipole as a configuration of two charges with equal amounts of charge and opposite signs (Figure 4.7.3) was introduced in Chapter 1.

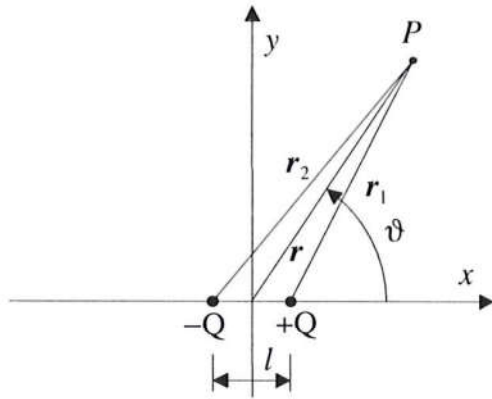


Figure 4.7.3. Electric dipole

The relationship between r and r_1 is

$$r_1^2 = r^2 + \left(\frac{l}{2}\right)^2 + rl \cos \vartheta \Rightarrow \frac{r}{r_1} = \frac{1}{\sqrt{1 + \left(\frac{l}{2r}\right)^2 + \frac{l}{r} \cos \vartheta}} \quad (4.7.8)$$

We can expand this expression into a series expansion of the function $(1 + x)^{-1}$. We study the dipole only in cases where $r \gg l$. In this case, the convergence criterion of the series is satisfied and we can neglect all terms of the series that contain powers higher than $(l/r)^2$.

The potential of a dipole at a point P is

$$\varphi = \frac{1}{4\pi\epsilon} \left(\frac{Q}{r_1} - \frac{Q}{r_2} \right) \approx \frac{1}{4\pi\epsilon} \frac{Ql \cos \vartheta}{r^2} = \frac{1}{4\pi\epsilon} \frac{\mathbf{p} \cdot \mathbf{r}}{r^3}, \quad (4.7.9)$$

where $\mathbf{p}$ is the *dipole moment*, a vector directed from negative to positive charge with a magnitude equal to Ql . Equipotential lines are defined by

$$\varphi = C = \frac{1}{4\pi\epsilon} \frac{p \cos \vartheta}{r^2} \Rightarrow r^2 = \frac{p}{4\pi\epsilon C} \cos \vartheta, \quad (4.7.10)$$

where C is a chosen potential. The electric field strength of a dipole in a spherical coordinate system is

$$\mathbf{E} = -\nabla\varphi = -\frac{\partial\varphi}{\partial r} \mathbf{a}_r - \frac{1}{r} \frac{\partial\varphi}{\partial\vartheta} \mathbf{a}_\vartheta = \frac{2p}{4\pi\epsilon r^3} \cos \vartheta \mathbf{a}_r + \frac{p}{4\pi\epsilon} \quad (4.7.11)$$

and we can express it as a function of $\mathbf{p}$ and $\mathbf{r}$ with the help of

$$\mathbf{E} = -\frac{1}{4\pi\epsilon} \nabla \left(\frac{\mathbf{p} \cdot \mathbf{r}}{r^3} \right) = \frac{1}{4\pi\epsilon} \left[3 \frac{\mathbf{p} \cdot \mathbf{r}}{r^5} \mathbf{r} - \frac{\mathbf{p}}{r^3} \right]. \quad (4.7.12)$$

We determine the field lines of a dipole using

$$\frac{dr}{E_r} = \frac{r d\vartheta}{E_\vartheta} \Rightarrow \frac{dr}{r d\vartheta} = \frac{E_r}{E_\vartheta}, \quad (4.7.13)$$

and the equation for the field lines of a dipole is

$$r = k \sin^2 \vartheta. \quad (4.7.14)$$

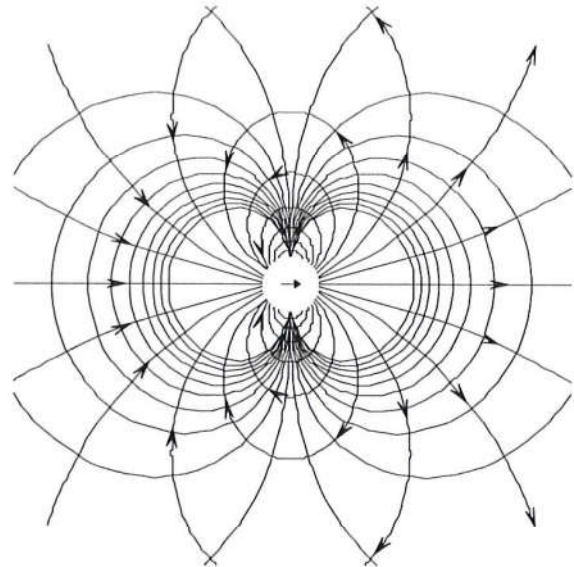


Figure 4.7.4. The static electric field of a dipole

4.7.2. Line charges

We introduced and defined a line charge in Section 1.2.4, and we calculate it with the help of the equation

$$Q = \int_l \lambda dl, \quad (4.7.15)$$

where λ is the line charge density and l is the line on which the charge is distributed. To a part of this line dl that carries the charge λdl we can assign the characteristics of a point charge, and we can express the potential of the line charge in an unbounded space using the principle of superposition as

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\epsilon} \int_l \frac{\lambda dl}{|\mathbf{r} - \mathbf{r}'|}, \quad (4.7.16)$$

and the electric field strength can be determined from the potential using (4.1.9).

Example 4.7.2.1. The field of a uniformly charged line segment

A segment of a line charged with a charge of uniform density λ is shown in Figure 4.7.5. We are solving the problem in a cylindrical coordinate system because of the axial symmetry of the arrangement.

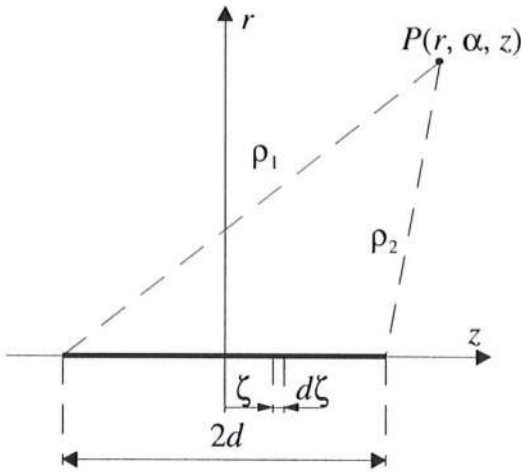


Figure 4.7.5. Uniformly charged line segment

According to equation (4.7.16), we calculate the potential at a point $P(r, \alpha, z)$ as

$$\varphi(r, \alpha, z) = \frac{1}{4\pi\epsilon} \int_{-d}^d \frac{\lambda d\zeta}{\sqrt{(z-\zeta)^2 + r^2}} = \frac{\lambda}{4\pi\epsilon} \ln \frac{z+d + \sqrt{(z+d)^2 + r^2}}{z-d + \sqrt{(z-d)^2 + r^2}}, \quad (4.7.17)$$

since the distance between elements of the line charge and the point is independent of the angle α . If we introduce distances ρ_1 and ρ_2 according to Figure 4.7.5, we can write the equation for equipotential lines in a plane of constant angle $\alpha = 0$ as

$$\frac{z+d+\rho_1}{z-d+\rho_2} = k = e^{\frac{4\pi\epsilon}{\lambda}\varphi}. \quad (4.7.18)$$

The expression $r^2 = \rho_1^2 - (z+d)^2 = \rho_2^2 - (z-d)^2$ is valid, and we obtain the equation for equipotential lines as being

$$\rho_1 + \rho_2 = 2d \frac{k+1}{k-1}. \quad (4.7.19)$$

According to this equation, the equipotential lines are *confocal ellipses* with foci at the ends of the segment charged with a uniform charge density, and *the equipotential surfaces are confocal rotational ellipsoids*. The axis of rotation is the z -axis, and on the same axis is the major axis of the ellipse a . For an individual equipotential line, the major axis a and the minor axis b are

$$a = d \frac{k+1}{k-1}; \quad b = 2d \frac{\sqrt{k}}{k-1}. \quad (4.7.20)$$

Since any equipotential surface can be taken as a new conductor surface, equation (4.7.17) also gives the potential distribution surrounding a rotational ellipsoid. If the major and minor axes a and b are given, we get $d^2 = a^2 - b^2$. The total charge on the ellipsoid is $Q = 2\lambda d$, and the potential on the surface of the ellipsoid is

$$\varphi_s = \frac{\lambda}{4\pi\epsilon} \ln \frac{a+d}{a-d} = \frac{1}{4\pi\epsilon} \frac{Q}{2d} \ln \frac{a+d}{a-d}, \quad (4.7.21)$$

and by usage of this equation we are able to determine the *capacitance of a solitary rotational ellipsoid* as

$$C = \frac{Q}{\varphi_s} = \frac{8\pi\epsilon d}{\ln \frac{a+d}{a-d}}. \quad (4.7.22)$$

For large ratios of a/b , the ellipsoid can be used as an approximation of the shape of a cylindrical rod with radius r and length l . If we choose the approximation using an ellipsoid for which $2a = l$ and $b = r$, for ratios $r/l < 0.05$ we obtain the expression

$$d = \sqrt{a^2 - b^2} = a \sqrt{1 - \frac{b^2}{a^2}} \approx a \left(1 - \frac{b^2}{2a^2} \right). \quad (4.7.23)$$

Substitution of this approximation into the expression for the potential of the ellipsoid (4.7.21) results in

$$\varphi_s = \frac{1}{4\pi\epsilon} \frac{Q}{2a} \ln \frac{2a}{\frac{b^2}{2a^2}} = \frac{1}{4\pi\epsilon} \frac{Q}{a} \ln \frac{2a}{b}, \quad (4.7.24)$$

and we can calculate the *capacitance of a rod approximated in this way* by usage of the

expression

$$C = \frac{Q}{\Phi_s} = \frac{2\pi\epsilon l}{\ln \frac{l}{r}}. \quad (4.7.25)$$

The electric field strength of a uniformly charged line segment is

$$\begin{aligned} \mathbf{E} &= -\nabla\phi = -\frac{\partial\phi}{\partial z}\mathbf{a}_z - \frac{\partial\phi}{\partial r}\mathbf{a}_r = \frac{\lambda}{4\pi\epsilon} \left[\left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) \mathbf{a}_z + \left(\frac{z+d}{r\rho_1} - \frac{z-d}{r\rho_2} \right) \mathbf{a}_r \right] = \\ &= \frac{\lambda}{4\pi\epsilon r} [(\sin\beta_1 - \sin\beta_2)\mathbf{a}_z + (\cos\beta_1 - \cos\beta_2)\mathbf{a}_r]. \end{aligned} \quad (4.7.26)$$

The field lines are defined by the equation

$$\sqrt{(z-d)^2 + r^2} - \sqrt{(z+d)^2 + r^2} = \rho_1 - \rho_2 = k. \quad (4.7.27)$$

This equation represents *confocal hyperbole* in the plane $\alpha = \text{const.}$

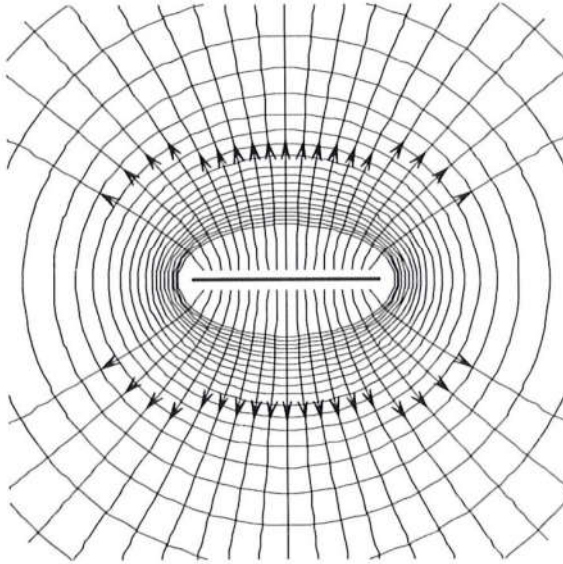


Figure 4.7.6. The static electric field of a uniformly charged line segment

Example 4.7.2.2. Infinite line charge

Let us imagine that a line is uniformly charged with a line charge density λ . The electric field strength in this case is determined from the electric field strength of a uniformly charged line segment if we allow the length of the segment to become infinite, i.e.

$$\begin{aligned} \mathbf{E} &= \frac{\lambda}{4\pi\epsilon} \lim_{d \rightarrow \infty} \left\{ \left[\frac{1}{\sqrt{(z+d)^2 + r^2}} - \frac{1}{\sqrt{(z-d)^2 + r^2}} \right] \mathbf{a}_z + \right. \\ &\quad \left. + \frac{1}{r} \left[\frac{z+d}{\sqrt{(z+d)^2 + r^2}} - \frac{z-d}{\sqrt{(z-d)^2 + r^2}} \right] \mathbf{a}_r \right\} = \frac{\lambda}{2\pi\epsilon r} \mathbf{a}_r. \end{aligned} \quad (4.7.28)$$

We determine potential with the help of equation (4.1.11) by direct integration

$$\varphi = - \int_{r_0}^r \mathbf{E} \cdot d\mathbf{l} = - \frac{\lambda}{2\pi\epsilon} \int_{r_0}^r \frac{\mathbf{a}_r}{r} \cdot \mathbf{a}_r dr = \frac{\lambda}{2\pi\epsilon} \ln \frac{r_0}{r} . \quad (4.7.29)$$

The position of the reference point to which we assign a zero potential is arbitrary. We can talk only about potential differences since the potential function becomes infinite for both boundary values $r = 0$ and $r \rightarrow \infty$, and the absolute potential is infinite at every point.

Such an arrangement is not possible because the total charge in it would be infinite, but it represents a very useful approximation for the analysis of a system of conductors with a length much larger than the dimensions of their cross-section

4.7.3. Surface charge

In Section 1.2.4. we introduced and defined a surface charge with density σ distributed over a surface S using the expression

$$Q = \int_S \sigma dS . \quad (4.7.30)$$

To a part of the surface dS that carries charge σdS we can assign the characteristics of a point charge, and by application of the superposition principle we can calculate the potential of the surface charge in an unbounded space as

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\epsilon} \int_S \frac{\sigma dS}{|\mathbf{r} - \mathbf{r}'|} , \quad (4.7.31)$$

and the electric field strength can be calculated using expression (4.1.9).

Example 4.7.3.1 The field of a uniformly charged rectangular plate

A rectangular plate uniformly charged with a surface charge density σ is shown in Figure 4.7.7.

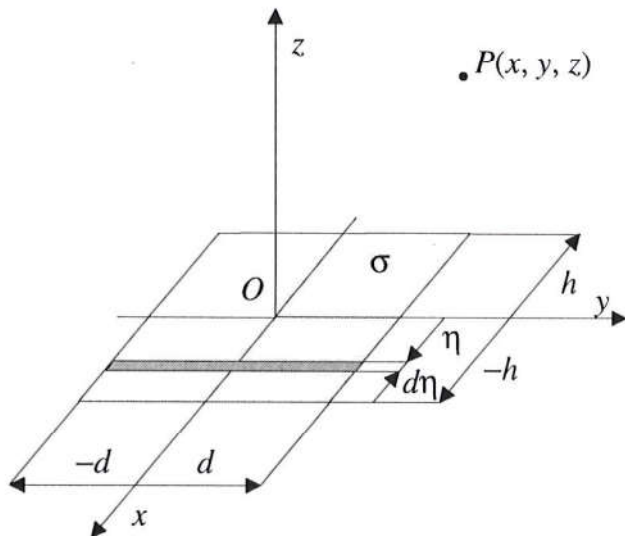


Figure 4.7.7. Uniformly charged rectangular plate

We determine the potential at a point $P(x,y,z)$ using the results of Example 4.7.2.1. The plate is divided into differentially thin parts to which the characteristics of a uniformly charged line segment may be assigned. The total potential is calculated by the superposition of all of these thin parts as

$$\varphi(x,y,z) = \frac{\sigma}{4\pi\epsilon} \int_{-h}^h \ln \frac{x+d+\sqrt{(x+d)^2+(y-\eta)^2+z^2}}{x-d+\sqrt{(x-d)^2+(y-\eta)^2+z^2}} d\eta. \quad (4.7.32)$$

After a relatively complex integration procedure, we obtain

$$\varphi(x,y,z) = \frac{\sigma}{4\pi\epsilon} [F(h) + G(h) - F(-h) - G(-h)], \quad (4.7.33)$$

where

$$\begin{aligned} F(\eta) = & -z \arctan \frac{-z(x+d)(y-\eta) - yz\sqrt{(x+d)^2+(y-\eta)^2+z^2}}{z^2\sqrt{(x+d)^2+(y-\eta)^2+z^2} - y(x+d)(y-\eta)^2} + \\ & + z \arctan \frac{-z(x-d)(y-\eta) - yz\sqrt{(x-d)^2+(y-\eta)^2+z^2}}{z^2\sqrt{(x-d)^2+(y-\eta)^2+z^2} - y(x-d)(y-\eta)^2} + \\ & + (x+d) \ln \left[-(y-\eta) + \sqrt{(x+d)^2+(y-\eta)^2+z^2} \right] - \\ & - (x-d) \ln \left[-(y-\eta) + \sqrt{(x-d)^2+(y-\eta)^2+z^2} \right], \\ G(\eta) = & \eta \ln \frac{(x+d) + \sqrt{(x+d)^2+(y-\eta)^2+z^2}}{(x-d) + \sqrt{(x-d)^2+(y-\eta)^2+z^2}} - \\ & - \frac{y}{2} \ln \left\{ [(x+d)^2+z^2] \left[2(x+d)^2+(y-\eta)^2+z^2 + 2(x+d)\sqrt{(x+d)^2+(y-\eta)^2+z^2} \right] \right\} + \\ & + \frac{y}{2} \ln \left\{ [(x-d)^2+z^2] \left[2(x-d)^2+(y-\eta)^2+z^2 + 2(x-d)\sqrt{(x-d)^2+(y-\eta)^2+z^2} \right] \right\}. \end{aligned}$$

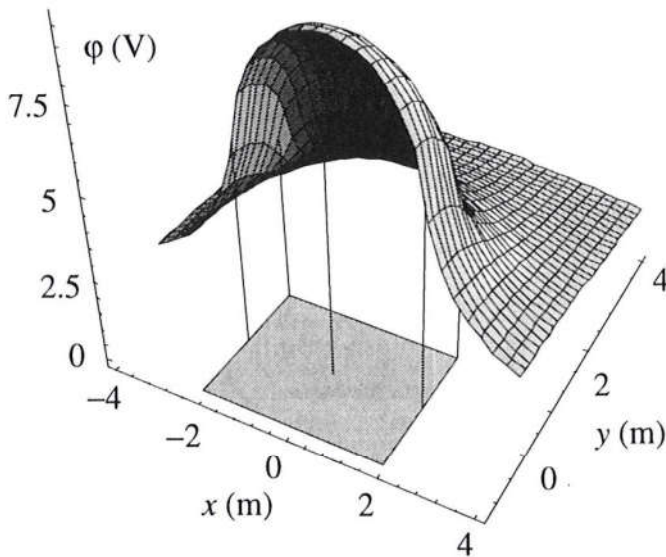


Figure 4.7.8. Distribution of potential of a uniformly charged plate in the plane of the plate

In Figure 4.7.8. we have showed the distribution of potential of a uniformly charged plate with surface charge density $\sigma = 4\pi\epsilon$ and dimensions $d = 2\text{ m}$ and $h = 1\text{ m}$ in the plane $z = 0$.

4.7.4. Volume charge and a multipole expansion

The potential of the volume charge is

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\epsilon} \int_V \frac{\rho(\mathbf{r}') dV}{|\mathbf{r} - \mathbf{r}'|} . \quad (4.7.35)$$

Calculation of a triple integral in (4.7.35) most frequently cannot be carried out using analytical methods and we therefore use numerical approximations.

The simplest approximation is a *multipole expansion of the potential*. Charge is distributed within volume V , and we choose the origin of the coordinate system inside this volume (Figure 4.7.9).

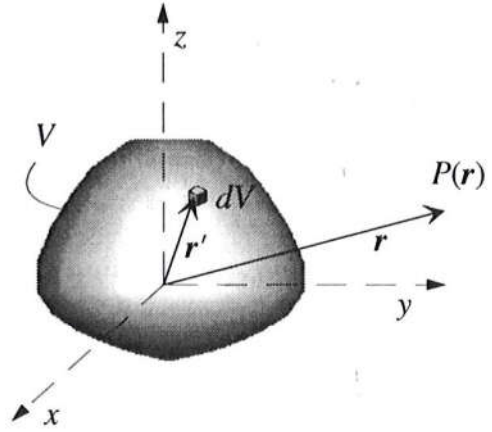


Figure 4.7.9. With the multipole expansion of the potential

We can determine the distance between points $\mathbf{r}$ and $\mathbf{r}'$ from a triangle defined by $\mathbf{r}$ and $\mathbf{r}'$

$$|\mathbf{r} - \mathbf{r}'| = \sqrt{r^2 + r'^2 - 2\mathbf{r} \cdot \mathbf{r}'} . \quad (4.7.36)$$

After the application of a series expansion of function $(1 + x)^{-1}$ and after neglecting the terms with indices r^{-3} , r^{-4} ... we obtain

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{r} \left\{ 1 + \frac{1}{r} \left(\frac{\mathbf{r} \cdot \mathbf{r}'}{r} \right) + \frac{1}{r^2} \left[\frac{3}{2} \left(\frac{\mathbf{r} \cdot \mathbf{r}'}{r} \right)^2 - \frac{1}{2} r'^2 \right] + \dots \right\} . \quad (4.7.37)$$

Substitution of this approximation into equation (4.7.35) results in

$$\begin{aligned} \varphi(\mathbf{r}) = & \frac{1}{4\pi\epsilon r} \int_V \rho(\mathbf{r}') dV + \frac{1}{4\pi\epsilon r^2} \int_V \frac{\mathbf{r} \cdot \mathbf{r}'}{r} \rho(\mathbf{r}') dV + \\ & + \frac{1}{4\pi\epsilon r^3} \int_V \left[\frac{3}{2} \left(\frac{\mathbf{r} \cdot \mathbf{r}'}{r} \right)^2 - \frac{1}{2} r'^2 \right] \rho(\mathbf{r}') dV + \dots \end{aligned} \quad (4.7.38)$$

Let us analyse terms of this approximation in sequence. The first term represents the

potential at point $\mathbf{r}$ caused by the total charge concentrated at the origin. We call this term *the monopole term*, and if all the charges are of the same sign, this term makes the greatest contribution to the total potential. The second term decreases with $1/r^2$ similar to the potential of a dipole and we call it *the dipole term*. The potential of the third term decreases with $1/r^3$ and we call it *the quadrupole term*.

Example 4.7.4.1. The field of a non-uniformly charged cube

A non-uniformly charged cube with dimensions $(2a \times 2a \times 2a)$ is placed at the origin of a Cartesian coordinate system. The volume charge density ρ is given as

$$\rho(z) = \begin{cases} \rho_0 & ; \quad -\frac{a}{2} \leq z \leq a ; \quad -a \leq y \leq a ; \quad -a \leq x \leq a , \\ -\rho_0 & ; \quad -a \leq z < -\frac{a}{2} ; \quad -a \leq y \leq a ; \quad -a \leq x \leq a \end{cases} \quad (4.7.39)$$

Even in this very simple case it is not possible to calculate the integral (4.7.35) analytically. We shall apply a multipole expansion in order to obtain an approximate solution. We calculate the contribution of the monopole term as

$$\int_V \rho dV = -2\rho_0 a^3 + 6\rho_0 a^3 = 4\rho_0 a^3 . \quad (4.7.40)$$

For the contribution of the dipole term we get

$$\begin{aligned} \int_V \left(\frac{\mathbf{r}}{r} \cdot \mathbf{r}' \right) \rho(\mathbf{r}') dV &= \int_{-\frac{a}{2}}^{\frac{a}{2}} dz' \int_{-a}^a dy' \int_{-a}^a dx' \rho_0 \frac{xx' + yy' + zz'}{\sqrt{x^2 + y^2 + z^2}} + \\ &+ \int_{-\frac{a}{2}}^{\frac{a}{2}} dz' \int_{-a}^a dy' \int_{-a}^a dx' (-\rho_0) \frac{xx' + yy' + zz'}{\sqrt{x^2 + y^2 + z^2}} = \frac{3a^4 z}{2\sqrt{x^2 + y^2 + z^2}} \rho_0 , \end{aligned} \quad (4.7.41)$$

and for the contribution of the quadrupole term we have

$$\int_V \left[\frac{3}{2} \frac{(xx' + yy' + zz')^2}{x^2 + y^2 + z^2} - \frac{1}{2} (x'^2 + y'^2 + z'^2) \right] \rho(x', y', z') dV = \frac{a^5 (x^2 + y^2 - 2z^2)}{2(x^2 + y^2 + z^2)} \rho_0 \quad (4.7.42)$$

and using expression (4.7.38) we can approximately calculate the potential at any point outside the cube by equation

$$\varphi(x, y, z) \approx \frac{\rho_0}{4\pi\epsilon} \left\{ \frac{4a^3}{\sqrt{x^2 + y^2 + z^2}} + \frac{3a^4 z}{(x^2 + y^2 + z^2)^{3/2}} + \frac{a^5 (x^2 + y^2 - 2z^2)}{(x^2 + y^2 + z^2)^{5/2}} \right\} . \quad (4.7.43)$$

4.8. DISTRIBUTION OF CHARGE IN A BOUNDED SPACE AND THE METHOD OF IMAGES

In Section 4.5.3, we established that in order to solve problems involving the static electric field of a given distribution of charge in a bounded space, one has to know Green's functions for a given geometry. In this case, we can apply the general integral solution of Poisson's equation that we determined by way of expression (4.5.8).

4.8.1. Green's functions of plane metal boundaries

4.8.1.1. A point charge in front of a conductive plane

We shall begin with the determination of the *three-dimensional Green's function of a conductive plane*. Let us analyse the simplest *Dirichlet's boundary value problem*: a point charge Q is placed in front of an infinite conductive plane which is at zero potential.

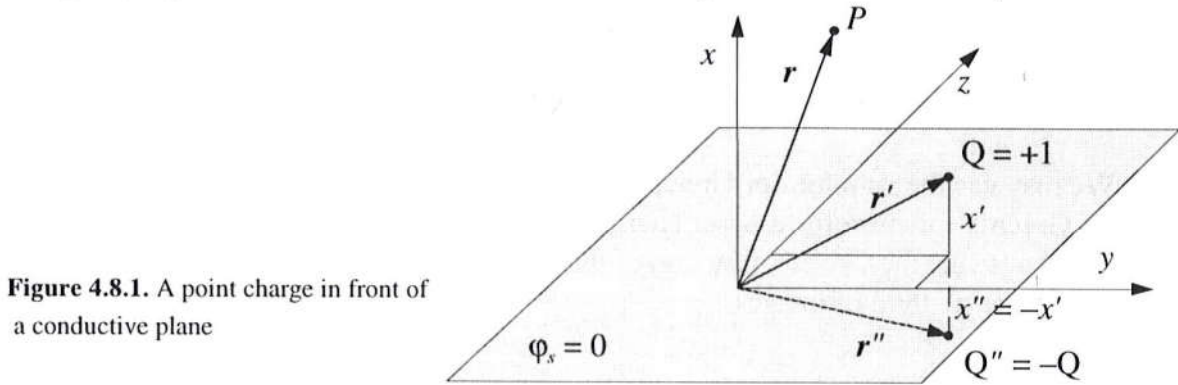


Figure 4.8.1. A point charge in front of a conductive plane

We shall try to construct Green's function for the given geometry. From equation (2.5.6) a form of Green's function follows, which is

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi\epsilon|\mathbf{r} - \mathbf{r}'|} + F(\mathbf{r}, \mathbf{r}') . \quad (4.8.1)$$

For Dirichlet's boundary value problem we prescribed the condition $G_D(\mathbf{r}, \mathbf{r}') = 0$ if $\mathbf{r}' \in S$, and we therefore obtain

$$G_D(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi\epsilon\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} + F = 0 \quad \text{for } x = 0 , \quad (4.8.2)$$

which means

$$F(x=0) = - \frac{1}{4\pi\epsilon\sqrt{(x')^2 + (y-y')^2 + (z-z')^2}} . \quad (4.8.3)$$

This expression is the potential due to a point charge $Q'' = -Q = -1$ in the plane $x = 0$. The charge Q'' is placed at a point $\mathbf{r}''$ where $x'' = \pm x'$, $y'' = y'$ and $z'' = z'$. If we choose

$x'' = x'$ Green's function shall be equal to zero everywhere. This is not the desired solution, and we therefore choose $x'' = -x'$. The final form of Green's function for the given geometry is then

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi\epsilon\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{4\pi\epsilon\sqrt{(x+x')^2+(y-y')^2+(z-z')^2}}, \quad (4.8.4)$$

and the potential of point charge Q in half space $x \geq 0$ can be calculated as

$$\varphi(\mathbf{r}) = \frac{Q}{4\pi\epsilon|\mathbf{r}-\mathbf{r}'|} + \frac{-Q}{4\pi\epsilon|\mathbf{r}-\mathbf{r}''|}. \quad (4.8.5)$$

This expression is equivalent to the expression for the potential of two point charges Q and $-Q$ placed symmetrically with respect to the plane $x = 0$. The charge $-Q$ is called the *image charge* because it represents a mirror image of charge Q through the plane $x = 0$. Such a procedure of constructing Green's functions is called *the method of images*.

On the basis of the expression (4.8.4) we can interpret Green's function G_D as the potential of a unit charge in the given geometry when the boundary surface is at zero potential. We shall exploit this interpretation for the construction of Green's functions in the continuation of this unit.

4.8.1.2. A point charge in a conductive angle

We may use the solution for Green's function from Section 4.8.1.1. for the construction of Green's function for a point charge placed in a conductive angle (Figure 4.8.2). We shall construct the Green's function of this geometry using the potential of a unit point charge and the method of images.

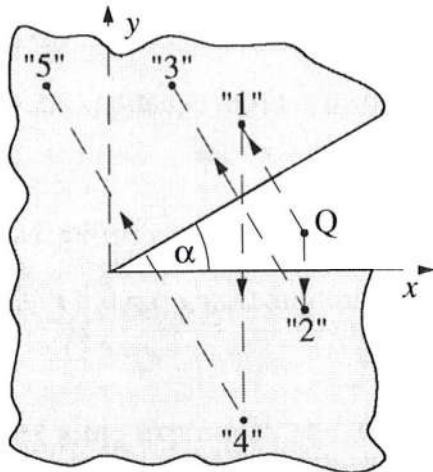


Figure 4.8.2. Point charge in a conductive angle

In order to set Green's function equal to zero in the plane $x = \cos \alpha$, $y = \sin \alpha$, we have to add a negative unit charge symmetrical to the unit charge at place "1". But now Green's function is not equal to zero in the plane $y = 0$, and we have to add opposite unit charges at places "2" and "4". This disturbs the boundary condition in the plane $x = \cos \alpha$, $y = \sin \alpha$, and we have to add opposite unit charges at places "3" and "5". This procedure can continue indefinitely, but if $\alpha = \pi/n$ (n is an integer) there will be a finite number

$(2n - 1)$ of image charges. All the charges (original and image) in such a case lie on a circle.

Example 4.8.1.1. The field of a point charge in a conductive right angle

Construction of Green's function is shown in Figure 4.8.3. The angle α is $\pi/2$, and $n = 2$, which results in three image charges.

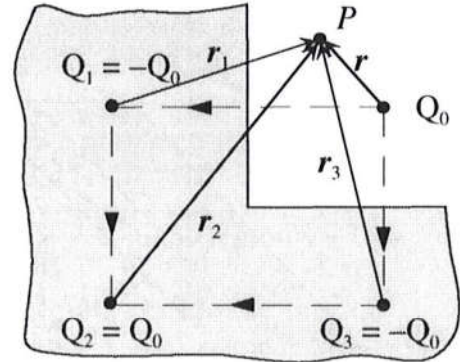


Figure 4.8.3. Point charge in a conductive right angle

The potential and field strength at all points of the rectangular space is calculated as the potential and field strength of four point charges in an unbounded space

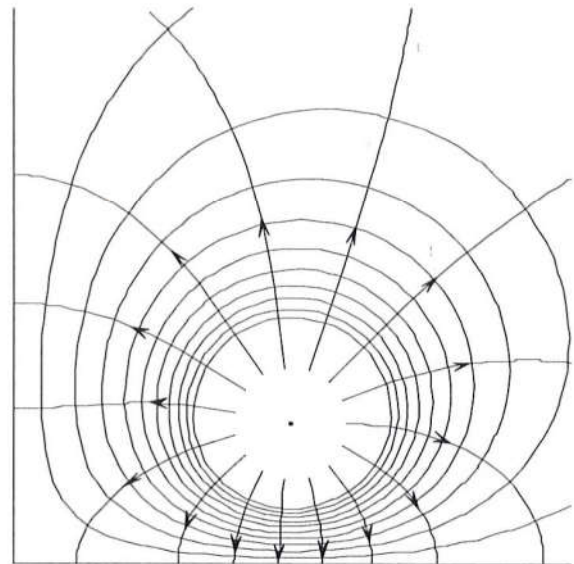


Figure 4.8.4. The field of a point charge in a conductive right angle

4.8.1.3. A line charge in front of a conductive plane

The next sort of problems for which we can determine Green's functions by the method of images are ones that involve infinite long distributions of charge independent of a longitudinal coordinate. In such cases, these problems are two-dimensional, and the place of a unit point charge in Section 4.8.1.1. is taken by a line charge of constant linear density $\lambda = 1$ C/m (oriented in the z direction) in front of a conductive plane (Figure 4.8.5).

We derived the potential of an infinitely long line charge of constant density in Section 4.7.2. If the line is placed at a point $\mathbf{r}'$, its potential at a point $\mathbf{r}$ is

$$\varphi(x,y) = -\frac{\lambda}{2\pi\epsilon} \ln \frac{|\mathbf{r} - \mathbf{r}'|}{|\mathbf{r}_0 - \mathbf{r}'|}, \quad (4.8.6)$$

where $\mathbf{r}_0$ is an arbitrary vector that defines the radius of the cylinder of reference at zero potential.

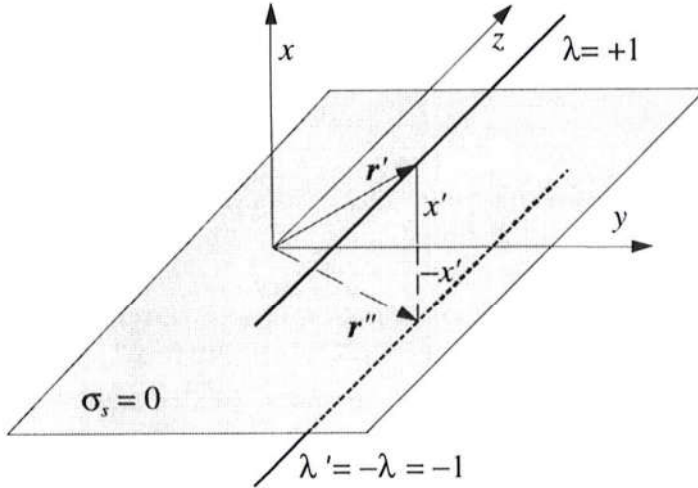


Figure 4.8.5. A line charge in front of a conductive plane

The Green's function (potential of a unit line charge) for this geometry must be equal to zero in the plane $x = 0$, and we can fulfil this requirement by adding an equal and opposite line charge in such a way that every point on the plane is equidistant from the two line charges. Accordingly, Green's function is equal to

$$G(\mathbf{r}, \mathbf{r}', \mathbf{r}'') = \frac{1}{2\pi\epsilon} \left\{ \ln \frac{|\mathbf{r}_0 - \mathbf{r}'|}{|\mathbf{r} - \mathbf{r}'|} - \ln \frac{|\mathbf{r}_0 - \mathbf{r}'|}{|\mathbf{r} - \mathbf{r}''|} \right\}, \quad (4.8.7)$$

This expression shows that the two-dimensional and three-dimensional Green's functions are similar in form. By the substitution

$$\frac{1}{4\pi\epsilon |\mathbf{r} - \mathbf{r}'|} \rightarrow \frac{1}{2\pi\epsilon} \ln \frac{|\mathbf{r}_0 - \mathbf{r}'|}{|\mathbf{r} - \mathbf{r}'|}. \quad (4.8.8)$$

we can write the two-dimensional form of Green's functions directly from the three-dimensional Green's functions.

Example 4.8.1.2. A two-wire line in front of a conductive plane

We approximate a two-wire line using a distribution of line charges at the centres of the wires because the radii of the wires ρ_0 are considerably smaller than the distance between the wires $2a$ and the distance between the wires and a conductive plane d (Figure 4.8.6).

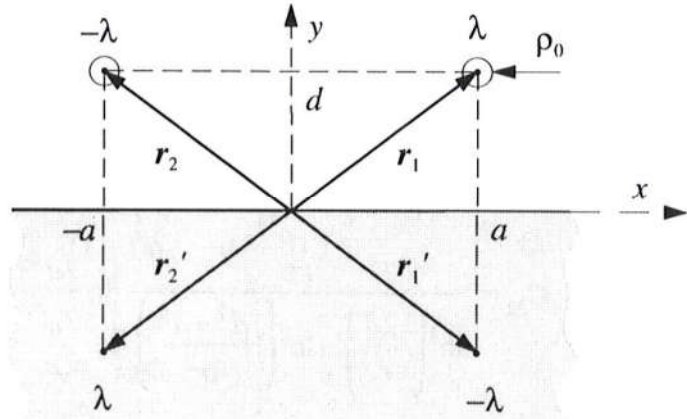


Figure 4.8.6. A two-wire line in front of a conductive plane

According to equations (4.8.6 and 7) we calculate the potential in the region $y \geq 0$ as

$$\begin{aligned} \varphi(x,y) &= \frac{\lambda}{2\pi\epsilon} \left\{ \ln \frac{|r_0 - r_1|}{|r - r_1|} - \ln \frac{|r_0 - r_1'|}{|r - r_1'|} - \ln \frac{|r_0 - r_2|}{|r - r_2|} + \ln \frac{|r_0 - r_2'|}{|r - r_2'|} \right\} = \\ &= \frac{\lambda}{4\pi\epsilon} \ln \frac{(x-a)^2 + (y+d)^2}{(x-a)^2 + (y-d)^2} \frac{(x+a)^2 + (y-d)^2}{(x+a)^2 + (y+d)^2} . \end{aligned} \quad (4.8.9)$$

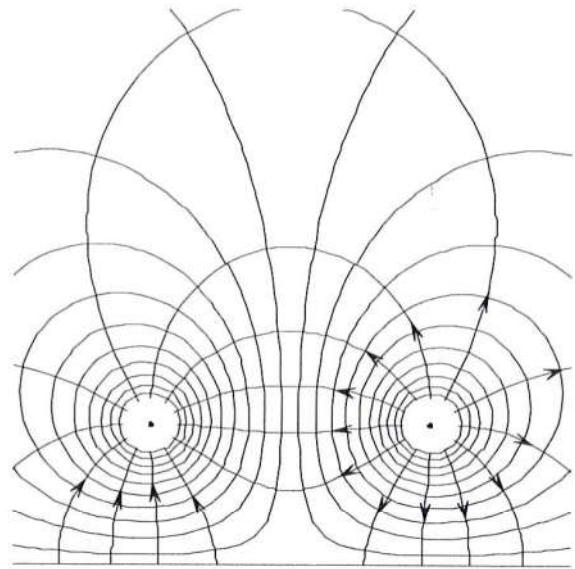


Figure 4.8.7. The field of a two-wire line in front of a conductive plane

We can use this solution to determine the partial capacitances of such a line. The potentials of the wires are

$$\begin{aligned} \varphi_1 &= \frac{\lambda}{4\pi\epsilon} \ln \frac{2d}{r_0} - \frac{\lambda}{4\pi\epsilon} \ln \frac{d^2 + a^2}{a^2} , \\ \varphi_2 &= \frac{\lambda}{4\pi\epsilon} \ln \frac{d^2 + a^2}{a^2} - \frac{\lambda}{4\pi\epsilon} \ln \frac{2d}{r_0} , \end{aligned} \quad (4.8.10)$$

and an application of the procedure for determining the partial capacitances described in

Section 4.4.2. results in

$$C_{12} = C_{21} = \frac{\pi\epsilon}{\ln^2\left(\frac{2d}{r_0}\right) - \ln^2\left(\frac{d^2 + a^2}{a^2}\right)} \ln \frac{d^2 + a^2}{a^2},$$

$$C_{11} = C_{22} = \frac{\pi\epsilon}{\ln^2\left(\frac{2d}{r_0}\right) - \ln^2\left(\frac{d^2 + a^2}{a^2}\right)} \left\{ \ln \frac{2d}{r_0} - \ln \frac{d^2 + a^2}{a^2} \right\}. \quad (4.8.11a,b)$$

An equivalent circuit of the electric field of such a line is shown in Figure 4.8.8.

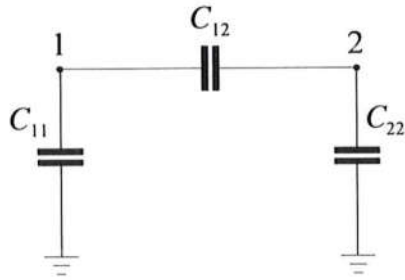


Figure 4.8.8. An equivalent circuit of the static electric field of a line in front of a conductive plane (earth)

4.8.2. Green's functions of a plane dielectric boundary

4.8.2.1. A point charge in front of a plane boundary between dielectrics

The procedure that we developed in the previous section can be applied in determining the Green's function of an unbounded space that consists of two semi-infinite regions with different permittivities.

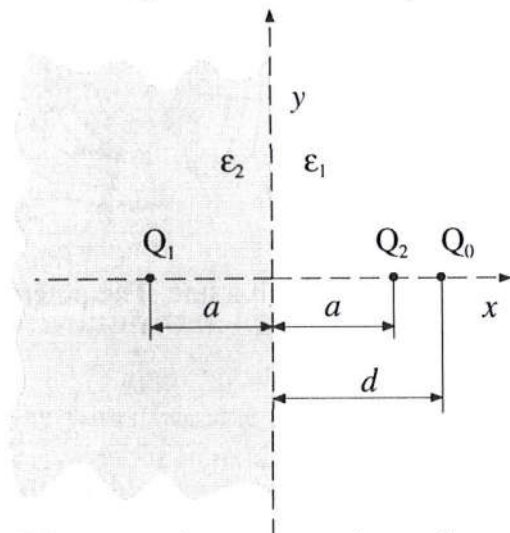


Figure 4.8.9. A point charge in front of a dielectric half space

This means that we are going to determine the potential of a point charge Q_0 that is placed in region "1" with permittivity ϵ_1 in front of a dielectric half space with permittivity ϵ_2 (Figure 4.8.9). Let us assume that the potential and field strength in half space "1" can be

calculated as the sum of the contributions of the given charge Q_0 and its image charge Q_1

$$\begin{aligned}\varphi_1(x,y,z) &= \frac{1}{4\pi\epsilon_1} \left(\frac{Q_0}{\sqrt{(x-d)^2+y^2+z^2}} + \frac{Q_1}{\sqrt{(x+a)^2+y^2+z^2}} \right), \\ E_1(x,y,z) &= \frac{1}{4\pi\epsilon_1} \left\{ \frac{Q_0}{[(x-d)^2+y^2+z^2]^{3/2}} [(x-d)\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z] + \right. \\ &\quad \left. + \frac{Q_1}{[(x+a)^2+y^2+z^2]^{3/2}} [(x+a)\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z] \right\},\end{aligned}\quad (4.8.12,13)$$

and in half space "2" as the sum of the contributions of the given charge Q_0 and the second image charge Q_2

$$\begin{aligned}\varphi_2(x,y,z) &= \frac{1}{4\pi\epsilon_2} \left(\frac{Q_0}{\sqrt{(x-d)^2+y^2+z^2}} + \frac{Q_2}{\sqrt{(x-a)^2+y^2+z^2}} \right), \\ E_2(x,y,z) &= \frac{1}{4\pi\epsilon_2} \left\{ \frac{Q_0}{[(x-d)^2+y^2+z^2]^{3/2}} [(x-d)\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z] + \right. \\ &\quad \left. + \frac{Q_2}{[(x-a)^2+y^2+z^2]^{3/2}} [(x-a)\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z] \right\}.\end{aligned}\quad (4.8.14,15)$$

We have assumed that the image and original charges are placed on a line perpendicular to the boundary plane because only such an arrangement can satisfy the condition of continuity of potential on the boundary. If we succeed in determining the values of Q_1 , Q_2 and a that satisfy all boundary conditions on the boundary between the two dielectrics, then according to the theorem of uniqueness from Poisson's equation, this is the solution of this problem. Conditions on the boundary plane $x = 0$ are

$$\begin{aligned}\varphi_1(0,y,z) &= \varphi_2(0,y,z), \\ D_{x1}(0,y,z) &= D_{x2}(0,y,z), \\ E_{y1}(0,y,z)\mathbf{a}_y + E_{z1}(0,y,z)\mathbf{a}_z &= E_{y2}(0,y,z)\mathbf{a}_y + E_{z2}(0,y,z)\mathbf{a}_z.\end{aligned}\quad (4.8.16 - 18)$$

From (4.8.17) it follows that $Q_2 = -Q_1$. Substitution of this condition into expressions (4.8.16) and (4.8.18) results in

$$a = d \quad ; \quad Q_2 = \frac{\epsilon_2 - \epsilon_1}{\epsilon_1 + \epsilon_2} Q_0 \quad ; \quad Q_1 = \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2} Q_0,$$

and we can finally conclude that we can calculate the field in half space "1" by usage of two charges $Q' = Q_0$ and $Q'' = Q_1$, and that we can calculate the field in half space "2" using one charge $Q = Q_0 + Q_2$ at a distance d from the boundary plane. The three-dimensional Green's functions for half spaces "1" and "2" are

$$G_1(x,y,z) = \frac{1}{4\pi\epsilon_1} \left[\frac{1}{\sqrt{(x-d)^2 + y^2 + z^2}} + \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2} \frac{1}{\sqrt{(x+d)^2 + y^2 + z^2}} \right], \quad (4.8.19,20)$$

$$G_2(x,y,z) = \frac{1}{4\pi\epsilon_2} \frac{2\epsilon_2}{\epsilon_1 + \epsilon_2} \frac{1}{\sqrt{(x-d)^2 + y^2 + z^2}}.$$

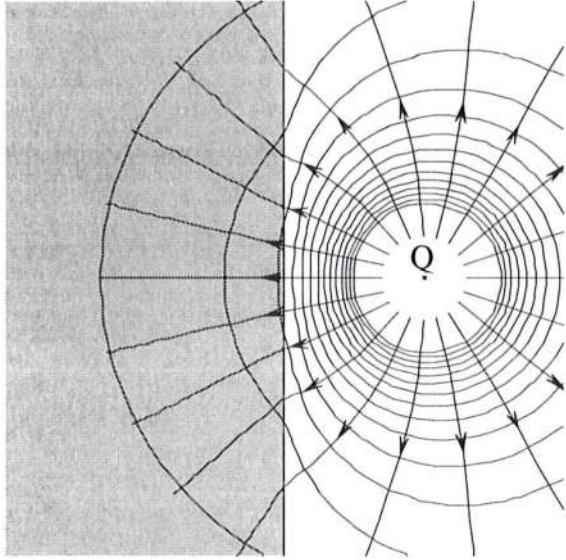


Figure 4.8.10. The field of a point charge in front of a dielectric half space

Example 4.8.2.1. A two-wire line in front of a dielectric half space

A two-wire line in front of a half space with permittivity ϵ is shown in Figure 4.8.11.

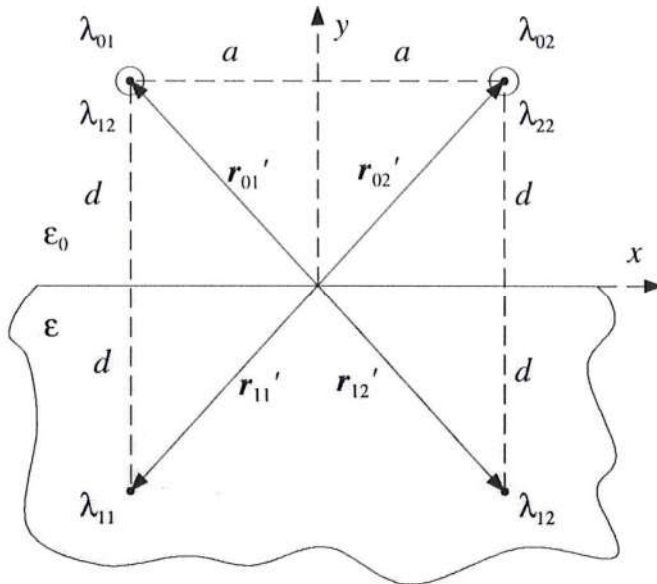


Figure 4.8.11. A two-wire line in front of a dielectric half space

If the conditions $a \gg \rho_0$ and $d \gg \rho_0$ are satisfied, we can replace charge distributed on the surface of the conductors with line charges in the centres of the conductors. The problem is thus two-dimensional and we derive Green's functions for both half spaces from equations (4.8.19,20) using the substitution (4.8.8)

$$G_1 = \frac{1}{2\pi\epsilon_0} \left[\ln \frac{|\mathbf{r}_1 - \mathbf{r}_{02}'|}{|\mathbf{r}_1 - \mathbf{r}_{01}'|} + \frac{\epsilon_0 - \epsilon}{\epsilon_0 + \epsilon} \ln \frac{|\mathbf{r}_1 - \mathbf{r}_{12}'|}{|\mathbf{r}_1 - \mathbf{r}_{11}'|} \right],$$

$$G_2 = \frac{1}{2\pi\epsilon} \frac{2\epsilon_0}{\epsilon_0 + \epsilon} \ln \frac{|\mathbf{r}_2 - \mathbf{r}_{01}'|}{|\mathbf{r}_2 - \mathbf{r}_{02}'|}.$$
(4.8.21,22)

The expressions for the potential in air and in the dielectric are

$$\varphi_1 = \frac{\lambda}{4\pi\epsilon_0} \left\{ \ln \frac{(x-a)^2 + (y-d)^2}{(x-a)^2 + (y+d)^2} + \frac{\epsilon_0 - \epsilon}{\epsilon_0 + \epsilon} \ln \frac{(x+a)^2 + (y-d)^2}{(x+a)^2 + (y+a)^2} \right\},$$

$$\varphi_2 = \frac{\lambda}{4\pi\epsilon_0} \frac{2\epsilon}{\epsilon_0 + \epsilon} \ln \frac{(x-a)^2 + (y-d)^2}{(x-a)^2 + (y+d)^2},$$
(4.8.23,24)

If the permittivity $\epsilon \rightarrow \infty$, then $\lambda_1 = -\lambda$ and $\lambda_2 = 0$. The characteristics of the dielectric are increasingly similar to the characteristics of a metal, and the result of this example becomes equal to the result of Example 4.8.1.2.

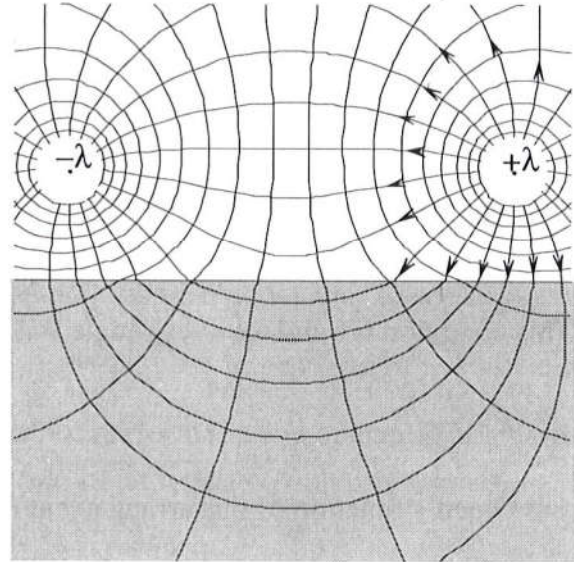


Figure 4.8.12. The field of a two-wire line in front of a dielectric half space

4.8.3. Green's functions of a cylindrical boundary

Green's functions of problems with cylindrical boundaries can be determined by the simple addition of image charges in only two-dimensional cases because the hyperbolic potential function of a point charge cannot be linked to the logarithmic potential function of an infinite cylinder.

4.8.3.1. A conductive cylindrical boundary

An infinite line charge with constant density λ is placed inside a conductive cylinder as shown in Figure 4.8.13.

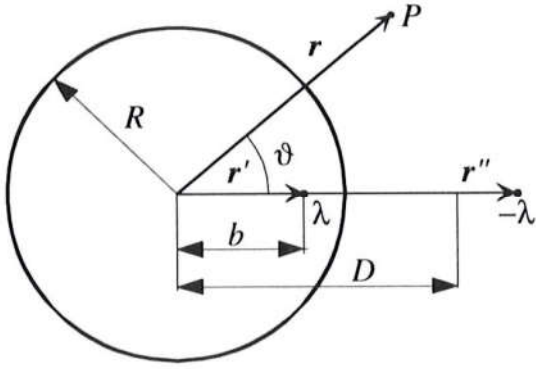


Figure 4.8.13. A line charge inside a conductive cylinder

We shall construct a two-dimensional Green's function of such an arrangement using the potential of a unit line charge inside it. We shall use a negative unit image line charge placed at $\mathbf{r}''$ ($\mathbf{r}'' = k\mathbf{r}'$) of a unit line charge. The potential at a point $\mathbf{r}$ is

$$\varphi(\mathbf{r}) = \frac{\lambda}{2\pi\epsilon} \ln \frac{|\mathbf{r}_0 - \mathbf{r}'|}{|\mathbf{r} - \mathbf{r}'|} + \frac{-\lambda}{2\pi\epsilon} \ln \frac{|\mathbf{r}_0 - \mathbf{r}''|}{|\mathbf{r} - \mathbf{r}''|} = \frac{\lambda}{2\pi\epsilon} \ln \frac{|\mathbf{r} - \mathbf{r}'|}{|\mathbf{r} - k\mathbf{r}'|} . \quad (4.8.25)$$

For points on the cylinder of radius $r = R$ we get that

$$\begin{aligned} |\mathbf{r} - \mathbf{r}'| &= R^2 + r'^2 - 2Rr' \cos \vartheta , \\ |\mathbf{r} - \mathbf{r}''| &= R^2 + k^2 r'^2 - 2Rkr' \cos \vartheta . \end{aligned} \quad (4.8.26a,b)$$

If we substitute (4.8.26a,b) into (4.8.25) and suppose that the potential of the cylinder is constant and amounts to $\ln C$, we obtain

$$R^2 + k^2 r'^2 - 2Rkr' \cos \vartheta = C(R^2 + r'^2 - 2Rr' \cos \vartheta) . \quad (4.8.27)$$

This condition is valid for every angle ϑ if

$$k = \frac{R^2}{r'^2} , \quad (4.8.28)$$

and Green's function of this arrangement is

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{2\pi\epsilon} \ln \frac{\left| \mathbf{r} - \frac{R^2}{b^2} \mathbf{r}' \right|}{|\mathbf{r} - \mathbf{r}'|} + \ln C = \frac{1}{2\pi\epsilon} \ln \frac{C \left| \mathbf{r} - \frac{R^2}{b^2} \mathbf{r}' \right|}{|\mathbf{r} - \mathbf{r}'|} . \quad (4.8.29)$$

Since $G(\mathbf{r}, \mathbf{r}')$ should equal zero on the cylinder $r = R$ we obtain $C = b/R$, and the final form of Green's function for a metal cylindrical boundary is

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{2\pi\epsilon} \ln \frac{\left| \frac{b}{R} \mathbf{r} - \frac{R}{b} \mathbf{r}' \right|}{|\mathbf{r} - \mathbf{r}'|} . \quad (4.8.30)$$

We should emphasise that the position of the original charge is irrelevant. If the original charge is inside the cylinder, then G is Green's function for the region inside the cylinder,

and vice versa, if the original charge is outside the cylinder, then G is Green's function outside the cylinder.

Example 4.8.3.1. The field of a strip inside a conductive cylinder

An infinite strip uniformly charged with a surface charge density σ is placed inside a grounded metal cylinder as shown in Figure 4.8.14.

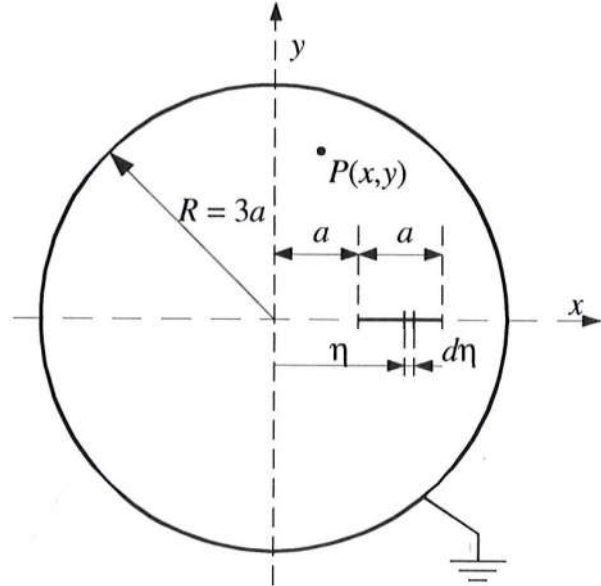


Figure 4.8.14. A strip inside a grounded metal cylinder

According to equation (4.5.9), the potential inside the cylinder is

$$\varphi(x, y) = \int_a^{2a} \sigma d\eta \frac{1}{2\pi\epsilon} \ln \left| \frac{\frac{\eta}{R}(xa_x + ya_y) - \frac{R}{\eta}a_x}{|xa_x + ya_y - \eta a_x|} \right| = \frac{\sigma}{4\pi\epsilon} \int_a^{2a} \ln \frac{\left(\eta \frac{x}{R} - R\right)^2 + \frac{\eta^2}{R^2} y^2}{(x - \eta)^2 + y^2} d\eta.$$

This integration is a relatively complex one, and we perform it using integration by parts and Euler's substitutions. The final result is

$$\begin{aligned} \varphi(x, y) = & \frac{\sigma}{2\pi\epsilon} \left\{ -2y \left(\arctan \frac{2a-x}{y} - \arctan \frac{a-x}{y} \right) + \right. \\ & + \frac{2R^2 y}{x^2 + y^2} \left[\arctan \frac{-R^2 x + 2a(x^2 + y^2)}{R^2 y} - \arctan \frac{-R^2 x + a(x^2 + y^2)}{R^2 y} \right] + \\ & + x \ln \frac{4a^2 - 4ax + x^2 + y^2}{a^2 - 2ax + x^2 + y^2} - \frac{R^2 x}{x^2 + y^2} \ln \frac{R^4 - 4aR^2 x + 4a^2(x^2 + y^2)}{R^4 - 2aR^2 x + a^2(x^2 + y^2)} + \\ & \left. + 2a \ln \frac{\left(-R + \frac{2ax}{R}\right)^2 + \frac{4a^2 y^2}{R^2}}{(x-2a)^2 + y^2} - a \ln \frac{\left(-R + \frac{ax}{R}\right)^2 + \frac{a^2 y^2}{R^2}}{(x-a)^2 + y^2} \right\}. \end{aligned} \quad (4.8.31)$$

The distribution of potential inside the cylinder for $R = 3$ m, $a = 1$ m and $\sigma = 2\pi\epsilon$ is shown

in Figure 4.8.15.

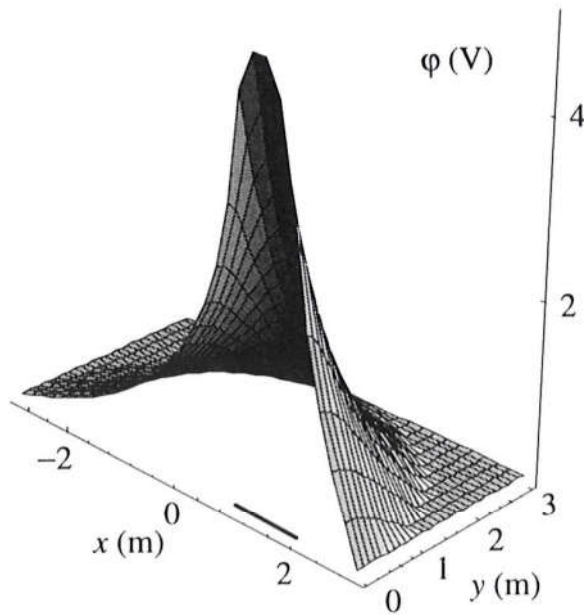


Figure 4.8.15. The distribution of potential of an uniformly charged strip inside a grounded metal cylinder

4.8.3.2. A dielectric cylindrical boundary

We shall seek the solution of the field of a line charge in front of a dielectric cylinder (Figure 4.8.16) by analogy with the solution of a line charge in front a dielectric half space.

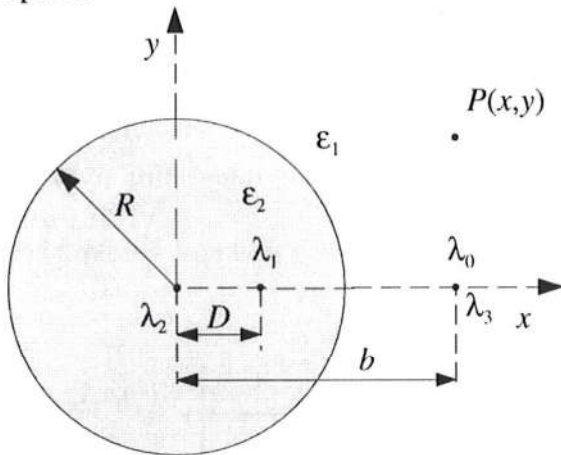


Figure 4.8.16. A line charge in front of a dielectric cylinder

We add an image line charge λ_1 placed inside the cylinder at a distance $D = R^2/b$ from the centre of the cylinder, to the original line charge λ_0 placed in the region with permittivity ϵ_1 at a distance b from the centre of the cylinder, as was the case with a metal cylindrical boundary. This introduces a charge into the region without any charge and we thus have to add another line charge $\lambda_2 = -\lambda_1$ at the centre of the cylinder. For calculation of the field inside the cylinder, we use an additional charge $\lambda_3 = -\lambda_1$ at the place of original charge λ_0 . Image charges are calculated as

$$\lambda_1 = \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2} \lambda_0 \quad ; \quad \lambda_2 = -\lambda_1 \quad ; \quad \lambda_3 = -\lambda_1 . \quad (4.8.32)$$

The potential outside and inside the cylinder is calculated with the help of

$$\varphi_1(x,y) = \frac{\lambda_0}{4\pi\epsilon_1} \left\{ \ln \frac{r_0^2}{(x-b)^2 + y^2} + \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2} \ln \frac{x^2 + y^2}{\left(x - \frac{R^2}{b}\right)^2 + y^2} \right\}, \quad (4.8.33,34)$$

$$\varphi_2(x,y) = \frac{\lambda_0}{4\pi} \frac{2}{\epsilon_1 + \epsilon_2} \ln \frac{r_0^2}{(x-b)^2 + y^2} .$$

When a point $P(x,y)$ is placed on the cylindrical surface, we get that $x^2 + y^2 = R^2$ and $\varphi_1 = \varphi_2$ should be satisfied. Therefore, the cylinder of reference has radius $r_0 = b$. Proof that such an arrangement satisfies all boundary conditions on the cylindrical boundary between two regions with different permittivities has been left to the reader.

Example 4.8.3.2. Two line charges and a dielectric cylinder

Two line charges λ and $-\lambda$ are placed symmetrical to the axis of a dielectric cylinder with permittivity ϵ_2 in a region with permittivity ϵ_1 as shown in Figure 4.8.17.

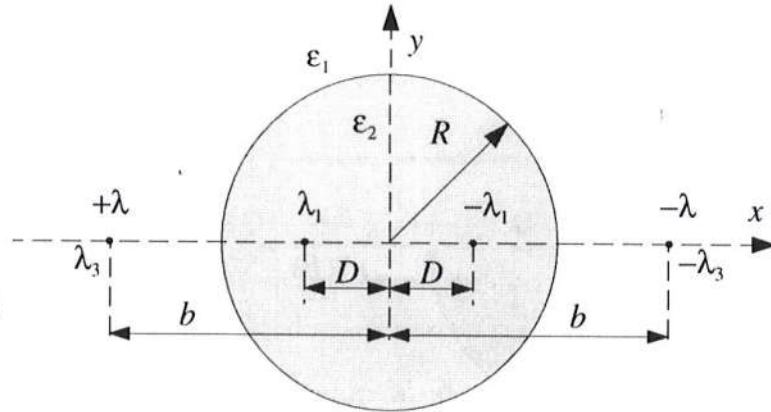


Figure 4.8.17. Two line charges and a dielectric cylinder

Line charges that should be added to the centre of the cylinder neutralize each other and we calculate the potential outside and inside the cylinder as being

$$\varphi_1(x,y) = \frac{\lambda}{4\pi\epsilon_1} \left(\ln \frac{(x-b)^2 + y^2}{(x+b)^2 + y^2} + \frac{\epsilon_2 - \epsilon_1}{\epsilon_1 + \epsilon_2} \ln \frac{(x-D)^2 + y^2}{(x+D)^2 + y^2} \right), \quad (4.8.35,36)$$

$$\varphi_2(x,y) = \frac{\lambda}{4\pi} \frac{2}{\epsilon_1 + \epsilon_2} \ln \frac{(x-b)^2 + y^2}{(x+b)^2 + y^2},$$

If $\epsilon_2 \rightarrow \infty$, this solution is transformed into the solution for two line charges in front of a metal cylinder.

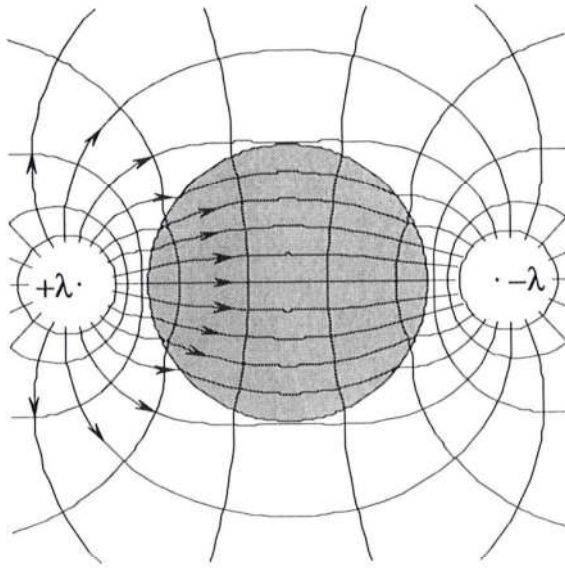


Figure 4.8.18. The field of two line charges in front of a dielectric cylinder

4.8.4. Green's function of a conductive spherical boundary

A point charge Q is placed in front of a conductive sphere with radius R at zero potential as shown in figure 4.8.19.

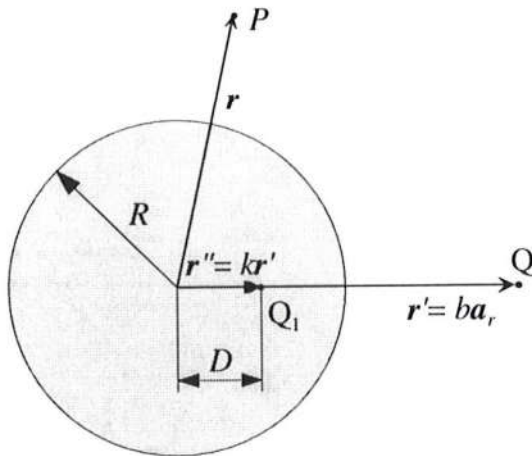


Figure 4.8.19. A point charge Q in front of a conductive sphere at zero potential

We shall determine the three-dimensional Green's function of a conductive spherical boundary as the potential of a unit point charge $Q = 1$ at a point $\mathbf{r}' = ba_r$. In solving this problem we shall use the method of images. In order to make the potential of the sphere equal to zero, we have to add a negative image charge $-Q_1$ at a place $\mathbf{r}'' = k\mathbf{r}'$. Green's function of this arrangement is

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi\epsilon} \left[\frac{Q}{|\mathbf{r} - \mathbf{r}'|} - \frac{Q_1}{|\mathbf{r} - k\mathbf{r}'|} \right]. \quad (4.8.37)$$

On the surface of the sphere where $r = R$ this Green's function should be equal to zero and we thus obtain

$$Q^2(R^2 + k^2 b^2 - 2Rkb \cos \vartheta) \equiv Q_1^2(R^2 + b^2 - 2Rb \cos \vartheta) . \quad (4.8.38)$$

This identity is satisfied for every angle ϑ if

$$Q_1 = Q \frac{R}{b} \quad ; \quad k = \frac{R^2}{b^2} , \quad (4.8.39)$$

and Green's function for a spherical metal boundary is

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi\epsilon |\mathbf{r} - \mathbf{r}'|} - \frac{\frac{R}{b}}{4\pi\epsilon \left| \mathbf{r} - \frac{R^2}{b^2} \mathbf{r}' \right|} . \quad (4.8.40)$$

This function is valid for any region outside or inside the sphere, depending on the position of the original charge.

Example 4.8.4.1. A uniformly charged line segment and a sphere

A uniformly charged line segment is placed in front of a sphere at zero potential as shown in Figure 4.8.20.

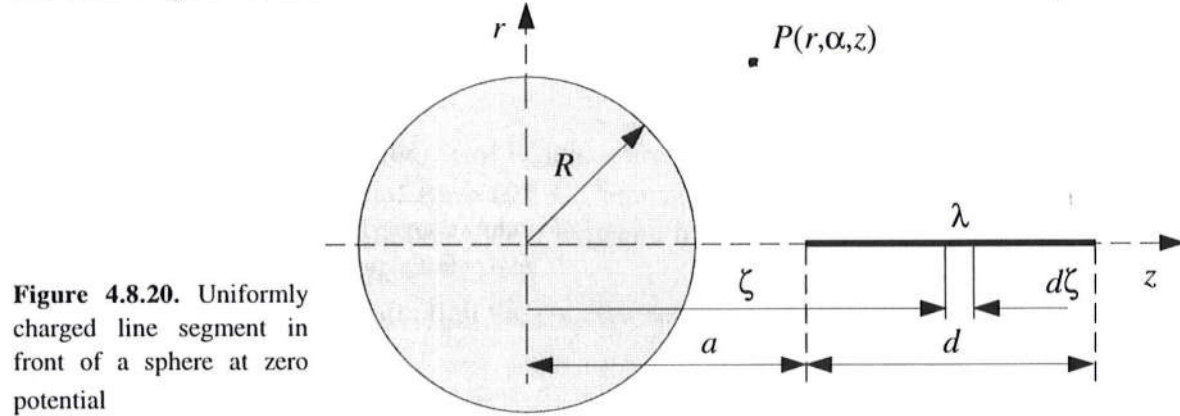


Figure 4.8.20. Uniformly charged line segment in front of a sphere at zero potential

Due to axial symmetry, we solve this problem in a cylindrical coordinate system. Green's function is

$$G(\mathbf{r}, \mathbf{r}') = G(r, z, z') = \frac{1}{4\pi\epsilon} \left[\frac{1}{\sqrt{(z - \zeta)^2 + r^2}} - \frac{R}{\zeta} \frac{1}{\sqrt{\left(z - \frac{R^2}{\zeta}\right)^2 + r^2}} \right] , \quad (4.8.41)$$

and we calculate the potential at a point $P(r, z)$ as being

$$\varphi(\rho, z) = \frac{\lambda}{4\pi\epsilon} \int_a^{a+d} \left[\frac{1}{\sqrt{(z - \zeta)^2 + r^2}} - \frac{R}{\sqrt{(z\zeta - R^2)^2 + r^2 \zeta^2}} \right] d\zeta . \quad (4.8.42)$$

The final result for the potential at points outside the sphere is

$$\varphi(r,z) = \frac{\lambda}{4\pi\epsilon} \left\{ \ln \frac{a+d-z+\sqrt{(a+d)^2-2z(a+d)+r^2+z^2}}{a-z+\sqrt{a^2-2za+r^2+z^2}} - \frac{R}{r^2+z^2} \ln \frac{\sqrt{r^2+z^2}\sqrt{(r^2+z^2)(a+d)^2-2zR^2(a+d)+R^4+(r^2+z^2)(a+d)-zR^2}}{\sqrt{r^2+z^2}\sqrt{(r^2+z^2)a^2-2zR^2a+R^4+(r^2+z^2)a-zR^2}} \right\}. \quad (4.8.43)$$

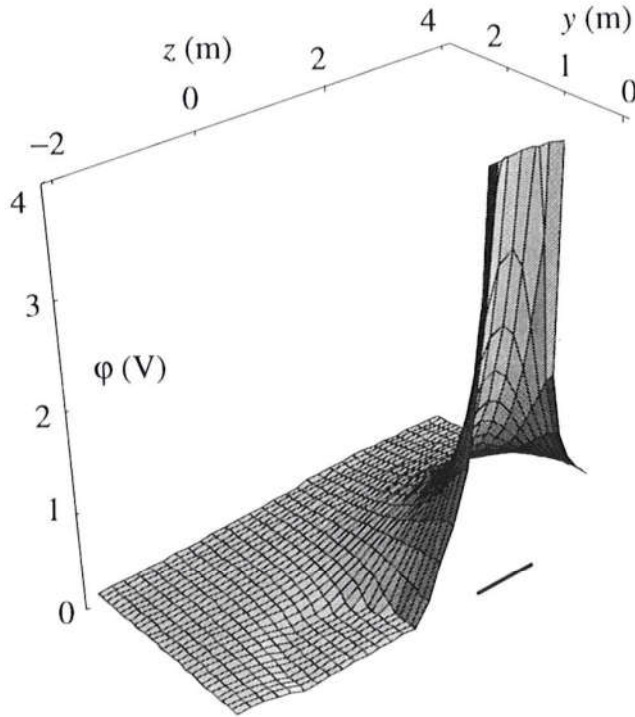


Figure 4.8.21. Distribution of potential of a uniformly charged line segment in front of a sphere at zero potential in the plane $\alpha = 0$

PROBLEMS

- 4.1. Determine the equations of field lines and equipotential surfaces of a unit point charge in free space obeying the rules given in Unit 4.2 !
- 4.2. Determine the force on a dielectric brick ($\epsilon_r > 1$) that is partially inserted into an ideal plate condenser filled with air ($\epsilon_r = 1$) !

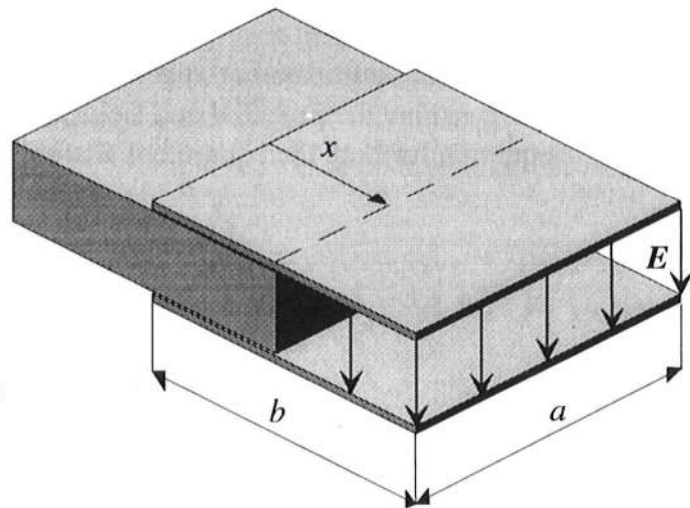


Figure P4.2. Dielectric brick inside the plate condenser

- 4.3. Three conductive spheres of radius r are placed at the vertices of an equilateral triangle, as shown in Figure P.4.3. Determine the partial capacitances of this system assuming that the distance d between the spheres is significantly greater than the radius of the spheres !
- Hint: under this assumption the spheres act as point charges at their centres.

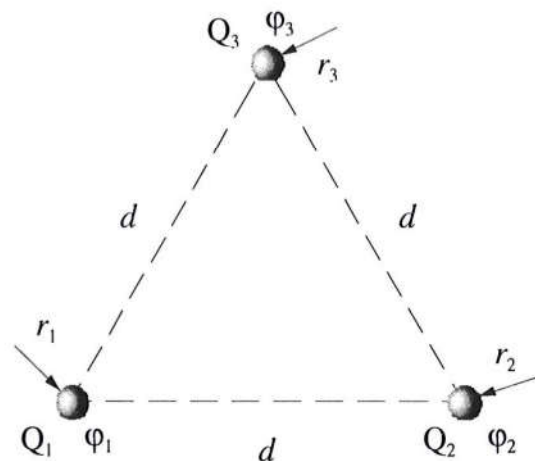


Figure P4.3. Three conductive spheres

- 4.4. Derive expressions for the electric field strength of the cylindrical conductor enclosed by a dielectric layer in an external field described in Example 4.6.2.1 !

- 4.5. Derive expressions for the electric field strength of the dielectric sphere in an external homogeneous field described in Example 4.6.3.1 !
- 4.6. Prove that the field outside a dielectric sphere in an external homogeneous field is composed of an external field and a dipole type field and determine the moment and direction of the dipole that can substitute for the disturbance into the homogeneous field caused by the dielectric sphere !
- 4.7. Derive the equation for the field lines of two point charges given in Example 4.7.1.1.
- 4.8. A linear quadrupole is an arrangement of four point charges as shown in Figure P4.8. Determine the potential and field strength of a linear quadrupole as well as the equation for both the equipotential and field lines !

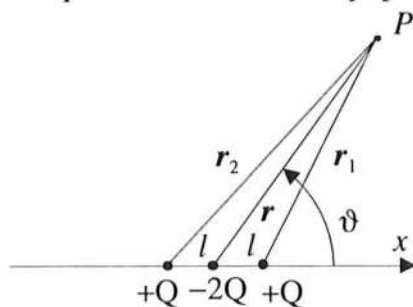


Figure P4.8. Linear quadrupole

- 4.9. Determine the electric field strength of the uniformly charged rectangular plate described in Example 4.7.3.1 !
- 4.10. Determine the potential of the non-uniformly charged cube described in Example 4.7.4.1. using a division of the cube into N plates with finite width $\Delta\zeta = 2a/N$ and replacing each of these plates of finite width with a uniformly charged surface in the middle of the plate!
- 4.11. Derive the expression for the potential of an arbitrary charge distribution $\rho(\mathbf{r}')$ in half space $x > 0$ if the distribution of potential $\phi(0, y', z')$ is given in the plane $x = 0$!
- 4.12. Determine the electric field strength of the two-wire line in front of a dielectric half space described in Example 4.8.2.1 !
- 4.13. A line dipole is a configuration of two equal and opposite line charges $+\lambda$ and $-\lambda$ that are placed at a small distance l apart, and we observe the field at points whose distance r from the charges is much larger than the distance l between the charges. Derive expressions for the potential and field strength of this line dipole !
- 4.14. Analyse the case where $b \rightarrow \infty$ in the solution of the field of two line charges in front of a dielectric cylinder given in Example 4.8.3.2. and prove that with the help of this solution, we can solve the field of a dielectric cylinder in an external homogeneous field !

5. A STATIC CURRENT FIELD IN A CONDUCTOR

In the previous chapter we analysed the static electric field caused by charge at rest. The next logical step in the study of electromagnetism is the analysis of an electromagnetic field caused by charge in uniform motion. The fundamental characteristic of such a field is that it may be split into two independent fields: a static current and static magnetic field. In this chapter we shall study the static current field in conductive materials and prove its analogy with a static electric field.

5.1. THE FIELD OF CHARGE IN UNIFORM MOTION

We assume that, in a conductive region, there exists a current independent of time. This means that charge is moving uniformly and that all electromagnetic field quantities are independent of time. The system of Maxwell's equations in this case reduces to

$$\begin{aligned} \nabla \times \mathbf{E} &= 0 & ; & & \nabla \cdot \mathbf{D} &= \rho , \\ \nabla \times \mathbf{H} &= \mathbf{J} & ; & & \nabla \cdot \mathbf{B} &= 0 . \end{aligned} \tag{5.1.1 - 4}$$

Equations (5.1.1,2) govern an electric field and equations (5.1.3,4) govern the magnetic field. These pairs of equations are mutually dependent since the current density $\mathbf{J}$ is a function of $\mathbf{E}$ and $\mathbf{B}$. If N is the density of free carriers of charge moving with velocity $\mathbf{v}$, and if their charge is q , we arrive at $\mathbf{J} = Nq\mathbf{v}$. Thus, the force on each carrier of charge is

$$\mathbf{F} = q \left(\mathbf{E} + \frac{1}{Nq} \mathbf{J} \times \mathbf{B} \right) = q \mathbf{E}_u , \tag{5.1.5}$$

where $\mathbf{E}_u$ is the total electric field. If we restrict our study to solid and liquid conductive materials with conductivity κ , the expression

$$\mathbf{J} = \kappa \mathbf{E}_u = \kappa \left(\mathbf{E} + \frac{1}{Nq} \mathbf{J} \times \mathbf{B} \right) , \tag{5.1.6}$$

is valid and we can write that

$$\mathbf{E} = \frac{\mathbf{J}}{\kappa} - \mathbf{J} \times \frac{\mathbf{B}}{Nq} . \tag{5.1.7}$$

If the influence of the second term in equation (5.1.7) is negligible, we can consider the pairs of equations (5.1.1,2) and (5.1.3,4) as two uncoupled pairs. This is the case in all

conductors, and we can observe the electric field in them independently of their magnetic field. This means that we can determine the electric field without any knowledge about the magnetic field. Such an electric field is called a *static current field in a conductor*.

5.2. EQUATIONS OF A CURRENT FIELD AND FIELD ANALOGIES

On the basis of the previous unit, we can conclude that equations of this field are equivalent to those of a static electric field. But, in addition to equations (5.1.1,2), we need to take into consideration the continuity equation, which in this case is $\nabla \cdot \mathbf{J} = 0$ since all derivatives with respect to time are equal to zero. In isotropic conductors the relationship between the vector of current density $\mathbf{J}$ and the vector of electric field strength $\mathbf{E}$ is $\mathbf{J} = \mathbf{J}_i + \kappa \mathbf{E}$, where $\mathbf{J}_i$ is the density of source current. In source-free regions, this relationship is $\mathbf{J} = \kappa \mathbf{E}$, and

$$\nabla \cdot \mathbf{D} = \rho_s = \epsilon \nabla \cdot \mathbf{E} = \frac{\epsilon}{\kappa} \nabla \cdot \mathbf{J} = 0 , \quad (5.2.1)$$

which means that the density of free immovable charge in a source-free conductor in which a time-independent current flows, is equal to zero. Free charge can exist only on the boundaries between two different conductors. The conditions for current density on a boundary between two conductors with different conductivities are

$$\mathbf{n} \times \left(\frac{\mathbf{J}_2}{\kappa_2} - \frac{\mathbf{J}_1}{\kappa_1} \right) = 0 , \quad (5.2.2,3)$$

$$\mathbf{n} \cdot (\mathbf{J}_2 - \mathbf{J}_1) = \sigma_s .$$

For typical conductors, conductivity κ is much greater than permittivity ϵ , and we can say that in a static current field only a current flow exists, while the electric flux is equal to zero. Accordingly, the equations that describe the static current field are

$$\begin{aligned} \nabla \times \mathbf{E} &= 0 , \\ \nabla \cdot \mathbf{J} &= 0 , \end{aligned} \quad (5.2.4,5)$$

together with the constitutive relationship $\mathbf{J} = \kappa \mathbf{E}$.

When we compare equations (5.2.4,5) and (4.1.1,2), we notice *the analogy between the static electric field and the static current field in conductors*. Both fields are defined by equations of the same form, and if we perform the substitutions

$$\kappa \leftrightarrow \epsilon ; \quad \mathbf{J} \leftrightarrow \mathbf{D} , \quad (5.2.6)$$

we can apply all procedures and results of problems in static electric fields to the solution of problems in static current fields and vice versa.

The scalar electric potential in static current fields in source-free regions satisfies Laplace's equation. Therefore, if perfect conductors (electrodes, $\kappa \rightarrow \infty$) exist in a current field, their potential is constant and the problem of a system of electrodes in a conductor

is analogous to the problem of a system of conductors in a dielectric. Analogous quantities include the charge of a conductor and the current flowing from an electrode. The relationship between the potential of the electrodes and currents that are flowing from them in linear conductive materials is linear

$$\varphi_i = \sum_{j=1}^N R_{ij} I_j \quad ; \quad i = 1, \dots, N. \quad (5.2.7)$$

Here, R_{ij} are *coefficients of resistance*, which are analogous to potential coefficients, and we can represent the static current field of a system of electrodes in a conductor as an *equivalent resistance mesh*. We particularly define the resistance R of a system of two solitary electrodes in which the current flowing out of the electrode whose potential is φ_1 completely flows into the other electrode whose potential is φ_2 as

$$R = \frac{\varphi_1 - \varphi_2}{I} = \frac{1}{G}, \quad (5.2.8)$$

where G is the *conductance* of such a system. Capacitance is an analogous quantity to conductance and if we know the capacitance of a specific geometry of a static electric field along with the permittivity ϵ , we can calculate the conductance of the same geometry of static current field with conductivity κ using the expression

$$G = \frac{\kappa}{\epsilon} C = \frac{1}{R}, \quad (5.2.9)$$

where G is measured in *siemens* (S).

Example 5.2.1. A spherical cavity in a metal casting

A large metal casting with conductivity κ containing a spherical cavity of radius R is shown in Figure 5.2.1.

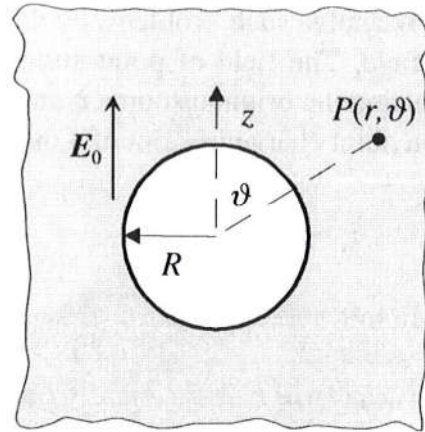


Figure 5.2.1. Spherical cavity in a metal casting

Without the cavity, the electric field would be uniform with field strength E_0 . We need to determine the distribution of current density $\mathbf{J}$ in the presence of the air cavity of conductivity $\kappa_0 = 0$. In order to determine the scalar electric potential and the electric field strength we shall use the analogy with the static electric field for the dielectric sphere in an external homogeneous field (Example 4.6.3.1), using the substitutions $\epsilon_0 \rightarrow \kappa$ and

$\varepsilon_1 \rightarrow 0$. The current density in the metal is

$$\mathbf{J} = \kappa \mathbf{E} = -\kappa E_0 \left[1 - \left(\frac{R}{r} \right)^3 \right] \cos \vartheta \mathbf{a}_r + \kappa E_0 \left[1 + \frac{1}{2} \left(\frac{R}{r} \right)^3 \right] \sin \vartheta \mathbf{a}_\vartheta . \quad (5.2.10)$$

A maximum in current density occurs in the metal at the boundary with the cavity and is equal to $1.5\kappa E$, while in the cavity no current flows due to the fact that $\kappa_0 = 0$.

5.3. THE SOLUTION OF A CURRENT FIELD BY THE METHOD OF IMAGES

Problems of a static current field are most frequently encountered when solving problems in the area of earthing systems. An earthing system consists of one or more electrodes buried in the earth (Figure 5.3.1).

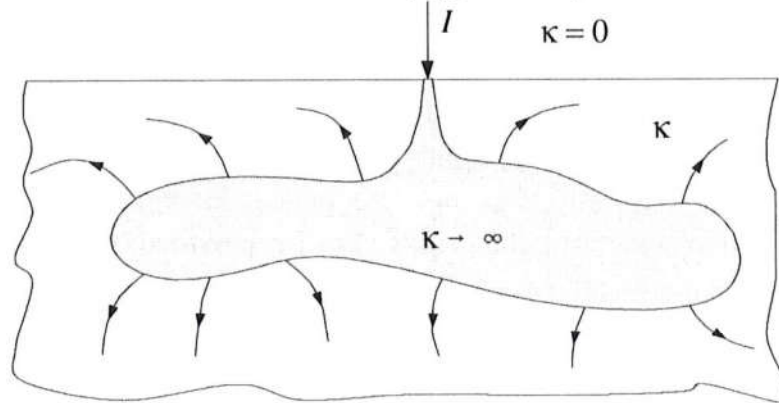


Figure 5.3.1. An electrode in a homogeneous earth

In the first approximation, we imagine that the earth is an infinite homogeneous half space of conductivity κ_1 , and that the air is an ideal dielectric of conductivity κ_2 equal to zero. We solve such problems by the method of images using the analogy with a static electric field. The field of point current source i_0 in the earth of conductivity κ_1 is determined using the original source i_0 and its image i_1 . From an analogy with a static electric field of a point charge in front of a dielectric half space it follows that

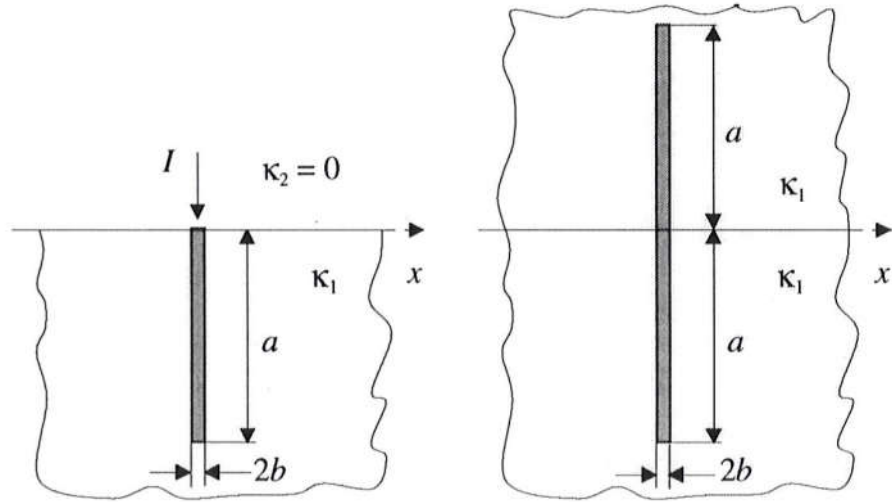
$$i_1 = \frac{\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2} i_0 = i_0 . \quad (5.3.1)$$

In more complex arrangements we use the principle of superposition.

Example 5.3.1. An earthing rod

Simple earthing systems often consist of metal rods that are buried vertically in the earth, as shown in Figure 5.3.2. By application of the method of images, we find that this problem is equivalent to the problem of a rod of double length in a homogeneous material of conductivity κ_1 .

Figure 5.3.2. Vertical earthing rod



In order to solve this problem, we use the analogy with the field of a rotational ellipsoid (Example 4.7.2.1.), which means that we approximate the rod to be a rotational ellipsoid inside the rod. Most frequently we are interested in the distribution of potential on the surface of the earth as well as the resistance of the earthing system. We determine the potential on the surface of the earth using expression (4.7.17) for $z = 0$

$$\varphi(r) = \frac{I}{4\pi\kappa_1 a} \ln \frac{a + \sqrt{a^2 + r^2}}{-a + \sqrt{a^2 + r^2}}. \quad (5.3.2)$$

We can calculate the resistance of this earthing rod approximately, using expression (4.7.25), as

$$R = \frac{1}{2\pi\kappa_1 a} \ln \frac{2a}{b}. \quad (5.3.3)$$

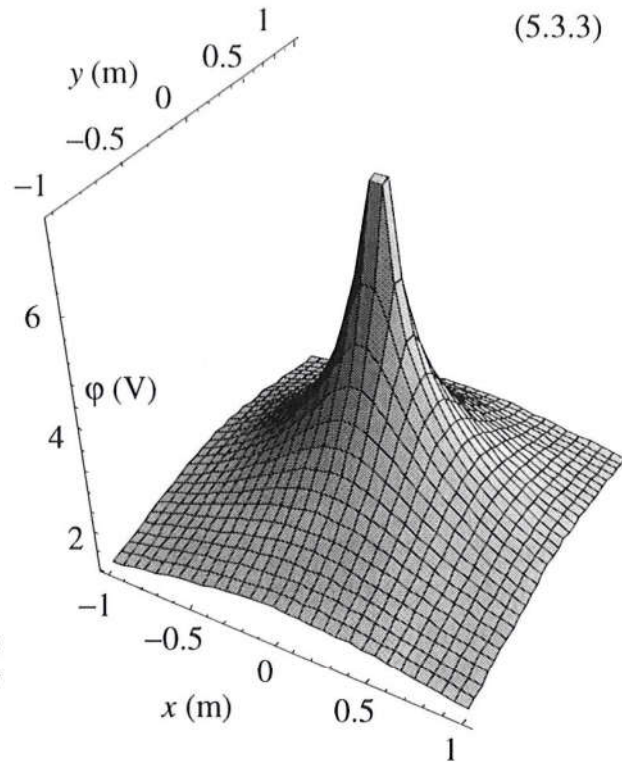


Figure 5.3.3. The distribution of potential of an earthing rod on the surface of the earth for $I/(4\pi\kappa_1) = 1 \text{ Vm}$, $a = 1 \text{ m}$, $r^2 = x^2 + y^2$

PROBLEMS

- 5.1. Define the analogous quantities in a static temperature field, gravitational field, fluid dynamic field and in a static electric field !
- 5.2. Determine the ratio between terms in equation (5.1.7) in copper using one free electron per atom and assuming that induction is 2T !
($\kappa = 57 \text{ MS/m}$, $q = e = 1.602 \cdot 10^{-19} \text{ C}$, and $N = 8.47 \cdot 10^{28} \text{ electrons/m}^3$)
- 5.3. The earthing rod described in Example 5.3.1. is buried horizontally in the earth at a depth of a . Determine the potential on the surface of the earth and the resistance of this rod by approximating the rod to be a rotational ellipsoid inside the rod!

6. A STATIC MAGNETIC FIELD

In this chapter, we shall continue the study of an electromagnetic field caused by charge in uniform motion. In Unit 5.1, we showed that in such cases, the distribution of electric currents in conductors is practically unaffected by a static magnetic field. Consequently, we can analyse a static magnetic field independently of any static current field.

Sources of a static magnetic field can be direct currents and permanent magnets. In Chapter 1, we showed that permanent magnets can be modelled using equivalent macroscopic currents. Therefore, we solve problems involving static magnetic fields by assuming that currents, which are sources of the field, are known.

We shall begin with an overview of the fundamental equations of a static magnetic field. Afterwards we shall study the energy stored in such a field. The fundamental integral characteristics of a static magnetic field are inductances and we shall define them using the energy stored within the static magnetic field. In continuation, we shall present the solving of problems in static magnetic fields using analytical procedures.

6.1. EQUATIONS OF A STATIC MAGNETIC FIELD

We derived the equations of a static magnetic field in Unit 5.1. as being

$$\begin{aligned}\nabla \times \mathbf{H} &= \mathbf{J} , \\ \nabla \cdot \mathbf{B} &= 0 ,\end{aligned}\tag{6.1.1,2}$$

where $\mathbf{J}$ is a *known* function. The relationship between the vectors $\mathbf{B}$ and $\mathbf{H}$ can be expressed as $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$, which, in isotropic materials, reduces to the constitutive relationship $\mathbf{B} = \mu_0(1 + \chi_m)\mathbf{H} = \mu\mathbf{H}$, where μ is the permeability of a material.

Within the scope of this book, we shall study problems in which μ is a positive scalar function of magnitude of the vector $\mathbf{H}$ (or $\mathbf{B}$). The integral form of equations (6.1.1,2) is

$$\begin{aligned}\oint_C \mathbf{H} \cdot d\mathbf{l} &= \int_S \mathbf{J} \cdot \mathbf{n} dS , \\ \oint_S \mathbf{B} \cdot \mathbf{n} dS &= 0 .\end{aligned}\tag{6.1.3,4}$$

We derived the conditions which magnetic field vectors $\mathbf{B}$ and $\mathbf{H}$ have to satisfy at a boundary surface between two materials with different permeability in Unit 1.5. The tangential component of the vector of magnetic field strength shoots up at the boundary by an amount of density of free surface currents $\mathbf{n} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{K}$, and the normal component of magnetic induction $\mathbf{B}$ is continuous at the boundary surface $\mathbf{n} \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0$. Here, $\mathbf{n}$ is a unit vector normal to the boundary surface and directed from region 2 to region 1.

6.2. POTENTIALS IN A STATIC MAGNETIC FIELD

6.2.1. Magnetic vector potential

Equation (6.1.2) displays the *solenoidal nature of a magnetic field*. Therefore, we can express the vector $\mathbf{B}$ as the curl of another vector $\mathbf{A}$, $\mathbf{B} = \nabla \times \mathbf{A}$. Vector $\mathbf{A}$ is called the magnetic vector potential. From equations (6.1.1), it follows that

$$\nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A} \right) = \mathbf{J} . \quad (6.2.1)$$

For linear isotropic homogeneous materials, this *vector differential equation* reduces to $\nabla \times \nabla \times \mathbf{A} = \mu \mathbf{J}$. The introduction of Coulomb's gauge $\nabla \cdot \mathbf{A} = 0$, which is in this case equal to the Lorentz gauge, ensures the uniqueness of the magnetic vector potential. Therefore, the equation for the magnetic vector potential in such materials is

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \Delta \mathbf{A} = -\Delta \mathbf{A} = \mu \mathbf{J} . \quad (6.2.2)$$

The magnetic flux Φ through a surface S enclosed by a closed curve C can be calculated using the known magnetic vector potential as

$$\Phi = \int_S \nabla \times \mathbf{A} \cdot \mathbf{n} dS = \oint_C \mathbf{A} \cdot d\mathbf{l} . \quad (6.2.3)$$

This equation is a result of the application of Stokes's theorem.

In two-dimensional problems, field quantities are independent of one of the coordinates and currents flow in the direction of this coordinate. In such cases, the *vector differential equation* (6.2.2) transforms to a *scalar differential equation* for an unknown component of magnetic vector potential in the direction of current flow.

For example, if currents in a Cartesian coordinate system are $\mathbf{J} = J(x,y)\mathbf{a}_z$, the final form of the equation for magnetic vector potential in this two-dimensional case is the scalar differential equation $\Delta A(x,y) = -\mu J(x,y)$. A comparison with equation (4.1.7) highlights the analogy between a two-dimensional problem of a static magnetic field and the problem of a static electric field. Analogous quantities are $A(x,y)$ and ϕ , along with $1/\mu$ and ϵ . The fundamental difference lies in the determination of the field vectors from potentials. Vectors $\mathbf{E} = -\nabla\phi$ and $\mathbf{B} = \nabla \times \mathbf{A}$ have equal magnitudes at every point and they are mutually perpendicular.

6.2.2. Antipotential

A fundamental motive for the introduction of an antipotential in the solving of a static magnetic field is the reduction of the number of unknown functions. In current-free regions, equations of a static magnetic field are $\nabla \times \mathbf{H} = 0$ and $\nabla \cdot \mathbf{B} = 0$. Therefore, we can introduce the antipotential ϕ_m into these equations analogously to a static electric field and

reducing them to a scalar Laplace's equation $\Delta\phi_m = 0$. The relationship between ϕ_m and the magnetic field strength $\mathbf{H}$ is

$$\mathbf{H} = -\nabla\phi_m \quad ; \quad \phi_m(\mathbf{r}) = \int_r^{r_0} \mathbf{H} \cdot d\mathbf{l} \quad , \quad (6.2.4,5)$$

where $\mathbf{r}_0$ is the position vector of a reference point. Since we have introduced ϕ_m only for a source-free region, the line of integration in (6.2.5) should not pass through any region in which currents are flowing. This restriction does not ensure the uniqueness of ϕ_m .

The antipotential ϕ_m is unique when its value is independent of integration path, which means that integral (6.2.5) along a closed curve has to be equal to zero. In the case of a multiply connected region, one can close the loop that encloses the currents outside this region and ϕ_m is non-unique. Due to expression (6.2.3), the integral (6.2.5) along such a curve is equal to the sum of the enclosed currents.

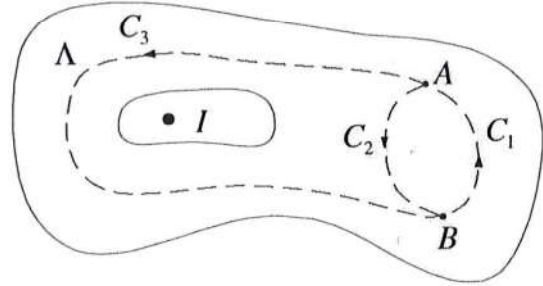


Figure 6.2.1. A multiply connected region and non-uniqueness of ϕ_m

In the region Λ from Figure 6.2.1, the integral (6.2.5) along the closed path that is formed by C_1 and C_2 is equal to zero, and along the closed path that is formed by C_1 and C_3 is equal to I . Therefore, $\phi_m(B) - \phi_m(A)$ is dependent on the integration path. Thus, we can introduce ϕ_m only in simply connected regions.

6.3. A VISUAL PRESENTATION OF A STATIC MAGNETIC FIELD

The graphical presentation of a field is of great help in understanding a static magnetic field, similarly to a static electric field. *A static magnetic field is non-conservative and we cannot present it using equipotential surfaces.* Therefore, the field plot consists of field lines ($\mathbf{B}$ -lines) that are defined as lines to which the vector $\mathbf{B}$ is tangential at every point. We determine the field lines using the differential equations

$$\frac{dx}{B_x} = \frac{dy}{B_y} = \frac{dz}{B_z} \quad . \quad (6.3.1)$$

In two-dimensional problems, we can determine the field lines directly from the magnetic vector potential. In a Cartesian coordinate system, the equation of a field line is $B_x dy - B_y dx = 0$. We calculate the vector $\mathbf{B}$ as

$$\mathbf{B} = \nabla \times A(x,y) \mathbf{a}_z = \frac{\partial A(x,y)}{\partial y} \mathbf{a}_x - \frac{\partial A(x,y)}{\partial x} \mathbf{a}_y . \quad (6.3.2)$$

Accordingly, we obtain the equation of field lines as

$$\frac{\partial A(x,y)}{\partial x} dx + \frac{\partial A(x,y)}{\partial y} dy = 0 \quad \Rightarrow \quad dA(x,y) = 0 \quad \Rightarrow \quad A(x,y) = \text{const.} , \quad (6.3.3)$$

meaning that lines of constant magnetic vector potential are identical to field lines in two-dimensional problems of a static magnetic field.

6.4. ENERGY STORED IN A STATIC MAGNETIC FIELD

In Section 3.3.4. we showed that in order to set up a magnetic field, one has to inject an amount of energy

$$W_m = \int_V \left\{ \int_0^B \mathbf{H} \cdot d\mathbf{B} \right\} dV . \quad (6.4.1)$$

The integrand in this relation can be rearranged using the identity concerning the divergence of a cross product along with equation (6.1.1) as

$$\mathbf{H} \cdot d\mathbf{B} = \mathbf{H} \cdot \nabla \times (d\mathbf{A}) = \nabla \cdot (d\mathbf{A} \times \mathbf{H}) + d\mathbf{A} \cdot \nabla \times \mathbf{H} = \nabla \cdot (d\mathbf{A} \times \mathbf{H}) + \mathbf{J} \cdot d\mathbf{A} . \quad (6.4.2)$$

Substitution of (6.4.2) into (6.4.1) and application of Gauss' divergence theorem results in

$$W_m = \int_V \left\{ \nabla \cdot \int_0^A d\mathbf{A} \times \mathbf{H} + \int_0^A \mathbf{J} \cdot d\mathbf{A} \right\} dV = \oint_S \left\{ \int_0^A d\mathbf{A} \times \mathbf{H} \right\} \mathbf{n} dS + \int_{V_j} \left\{ \int_0^A \mathbf{J} \cdot d\mathbf{A} \right\} dV , \quad (6.4.3)$$

where S is the surface that encloses the domain of the problem, and V_j is the volume of the region in which currents are flowing. If the domain of the problem encloses the complete space of a field, then on the boundary surface S $\mathbf{H} = 0$, and the surface integral in (6.4.3) is equal to zero. In this case, we can calculate the stored energy as being

$$W_m = \int_{V_j} \left\{ \int_0^A \mathbf{J} \cdot d\mathbf{A} \right\} dV . \quad (6.4.4)$$

In linear materials, expression (6.4.1) reduces to

$$W_m = \frac{1}{2} \int_V \mathbf{H} \cdot \mathbf{B} dV = \frac{1}{2} \oint_S (\mathbf{A} \times \mathbf{H}) \mathbf{n} dS + \frac{1}{2} \int_{V_j} \mathbf{J} \cdot \mathbf{A} dV . \quad (6.4.5)$$

If S encloses the complete space of a field, then this expression becomes

$$W_m = \frac{1}{2} \int_{V_j} \mathbf{J} \cdot \mathbf{A} dV . \quad (6.4.6)$$

An especially important case concerns the energy stored in a static magnetic field of a system of N current loops $C_1, \dots, C_N$ in which currents $I_1, \dots, I_N$ are flowing. The loops are in a linear material.

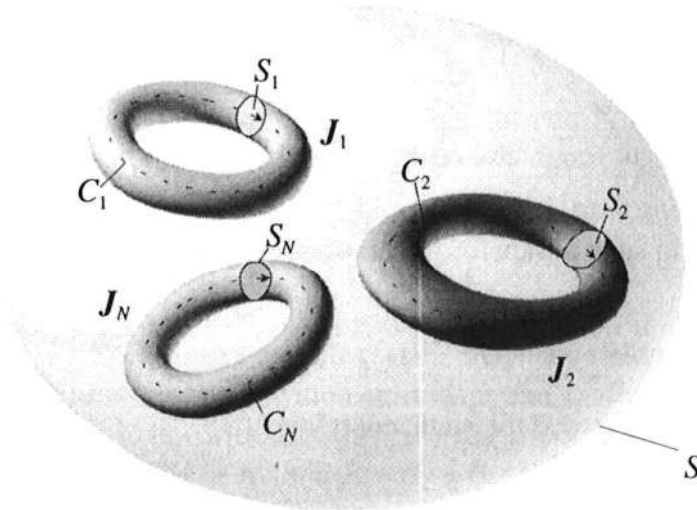


Figure 6.4.1. The static magnetic field of N current loops

In this case, equation (6.4.6) transforms to

$$W_m = \frac{1}{2} \sum_{i=1}^N \int_{V_i} \mathbf{J}_i \cdot \mathbf{A} dV_i . \quad (6.4.7)$$

If we assume that the current loops are thin and have constant cross-section, we can write

$$W_m = \frac{1}{2} \sum_{i=1}^N \int_{V_i} \mathbf{J}_i \cdot \mathbf{A} dV_i = \frac{1}{2} \sum_{i=1}^N \oint_{C_i} \mathbf{J}_i \cdot \mathbf{A} S_i dl_i = \frac{1}{2} \sum_{i=1}^N I_i \oint_{C_i} \mathbf{A} dl_i = \frac{1}{2} \sum_{i=1}^N I_i \Phi_i , \quad (6.4.8)$$

where Φ_i is the magnetic flux linked by curve C_i .

6.4.1. Energy-based determination of forces in a static magnetic field

In Unit 3.5, we studied in detail the determination of forces in a static magnetic field using field quantities and distributions of current. We can apply *the method of virtual displacements* here too, as was the case in static electric fields. Let us analyse a current loop or a body of linear material in a static magnetic field. If, because of the acting of a magnetic force, this loop or body moves by a small virtual displacement δs in the direction of the s -axis, this force has done work $\delta A_m = F_s \delta s$. Similarly to Section 4.3.1, we determine a component of magnetic force in the direction of the s -axis as

$$F_s = \frac{\delta A}{\delta s} . \quad (6.4.9)$$

The work done by the magnetic forces can be expressed as the increase in energy stored in the static magnetic field in two distinct cases:

1. *The magnetic vector potential remains constant at all points inside a conductor (insulated system).* Using equation (6.4.6) we conclude that, in this case, energy is not added to the system and that the total work is done by consuming energy stored in the system. Therefore, $\delta A_m = -\delta W_m$, which means

$$F_s = -\frac{\delta W_m}{\delta s} . \quad (6.4.10)$$

2. *The current density J remains constant at all points of a conductor (uninsulated system).* Energy has to be added in such a transformation of a system, and equals:

$$\delta W_{\text{displacement}} = \int_{V_j} \mathbf{J} \cdot d\mathbf{A} dV . \quad (6.4.11)$$

If materials are linear, the increase in stored magnetic energy is

$$\delta W_m = \frac{1}{2} \int_{V_j} \mathbf{J} \cdot d\mathbf{A} dV . \quad (6.4.12)$$

According to the law of conservation of energy, the added energy $W_{\text{displacement}}$ must have been consumed by the increase in stored magnetic energy, and by the work done by the magnetic forces. Therefore, $\delta A_m = \delta W_m$, and using the expression

$$F_s = \frac{\delta W_m}{\delta s} , \quad (6.4.13)$$

we can calculate the component of the resultant force in the direction of the s -axis in this case.

6.5. INDUCTANCES

Let us analyse the simplest system of a static magnetic field: a single current loop that carries a current I in a linear material. In this case, magnetic field strength and magnetic induction are proportional to the current I and we can express the energy stored in this system as

$$W_m = \frac{1}{2} \int_V \mathbf{H} \cdot \mathbf{B} dV = \frac{1}{2} LI^2 . \quad (6.5.1)$$

The constant L is called *the inductance* or *self-inductance* of the observed loop. The unit of inductance is *the henry* (H), ($1\text{H} = 1\text{Wb}/1\text{A}$). In linear materials, the inductance de-

depends on the geometry of the static magnetic field and on the permeability. We calculate it as being

$$L = \frac{2W_m}{I^2} = \frac{1}{I^2} \int_V \mathbf{H} \cdot \mathbf{B} dV . \quad (6.5.2)$$

We often write this expression in the form

$$L = \frac{1}{I^2} \int_V \mathbf{H} \cdot \mathbf{B} dV = \frac{1}{I^2} \int_{V_i} \mathbf{H} \cdot \mathbf{B} dV + \frac{1}{I^2} \int_{V_o} \mathbf{H} \cdot \mathbf{B} dV = L_i + L_e , \quad (6.5.3)$$

where V_i is the volume of the conductor and V_o is the volume outside the conductor. The coefficient L_i is called *the internal inductance*, and L_e is called *the external inductance*.

6.5.1. Mutual inductance

The next, more complex, static magnetic field that we are going to analyse consists of two current loops in which currents I_1 and I_2 are flowing. The loops are in a linear material. Due to linearity, we can apply the principle of superposition and express the resultant field as $\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2$ and $\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2$. Here $\mathbf{H}_1$ and $\mathbf{B}_1$ are field vectors caused by I_1 if there is no current in the second loop, and $\mathbf{H}_2$ and $\mathbf{B}_2$ are field vectors caused by I_2 if there is no current in the first loop. The total energy stored in such a system can be expressed as

$$W_m = \int_V (\mathbf{H}_1 + \mathbf{H}_2) \cdot (\mathbf{B}_1 + \mathbf{B}_2) dV = \frac{1}{2} \int_V \mathbf{H}_1 \cdot \mathbf{B}_1 dV + \frac{1}{2} \int_V \mathbf{H}_2 \cdot \mathbf{B}_2 dV + \int_V \mathbf{H}_1 \cdot \mathbf{B}_2 dV \quad (6.5.4)$$

According to this equation, we can express the total stored energy as the sum of their individual energies, and the energy of interaction between the loops. Since the vectors $\mathbf{H}_1$ and $\mathbf{B}_1$ are proportional to the current I_1 , and the vectors $\mathbf{H}_2$ and $\mathbf{B}_2$ are proportional to the current I_2 , we can write equation (6.5.4) as

$$W_m = \frac{1}{2} L_{11} I_1^2 + \frac{1}{2} L_{22} I_2^2 + L_{12} I_1 I_2 , \quad (6.5.5)$$

where

$$\begin{aligned} L_{11} &= \frac{1}{I_1^2} \int_V \mathbf{H}_1 \cdot \mathbf{B}_1 dV , \\ L_{22} &= \frac{1}{I_2^2} \int_V \mathbf{H}_2 \cdot \mathbf{B}_2 dV , \\ L_{12} &= \frac{1}{I_1 I_2} \int_V \mathbf{H}_1 \cdot \mathbf{B}_2 dV = \frac{1}{I_1 I_2} \int_V \mathbf{H}_2 \cdot \mathbf{B}_1 dV . \end{aligned} \quad (6.5.6a - c)$$

The coefficients L_{11} and L_{22} are called *the self-inductances* of the loops, and L_{12} is called *the mutual inductance*. Naturally, the integrals in equations (6.5.6a – c) can be determined

using the magnetic vector potential by application of equation (6.4.6)

$$\begin{aligned}
 L_{11} &= \frac{1}{I_1^2} \int_{V_1} \mathbf{J}_1 \cdot \mathbf{A}_1 dV, \\
 L_{22} &= \frac{1}{I_2^2} \int_{V_2} \mathbf{J}_2 \cdot \mathbf{A}_2 dV, \\
 L_{12} &= \frac{1}{I_1 I_2} \int_{V_1} \mathbf{J}_1 \cdot \mathbf{A}_2 dV = \frac{1}{I_1 I_2} \int_{V_2} \mathbf{J}_2 \cdot \mathbf{A}_1 dV.
 \end{aligned} \tag{6.5.7a - c}$$

In these equations V_1 is the volume of the conductor in which current I_1 is flowing, and V_2 is the volume of the conductor in which the current I_2 is flowing. Expression (6.5.7c) enables an essentially simpler calculation of the mutual inductance between thin current loops of constant cross-section. In this case, we can replace $\mathbf{J}dV$ with $I d\mathbf{l}$ and transform expression (6.5.7c) into

$$L_{12} = \frac{1}{I_2} \oint_{C_1} \mathbf{A}_2 \cdot d\mathbf{l}_1 = \frac{1}{I_1} \oint_{C_2} \mathbf{A}_1 \cdot d\mathbf{l}_2. \tag{6.5.8}$$

Using equation (6.2.3), we can write an expression for calculating mutual inductances as a function of magnetic flux

$$L_{12} = \frac{\Phi_{21}}{I_2} = \frac{\int_{S_1} \mathbf{B}_2 \cdot \mathbf{n} dS_1}{I_2} = \frac{\Phi_{12}}{I_1} = \frac{\int_{S_2} \mathbf{B}_1 \cdot \mathbf{n} dS_2}{I_1}. \tag{6.5.9}$$

The flux Φ_{21} is caused by current I_2 and linked by loop C_1 , and the flux Φ_{12} is caused by current I_1 and linked by loop C_2 . We use expression (6.5.9) for an alternative definition of the mutual inductance between two thin current loops as the ratio of the flux linked by one loop and the current in another loop that is causing the flux.

The same approach in which we compress a current distributed over a cross-section of a conductor into its centre can be applied to the calculation of the external self-inductances of thin current loops of constant cross-sections. The internal self-inductance has to be determined from the energy stored in the conductor.

6.5.2. The inductances of a system of current loops

A static magnetic field made up of N current loops through which currents I_i , $i = 1, \dots, N$, are flowing can be represented as the sum of the individual contributions of every single loop, each calculated as if the currents are not flowing in all other loops

$$\mathbf{H} = \sum_{i=1}^N \mathbf{H}_i \quad ; \quad \mathbf{B} = \sum_{i=1}^N \mathbf{B}_i. \tag{6.5.10}$$

The energy stored in such a system can be expressed as

$$W_m = \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \int_V \mathbf{H}_i \cdot \mathbf{B}_j dV , \quad (6.5.11)$$

and due to the linearity of the material, we can introduce the coefficients of proportionality L_{ij} as

$$W_m = \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N I_i I_j L_{ij} , \quad (6.5.12)$$

where

$$L_{ij} = \frac{1}{I_i I_j} \int_V \mathbf{H}_i \cdot \mathbf{B}_j dV . \quad (6.5.13)$$

The coefficient L_{ii} is called the *self-inductance* of i -th loop, and the coefficient L_{ij} is called the *mutual inductance* between the i -th and j -th loop.

6.6. A MAGNETIC FIELD IN A CURRENT-FREE REGION

In Unit 6.2, we showed that this kind of problem can be easily solved using an *antipotential* (magnetic scalar potential). In a simply connected current-free region the antipotential satisfies Laplace's differential equation. With the substitutions

$$\varphi \rightarrow \varphi_m ; \quad \mathbf{E} \rightarrow \mathbf{H} ; \quad \varepsilon \rightarrow \mu . \quad (6.6.1)$$

we can directly apply methods for solving this equation described in Unit 4.6. for solving static magnetic fields.

Example 6.6.1. A ferromagnetic sphere in an external homogeneous field

A sphere in an external homogeneous field in a spherical coordinate system is shown in Figure 6.6.1.

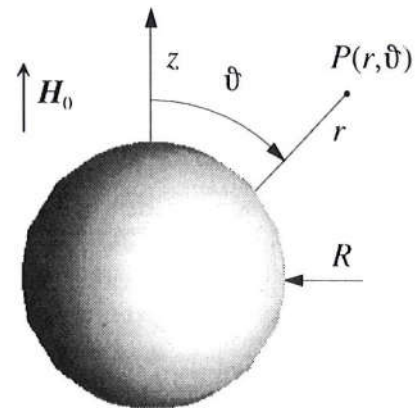


Figure 6.6.1. A ferromagnetic sphere in an external magnetic field

The analogous problem in a static electric field is a dielectric sphere in a homogeneous electric field. This problem was solved in Example 4.6.3.1, and by the application of

analogous substitutions (6.6.1), we can directly write the solution for linear materials. The field outside the sphere and the homogeneous field inside the sphere are

$$\begin{aligned} \mathbf{H}_o &= H_0 \left[1 + 2 \frac{\mu - \mu_0}{2\mu_0 + \mu} \left(\frac{R}{r} \right)^3 \right] \cos \vartheta \mathbf{a}_r + H_0 \left[-1 + \frac{\mu - \mu_0}{2\mu_0 + \mu} \left(\frac{R}{r} \right)^3 \right] \sin \vartheta \mathbf{a}_\vartheta, \\ \mathbf{H}_i &= \frac{3H_0\mu_0 \cos \vartheta}{2\mu_0 + \mu} \mathbf{a}_r - \frac{3H_0\mu_0 \sin \vartheta}{2\mu_0 + \mu} \mathbf{a}_\vartheta = \frac{3\mu_0}{2\mu_0 + \mu} \mathbf{H}_0. \end{aligned} \quad (6.6.2,3)$$

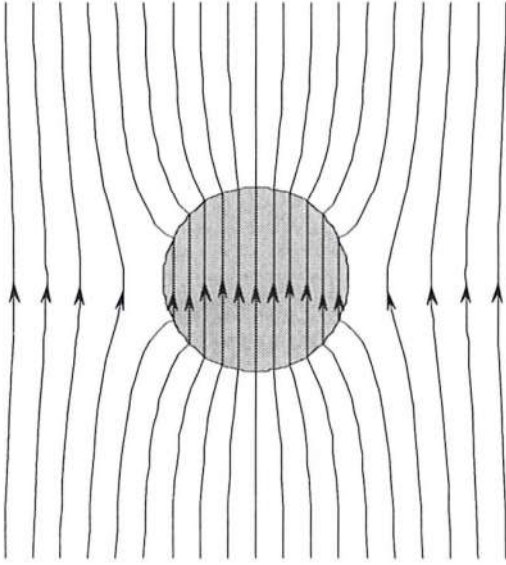


Figure 6.6.2. The static magnetic field of a ferromagnetic sphere in an external magnetic field ($\mu_r = 100$).

We can use this result in order to solve problems relating to the static magnetic field of a sphere made of a non-linear, but homogeneous and isotropic material in an external homogeneous field. The relationship between field strength and induction inside the sphere in this case is $\mathbf{B}_i = \mu_0(\mathbf{H}_i + \mathbf{M})$. Here, the field inside the sphere will be homogeneous and parallel to the external field, and the vectors $\mathbf{H}_0$, $\mathbf{H}_i$, $\mathbf{B}_i$ and $\mathbf{M}$ will be oriented in the same direction. Therefore, we may assume the form of the antipotential inside and outside the sphere to be

$$\begin{aligned} \varphi_{mi} &= A r \cos \vartheta, \\ \varphi_{mo} &= \left(-H_0 r + \frac{B}{r^2} \right) \cos \vartheta. \end{aligned} \quad (6.6.4,5)$$

On the surface of the sphere the radial components of the vector of magnetic induction must be equal to

$$-\left(\frac{\partial \varphi_{mo}}{\partial r} \right)_{r=R} = -\left(\frac{\partial \varphi_{mi}}{\partial r} \right)_{r=R} + M \cos \vartheta \Rightarrow H_0 + \frac{2B}{R^3} = -A + M. \quad (6.6.6)$$

Simultaneously, the tangential components of the vector of magnetic field strength on the surface of the sphere should be equal to

$$-\left(\frac{1}{r} \frac{\partial \varphi_{mo}}{\partial \vartheta}\right)_{r=R} = -\left(\frac{1}{r} \frac{\partial \varphi_{mi}}{\partial \vartheta}\right)_{r=R} \Rightarrow AR = -H_0 + \frac{B}{R^2}. \quad (6.6.7)$$

Using equations (6.6.6,7) we obtain constants A and B , and the final form of the antipotential inside and outside the sphere is

$$\begin{aligned} \varphi_{mi} &= \left(\frac{M}{3} - H_0\right) r \cos \vartheta, \\ \varphi_{mo} &= \left(-H_0 r + \frac{R^3 M}{3r^2}\right) \cos \vartheta. \end{aligned} \quad (6.6.8,9)$$

The field inside the sphere is homogeneous with field strength and with induction being

$$H_i = H_0 - \frac{M}{3} \quad ; \quad B_i = \mu_0(H_i + M) = \mu_0\left(H_0 + \frac{2}{3}M\right). \quad (6.6.10,11)$$

By eliminating M and by usage of equation (6.6.11), we obtain $B_i = \mu_0(3H_0 - 2H_i)$. This is the equation of line "a" in the H_i - B_i plane and its intersection with the magnetisation curve (Figure 6.6.3) defines H_i and B_i , or in another words, the working point of the ferromagnetic sphere.

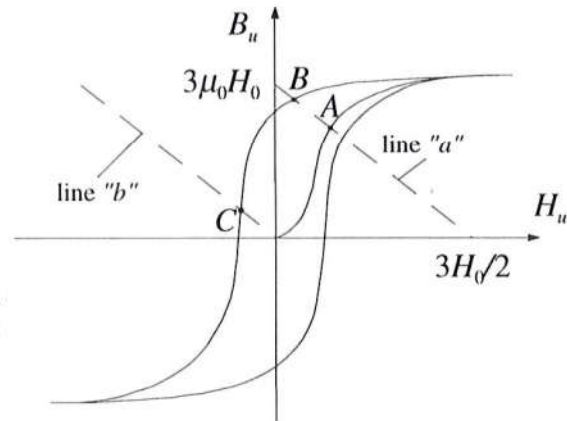


Figure 6.6.3. Determination of the working point of a ferromagnetic curve in an external homogeneous field and of a magnetised spherical permanent magnet

In order to clarify the presentation, we have plotted unrealistically steep slopes for the lines. If a material is pure iron, its retentivity is 0.6 T and the segment of the line "a" on the axis B_u for $H_0 = 200$ A/m is 0.25 mT. This means that in Figure 6.6.3, for pure iron, we would not be able to distinguish the lines "a" and "b" from the axis H_u .

If the sphere was not magnetised before insertion into the homogeneous field, the working point would be A on the curve of first magnetisation. If the sphere was magnetised to saturation, the working point would be point B on the hysteresis.

Example 6.6.2. A homogeneously magnetised spherical permanent magnet

Let us imagine that after the magnetisation of the sphere from the previous example, the magnetisation M remained after the vanishing of the external field H_0 . In such a way, we obtain a permanent magnet with a field H_p and induction B_p . We can determine H_p

from expression (6.6.10) and B_p from expression (6.6.11) as

$$H_p = \lim_{H_0 \rightarrow 0} \frac{3H_0 - M}{3} = -\frac{M}{3} \quad ; \quad B_p = \lim_{H_0 \rightarrow 0} \mu_0 \frac{3H_0 + 2M}{3} \quad (6.6.12,13)$$

It is obvious from these equations that inside the permanent magnet, B_p and H_p are in opposite directions. We obtain the antipotential outside the magnet using expression (6.6.9)

$$\Phi_{mo} = \frac{R^3 M}{3r^2} \cos \vartheta \quad , \quad (6.6.14)$$

and the field strength outside the magnet is

$$\mathbf{H}_o = -\nabla \Phi_{mo} = \frac{R^3 M}{3r^3} (2 \cos \vartheta \mathbf{a}_r + \sin \vartheta \mathbf{a}_\vartheta) \quad ; \quad \mathbf{B}_o = \mu_0 \mathbf{H}_o \quad . \quad (6.6.15)$$

The relationship between field and induction is obtained from the equation $B_p = \mu_0(3H_0 - 2H_i)$ as $B_p = -2\mu_0 H_i$, and the working point of the magnet is point C at the intersection of this line (line "b" in Figure 6.6.3) and the hysteresis in the second quadrant.

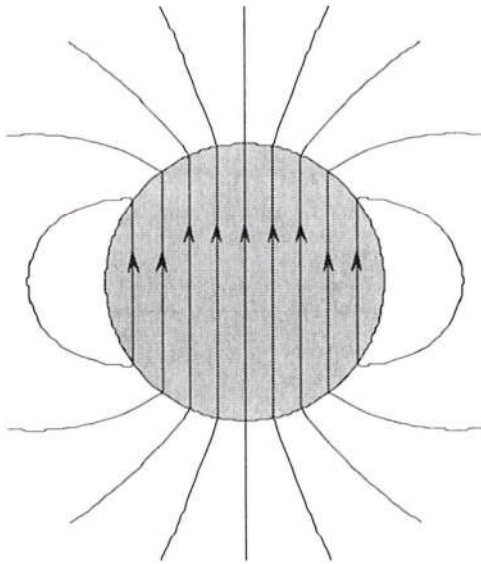


Figure 6.6.4. The field of a homogeneously magnetised spherical permanent magnet

Example 6.6.3. A spherical shell in a homogeneous field

With this example, we want to demonstrate an application of magnetic shields with high permeability for the protection of magnetic fields. We shall analyse a simple form of a shield: a spherical shell of inner radius R_1 and outer radius R_2 placed in a homogeneous field $\mathbf{H}_0$. The shell is made of a linear material with permeability μ .

We shall find the solution of this problem using Laplace's equation for an antipotential by application of a separation of variables in a spherical coordinate system. We assume the form of the solution for the antipotential in the three regions of this problem to be a linear combination of zero and first order zonal harmonics like so:

$$\begin{aligned}
\varphi_{m3} &= A_3 + \frac{B_3}{r} + C_3 r \cos \vartheta + \frac{D_3}{r^2} \cos \vartheta \quad ; \quad r \geq R_2 , \\
\varphi_{m2} &= A_2 + \frac{B_2}{r} + C_2 r \cos \vartheta + \frac{D_2}{r^2} \cos \vartheta \quad ; \quad R_1 \geq r \geq R_2 , \\
\varphi_{m1} &= A_1 + \frac{B_1}{r} + C_1 r \cos \vartheta + \frac{D_1}{r^2} \cos \vartheta \quad ; \quad R_1 \geq r \geq 0 .
\end{aligned} \tag{6.6.16a - c}$$

The antipotential must be finite at all points and therefore $B_1 = D_1 = 0$. At great distances from the shell, the antipotential φ_{m3} has to be equal to the antipotential of the homogeneous field in the direction of the z -axis

$$\varphi_{m3}(r \gg R_2) = -H_0 r \cos \vartheta = A_3 + C_3 r \cos \vartheta \quad \Rightarrow \quad A_3 = 0, \quad C_3 = -H_0 . \tag{6.6.17}$$

The antipotential should be continuous at the boundaries $r = R_1$ and $r = R_2$, and because $A_3 = 0$, the expression $A_2 = A_1 = 0$ is valid, and because $B_1 = 0$ the expression $B_2 = B_3 = 0$ is also valid. We determine the unknown coefficients C_1 , C_2 , D_2 and D_3 using the boundary conditions

$$\begin{aligned}
\varphi_{m1}(R_1) &= \varphi_{m2}(R_1) , \\
\varphi_{m2}(R_2) &= \varphi_{m3}(R_2) , \\
\mu_0 \left(\frac{\partial \varphi_{m1}}{\partial r} \right)_{r=R_1} &= \mu \left(\frac{\partial \varphi_{m2}}{\partial r} \right)_{r=R_1} , \\
\mu_0 \left(\frac{\partial \varphi_{m3}}{\partial r} \right)_{r=R_2} &= \mu \left(\frac{\partial \varphi_{m2}}{\partial r} \right)_{r=R_2} ,
\end{aligned} \tag{6.6.18 - 21}$$

The magnetic field strength in the cavity is

$$H_1 = \frac{1}{1 + \frac{2(\mu_r - 1)^2}{9\mu_r} \left[1 - \left(\frac{R_1}{R_2} \right)^3 \right]} H_0 . \tag{6.6.22}$$

The ratio

$$\eta = \frac{H_1}{H_0} = \frac{1}{1 + \frac{2(\mu_r - 1)^2}{9\mu_r} \left[1 - \left(\frac{R_1}{R_2} \right)^3 \right]} , \tag{6.6.23}$$

represents the quality of protection, or in other words it represents the decrease in external field caused by the shield in the form of a spherical shell.

It is obvious that we can effectively reduce the field with a relatively thin shell, if the material used possesses great permeability. In any case, if H_0 brings the magnetic material of the shell to saturation, effective reduction of the field cannot be achieved.

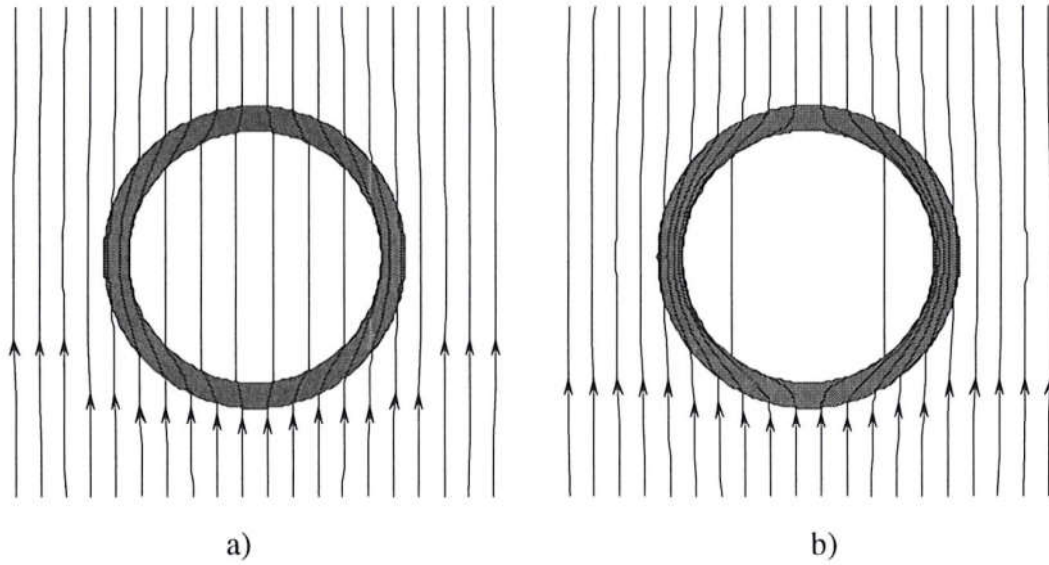


Figure 6.6.5. The magnetic field of a spherical shell in a homogeneous field for $k = R_2/R_1 = 1.2$; a) $\mu_r = 2$, and b) $\mu_r = 3$

6.7. THE FIELD OF GIVEN CURRENTS IN AN UNBOUNDED SPACE

We solve problems of this kind using an integral equation approach based around the magnetic vector potential. In Unit 6.2, we showed that the magnetic vector potential in a linear isotropic material satisfies the equation $\Delta \mathbf{A} = -\mu \mathbf{J}$. In Section 2.4.3, we derived the solution of this equation in an unbounded space as

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{V'} \frac{\mathbf{J}(\mathbf{r}') dV}{|\mathbf{r} - \mathbf{r}'|} . \quad (6.7.1)$$

In a Cartesian coordinate system this solution can be written in the form of three scalar solutions

$$A_j(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{V'} \frac{J_j(\mathbf{r}') dV}{|\mathbf{r} - \mathbf{r}'|} \quad ; \quad j=x, y, z , \quad (6.7.2)$$

as in this system, $\Delta \mathbf{A} = \Delta A_x \mathbf{a}_x + \Delta A_y \mathbf{a}_y + \Delta A_z \mathbf{a}_z = -\mu(J_x \mathbf{a}_x + J_y \mathbf{a}_y + J_z \mathbf{a}_z)$. It is not so simple to separate equation (6.7.1) into three scalar differential equations with respect to components in the directions of other rectangular coordinate systems. Therefore, one should be very careful while applying expression (6.7.2) in these systems. Using the identity regarding the curl of a product of a scalar function and a vector constant $\nabla \times (f\mathbf{C}) = \nabla f \times \mathbf{C}$, and

$$\nabla \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = -\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}, \quad (6.7.3)$$

we can write

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A} = \frac{\mu_0}{4\pi} \int_{V'} \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dV'. \quad (6.7.4)$$

This equation is known as the *Biot-Savart law*.

6.7.1. The field of line currents in an unbounded space

We introduced the concept of a line current in Unit 1.2. as an idealised conductor in which the complete current is concentrated in a geometrical line. In this case, the relationship $\mathbf{J}dV = I d\mathbf{l}$ is valid and the expressions

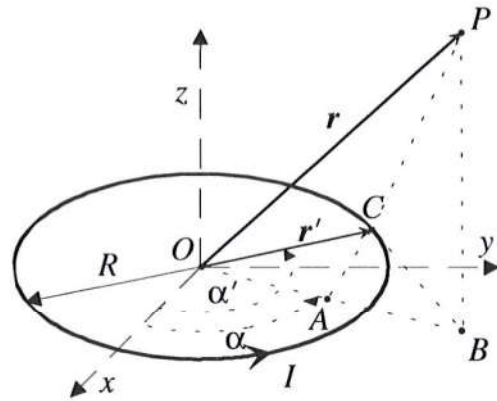
$$\mathbf{A} = \frac{\mu_0 I}{4\pi} \int_C \frac{d\mathbf{l}}{|\mathbf{r} - \mathbf{r}'|} \quad ; \quad \mathbf{B} = \frac{\mu_0 I}{4\pi} \int_C \frac{d\mathbf{l} \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}, \quad (6.7.5,6)$$

enable the calculation of the magnetic vector potential and induction of line currents.

Example 6.7.1.1. The static magnetic field of a circular current loop

A circular current loop of radius R is shown in Figure 6.7.1. in a cylindrical coordinate system. Current I is flowing through the loop.

Figure 6.7.1. Circular current loop



We calculate the magnetic vector potential of the circular current loop at a point P using expression (6.7.5). The contribution of $d\mathbf{l}$ to $d\mathbf{A}$ at point P changes direction as we move along the loop, so we must express it as a function of cylindrical coordinates at point $P(r, \alpha, z)$

$$\mathbf{A}_P = \frac{\mu_0 I}{4\pi} \oint_C \frac{R [-\sin(\alpha - \alpha') \mathbf{a}_r + \cos(\alpha - \alpha') \mathbf{a}_\alpha]}{|\mathbf{r} - \mathbf{r}'|} d\alpha. \quad (6.7.7)$$

From triangles OCA , CAB and CBP it follows that

$$|\mathbf{r} - \mathbf{r}'| = \sqrt{[r - R \cos(\alpha - \alpha')]^2 + R^2 \sin^2(\alpha - \alpha') + z^2}, \quad (6.7.8)$$

and therefore

$$A_P = \frac{\mu_0 I R}{4\pi} \left(\mathbf{a}_r \int_0^{2\pi} \frac{-\sin \vartheta d\vartheta}{\sqrt{R^2 + r^2 + z^2 - 2Rr \cos \vartheta}} + \mathbf{a}_\alpha \int_0^{2\pi} \frac{\cos \vartheta d\vartheta}{\sqrt{R^2 + r^2 + z^2 - 2Rr \cos \vartheta}} \right), \quad (6.7.9)$$

where $\vartheta = \alpha - \alpha'$. The integral in the direction of the coordinate $\mathbf{a}_r$ can be solved analytically, and within the boundaries from 0 to 2π is equal to zero. Therefore, the magnetic vector potential possesses only a component in the direction of the coordinate $\mathbf{a}_\alpha$. By the substitution

$$\vartheta = \pi + 2\beta \quad \Rightarrow \quad d\vartheta = 2d\beta \quad ; \quad \cos \vartheta = 2\sin^2 \beta - 1, \quad (6.7.10)$$

we can transform the integral (6.7.9) to the form

$$A_\alpha = \frac{\mu_0 I R^{\frac{2}{2}}}{\pi} \int_0^{\frac{\pi}{2}} \frac{(2\sin^2 \beta - 1) d\beta}{\sqrt{(R+r)^2 + z^2 - 4Rr \sin^2 \beta}}. \quad (6.7.11)$$

If we substitute

$$k^2 = \frac{4Rr}{(R+r)^2 + z^2}, \quad (6.7.12)$$

into equation (6.7.11) we obtain

$$A_\alpha = \frac{\mu_0 k I}{2\pi} \sqrt{\frac{R}{r}} \left[\left(\frac{2}{k^2} - 1 \right) \int_0^{\frac{\pi}{2}} \frac{d\beta}{\sqrt{1 - k^2 \sin^2 \beta}} - \frac{2}{k^2} \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \beta} d\beta \right]. \quad (6.7.13)$$

The integrals in (6.7.13) cannot be solved analytically, and we call them complete elliptic integrals. The complete elliptic integral of the first kind $K(k)$ and the complete elliptic integral of the second kind $E(k)$ are

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\beta}{\sqrt{1 - k^2 \sin^2 \beta}}, \quad (6.7.14)$$

$$E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \beta} d\beta.$$

These elliptic integrals are calculated using an infinite series of the parameter k . We usually write the expression for the magnetic vector potential of a circular current loop as

$$A_\alpha = \frac{\mu_0 I}{\pi k} \sqrt{\frac{R}{r}} \left[\left(1 - \frac{k^2}{2} \right) K(k) - E(k) \right]. \quad (6.7.16)$$

We calculate the components of magnetic induction using the equation $\mathbf{B} = \nabla \times \mathbf{A}$ in a cylindrical coordinate system. The final expressions for the magnetic induction of a circular current loop at a point P are

$$B_r = \frac{\mu_0 I}{2\pi} \frac{z}{r} \frac{1}{\sqrt{(R+r)^2 + z^2}} \left[-K(k) + \frac{R^2 + r^2 + z^2}{(R-r)^2 + z^2} E(k) \right],$$

$$B_z = \frac{\mu_0 I}{2\pi} \frac{1}{\sqrt{(R+r)^2 + z^2}} \left[K(k) + \frac{R^2 - r^2 - z^2}{(R-r)^2 + z^2} E(k) \right]. \quad (6.7.17,18)$$

The field lines can be determined using the magnetic vector potential due to the axially symmetrical nature of the field. The equation for field lines independent of any circumferential coordinate in a cylindrical coordinate system is

$$\frac{B_r}{B_z} = \frac{dr}{dz}. \quad (6.7.19)$$

From this equation it follows that

$$B_z dr - B_r dz = \frac{1}{r} \frac{\partial(rA_\alpha)}{\partial r} dr + \frac{\partial A_\alpha}{\partial z} dz = d(rA_\alpha) = 0, \quad (6.7.20)$$

and field lines in such problems are defined as $rA_\alpha = \text{const.}$

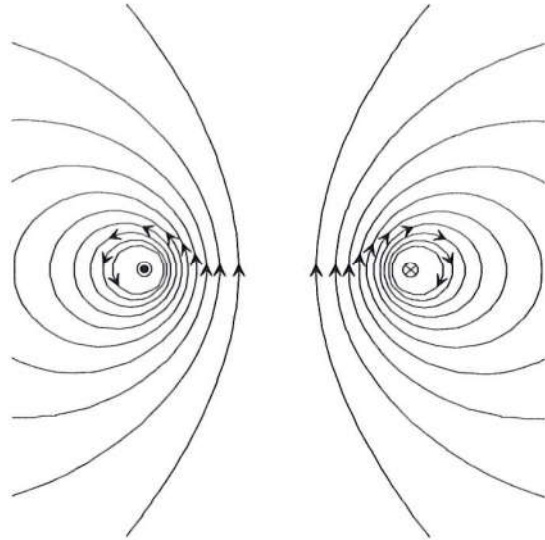


Figure 6.7.2. The static magnetic field of a circular current loop

The field of a circular current loop carrying current I flowing in a circumferential direction in the plane $\alpha = \text{const}$ is shown in Figure 6.7.2.

6.7.1.1. Magnetic dipole

A very small current loop C carrying current I is called a magnetic dipole.

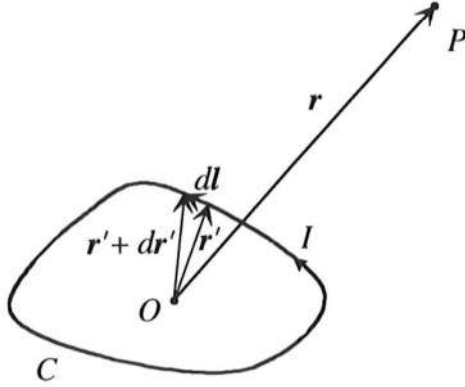


Figure 6.7.3. Magnetic dipole

The magnetic vector potential of such a current loop is defined by equation (6.7.9). From the triangle defined by $\mathbf{r}$, $\mathbf{r}'$, $\mathbf{r} - \mathbf{r}'$ it follows that $|\mathbf{r} - \mathbf{r}'|^2 = r^2 + r'^2 - 2\mathbf{r} \cdot \mathbf{r}'$. If point P is far away from the loop, meaning $r \gg r'$, we can use the series expansion for $(1 - x)^{1/2}$ and approximately write

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{r} \sqrt{1 - \frac{2\mathbf{r} \cdot \mathbf{r}'}{r^2} + \frac{r'^2}{r^2}} \approx \frac{1}{r} + \frac{\mathbf{r} \cdot \mathbf{r}'}{r^3} . \quad (6.7.21)$$

If we substitute this approximation into equation (6.7.5) we obtain

$$\mathbf{A} = \frac{\mu_0 I}{4\pi r} \oint_C d\mathbf{l} + \frac{\mu_0 I}{4\pi r^3} \oint_C (\mathbf{r} \cdot \mathbf{r}') d\mathbf{l} . \quad (6.7.22)$$

The first term in this expression is equal to zero and from the equation $\mathbf{r}' + d\mathbf{r}' = \mathbf{r}' + d\mathbf{l}$ it follows that $d\mathbf{l} = d\mathbf{r}'$ and

$$\mathbf{A} = \frac{\mu_0 I}{4\pi r^3} \oint_C (\mathbf{r} \cdot \mathbf{r}') d\mathbf{r}' . \quad (6.7.23)$$

After rearrangement of the integrand in this equation we finally obtain

$$\mathbf{A} = \frac{\mu_0 I}{4\pi r^3} \oint_C \frac{1}{2} (\mathbf{r}' \times d\mathbf{r}') \times \mathbf{r} . \quad (6.7.24)$$

Here, $\mathbf{r}$ is a constant vector and the product $(\mathbf{r}' \times d\mathbf{r}')/2$ is the surface vector of the triangle defined by $\mathbf{r}'$ and $d\mathbf{r}'$. Therefore,

$$\oint_C \frac{1}{2} (\mathbf{r}' \times d\mathbf{r}') = \mathbf{S} , \quad (6.7.25)$$

is the surface vector of the loop. The expression for magnetic vector potential of the

magnetic dipole is

$$\mathbf{A} = \frac{\mu_0}{4\pi} \frac{IS \times \mathbf{r}}{r^3} = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{r^3} . \quad (6.7.26)$$

From this expression it follows that the field of the magnetic dipole is independent of the shape of the current loop. It depends only on the quantity IS , called the *dipole moment*, and it is usually denoted by $\mathbf{m}$. Magnetic induction at a distant point is calculated from

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{\mu_0}{4\pi} \nabla \times \left[\frac{1}{r^3} (\mathbf{m} \times \mathbf{r}) \right] = \frac{\mu_0}{4\pi} \left[\frac{3(\mathbf{m} \cdot \mathbf{r}) \cdot \mathbf{r}}{r^5} - \frac{\mathbf{m}}{r^3} \right] \quad (6.7.27)$$

If we compare this equation with expression (4.6.32) for the electric field strength of an electric dipole, we can see a formal analogy between these two expressions, thus rationalizing the introduction of two fictitious magnetic charges as sources of the magnetic dipole field. Now we can write expression (6.7.27) as the negative gradient of the scalar function

$$\varphi_m = \frac{1}{4\pi} \frac{\mathbf{m} \cdot \mathbf{r}}{r^3} . \quad (6.7.28)$$

The function φ_m is called the antipotential or the magnetic scalar potential of the magnetic dipole.

6.7.2. The field of a surface current

We introduced the notion of a surface current in Unit 1.2. as the current that flows along a geometrical surface. In this case, the relationship $\mathbf{J}dV = \mathbf{K}dS$ is valid. Here $\mathbf{K}$ is the surface current density. The expressions

$$\begin{aligned} \mathbf{A} &= \frac{\mu_0}{4\pi} \int_S \frac{\mathbf{K} dS}{|\mathbf{r} - \mathbf{r}'|} , \\ \mathbf{B} &= \frac{\mu_0}{4\pi} \int_S \frac{\mathbf{K} \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dS , \end{aligned} \quad (6.7.29,30)$$

determine the magnetic vector potential and induction of the surface current.

Example 6.7.2.1. The static magnetic field of a current-carrying cylinder

A current cylinder of radius R and height $2h$ carrying a current with uniform surface density in a circumferential direction $\mathbf{a}_\alpha$ is placed in cylindrical coordinate system as shown in Figure 6.7.4. We solve this problem by the superposition of the solution of Example 6.7.1.1. Therefore, the magnetic vector potential possesses only a circumferential component

$$A_\alpha = \frac{\mu_0}{2\pi} \int_0^\pi R \cos \vartheta d\vartheta \int_{-h}^{+h} \frac{d\zeta}{\sqrt{R^2 + r^2 + (z - \zeta)^2 - 2Rr \cos \vartheta}}. \quad (6.7.31)$$

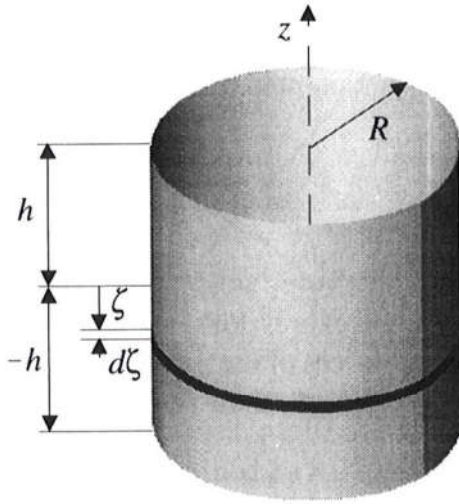


Figure 6.7.4. Current-carrying cylinder

Substitution of $u = z - \zeta$ and integration results in

$$A_\alpha = \frac{\mu_0 K R}{\pi} \int_0^{\frac{\pi}{2}} \left[\frac{k^2 \sin^2 2\beta}{\sqrt{1 - k^2 \sin^2 \beta} (a + \sqrt{1 - k^2 \sin^2 \beta})} \right]_{u=z-h}^{u=z+h} d\beta, \quad (6.7.32)$$

where

$$k^2 = \frac{4Rr}{(R+r)^2 + u^2}; \quad a = \frac{u}{\sqrt{(R+r)^2 + u^2}}. \quad (6.7.33)$$

The integral (6.7.32) can be separated into complete elliptic integrals. In contrast to Example 8.1.1.1, a complete elliptic integral of the third kind Π with parameters k and n occurs here

$$\Pi(n, k) = \int_0^{\frac{\pi}{2}} \frac{d\beta}{(1 + n \sin^2 \beta) \sqrt{1 - k^2 \sin^2 \beta}}. \quad (6.7.34)$$

The final result is

$$A_\alpha = \frac{\mu_0 K}{4\pi} \left\{ \frac{u}{r} \left[-\frac{u^2 + 2R^2 + 2r^2}{\sqrt{(R+r)^2 + u^2}} K(k) + \sqrt{(R+r)^2 + u^2} E(k) + \frac{(R-r)^2}{\sqrt{(R+r)^2 + u^2}} \Pi(n, k) \right] \right\}_{u=z-h}^{u=z+h} \quad (6.7.35)$$

and the parameter n of the complete elliptic integral of the third kind is

$$n = \frac{4Rr}{(R+r)^2} . \quad (6.7.36)$$

The components of magnetic induction are

$$B_r = \frac{\mu_0 K}{4\pi} \left\{ \frac{\sqrt{(R+r)^2 + u^2}}{r} \left[\left(2 - \frac{4Rr}{(R+r)^2 + u^2} \right) K(k) - 2E(k) \right] \right\}_{u=z-h}^{u=z+h} ,$$

$$B_z = \frac{\mu_0 K}{2\pi} \left\{ \frac{-u}{\sqrt{(R+r)^2 + u^2}} \left[K(k) + \frac{R-r}{R+r} \Pi(n, k) \right] \right\}_{u=z-h}^{u=z+h} . \quad (6.7.37)$$

The static magnetic field of a current cylinder with dimensions $R = h$ carrying a current with surface density K in a circumferential direction $\mathbf{a}_\alpha$ is shown in Figure 6.7.5.

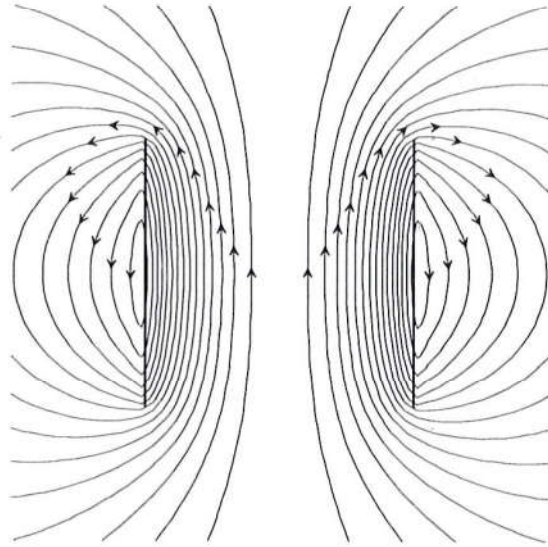


Figure 6.7.5. Static magnetic field of a current cylinder

6.7.3. The field of volume currents and a multipole expansion

We can calculate the magnetic vector potential and magnetic induction of volume currents in an unbounded space using equations (6.7.1) and (6.7.4). In most cases, it is not possible to calculate the triple integrals in these equations analytically and we often use numerical approximations. The simplest approximation is a multipole expansion of the magnetic vector potential similar to the multipole expansion of the electric scalar potential in a static electric field. If we substitute the expansion (4.7.37) into expression (6.7.1), we obtain

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi r} \int_V \mathbf{J}(\mathbf{r}') dV + \frac{\mu_0}{4\pi r^3} \int_V (\mathbf{r} \cdot \mathbf{r}') \mathbf{J}(\mathbf{r}') dV + \frac{\mu_0}{4\pi r^3} \int_V \left[\frac{3}{2} \left(\frac{\mathbf{r}}{r} \cdot \mathbf{r}' \right)^2 - \frac{1}{2} r'^2 \right] \mathbf{J}(\mathbf{r}') dV .$$

Let us analyse the first term in this expansion. For the x component of current density in a

Cartesian coordinate system we may write $J_x = \mathbf{J} \cdot \mathbf{a}_x = \mathbf{J} \cdot \nabla x = \nabla \cdot (x\mathbf{J}) - x\nabla \cdot \mathbf{J}$, and we obtain

$$\int_V J_x dV = \int_V \nabla \cdot (x\mathbf{J}) dV - \int_V x \nabla \cdot \mathbf{J} dV = \oint_S x \mathbf{J} \cdot \mathbf{n} dS - \int_V x \nabla \cdot \mathbf{J} dV . \quad (6.7.38)$$

The integral over the surface that encloses the volume of currents is equal to zero since on this surface the currents are tangential to the surface or equal to zero. The second integral is also equal to zero since $\nabla \cdot \mathbf{J} = 0$ in static fields. The proof for other components is identical. Accordingly, the term that corresponds to the monopole term in the electrostatic multipole expansion does not exist in the magnetostatic multipole expansion.

The magnetic vector potential in the second term of the expansion (6.7.38) decreases with the square of distance and it represents the greatest contribution. The integrand of this term can be written in a simpler form as being

$$\frac{\mu_0}{4\pi r^3} \frac{1}{2} \int_V \mathbf{r}' \times \mathbf{J} \times \mathbf{r} dV . \quad (6.7.39)$$

Vector $\mathbf{r}$ is constant, and we introduce the notion of the magnetic moment of a current distribution $\mathbf{m}$ as

$$\mathbf{m} = \frac{1}{2} \int_V \mathbf{r}' \times \mathbf{J} dV , \quad (6.7.40)$$

and we finally obtain an approximation of the magnetic vector potential of a given current distribution as

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{r^3} + \dots \quad (6.7.41)$$

Example 6.7.3.1. A cylindrical current-carrying tube

A cylindrical current-carrying tube of inner radius R_1 , outer radius R_2 and height $2h$ carrying a current with volume density $\mathbf{J} = J\mathbf{a}_z$ is shown in Figure 6.7.6.

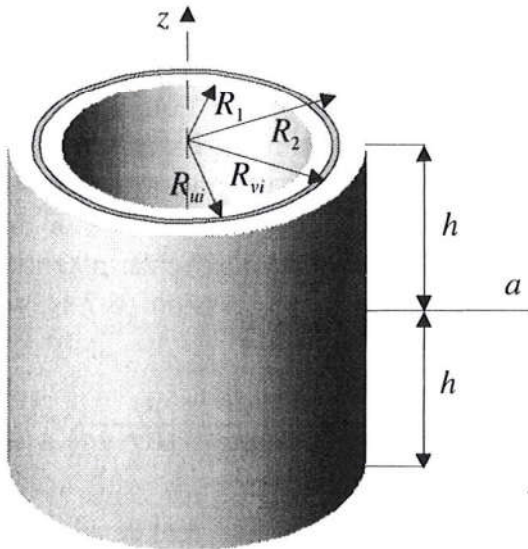


Figure 6.7.6. A cylindrical current-carrying tube

It is not possible to determine the magnetic vector potential and induction of such a tube analytically. We shall compare two approaches here:

1. *Determination of the magnetic vector potential by the superposition of contributions of current-carrying cylinders from Example 6.7.2.1. of finite width.*

With this approach, we divide the tube into N thick tubes of finite width $\Delta\eta = (R_2 - R_1)/N$ and we substitute each thick tube with a current-carrying cylinder at the centre of the tube's thickness. Inner and outer radii for each thick tube are

$$\left. \begin{aligned} R_{ui} &= R_1 + (i-1)\Delta\eta \\ R_{vi} &= R_1 + i\Delta\eta \end{aligned} \right\} \quad i = 1 \dots N . \quad (6.7.42)$$

The surface current density and radius of the i -th cylinder are

$$\left. \begin{aligned} K_i &= \frac{JS_i}{2h} = (R_{vi} - R_{ui})J \\ R_i &= R_1 + (i-0.5)\Delta\eta \end{aligned} \right\} \quad i = 1 \dots N . \quad (6.7.43)$$

We can calculate the approximate value of the magnetic vector potential as

$$\begin{aligned} A_a = \frac{\mu_0}{4\pi} \sum_{i=1}^N \left\{ \frac{K_i u}{r} \left[-\frac{u^2 + 2R_i^2 + 2r^2}{\sqrt{(R_i+r)^2 + u^2}} K(k_i) + \right. \right. \\ \left. \left. + \sqrt{(R_i+r)^2 + u^2} E(k_i) + \frac{(R_i-r)^2}{\sqrt{(R_i+r)^2 + u^2}} \Pi(n_i, k_i) \right] \right\} \Bigg|_{u=z+h}^{u=z-h}, \end{aligned} \quad (6.7.44)$$

where parameters k_i and n_i of the complete elliptic integrals are

$$\left. \begin{aligned} k_i^2 &= \frac{4R_i r}{(R_i+r)^2 + u^2} \\ n_i &= \frac{4R_i r}{(R_i+r)^2} \end{aligned} \right\} \quad i = 1 \dots N . \quad (6.7.45)$$

2. *A Multipole expansion*

The magnetic moment of such a distribution of current is

$$\begin{aligned} \mathbf{m} &= \frac{1}{2} \int_V \mathbf{r}' \times \mathbf{J} dV = \frac{1}{2} \int_{-h}^h d\zeta \int_{R_1}^{R_2} d\eta \int_0^{2\pi} (\eta \mathbf{a}_r + \zeta \mathbf{a}_z + \eta \alpha \mathbf{a}_\alpha) \times \mathbf{J} \mathbf{a}_\alpha \eta d\alpha = \\ &= \frac{2\pi J h}{3} (R_2^3 - R_1^3) \mathbf{a}_z, \end{aligned} \quad (6.7.46)$$

and the magnetic vector potential at a point $\mathbf{r}$ can be approximately calculated as

$$\mathbf{A}(\mathbf{r}) \approx \frac{\mu_0 J h (R_2^3 - R_1^3)}{6} \frac{\mathbf{a}_z \times \mathbf{r}}{r^3} . \quad (6.7.47)$$

The static magnetic field of a current-carrying tube with $R_1 = 1\text{ m}$, $R_2 = 2\text{ m}$ i $\mathbf{J} = 1\mathbf{a}_z\text{ A/m}^2$ determined by approach 1. with $N = 10$ divisions is shown in Figure 6.7.7.

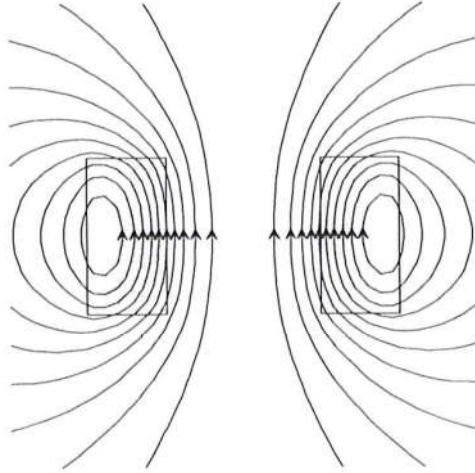


Figure 6.7.7. The static magnetic field of a current-carrying tube determined by the superposition of contributions of current-carrying cylinders ($N = 10$, $R_1 = 1\text{ m}$, $R_2 = 2\text{ m}$, $\mathbf{J} = 1\mathbf{a}_z\text{ A/m}^2$)

6.8. GIVEN CURRENTS IN A BOUNDED SPACE AND THE METHOD OF IMAGES

In Section 6.2.1, we presented an analogy between two-dimensional static magnetic fields and static electric fields. Both satisfy Poisson's equation and everything we said about the characteristics of this equation in Unit 4.5. and its solving using Green's functions can be applied to the solving of two-dimensional static magnetic fields. We can transform solutions of two-dimensional static electric fields in a Cartesian coordinate system to those of static magnetic fields by the substitutions

$$\varphi \rightarrow A_z \quad ; \quad \varepsilon \rightarrow \frac{1}{\mu} \quad ; \quad \rho \rightarrow j_z . \quad (6.8.1)$$

Accordingly, we can use two-dimensional Green's functions determined in Unit 4.8. for the determination of the magnetic vector potential possessing only one component in two-dimensional problems.

Of course, determination of fields from the potential function is different. We determine the magnetic induction in a two-dimensional static magnetic field using the expression

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{\partial A_z}{\partial y} \mathbf{a}_x - \frac{\partial A_z}{\partial x} \mathbf{a}_y . \quad (6.8.2)$$

Example 6.8.1. A two-wire line in front of a magnetic half space

The problem is illustrated in Figure 6.8.1.

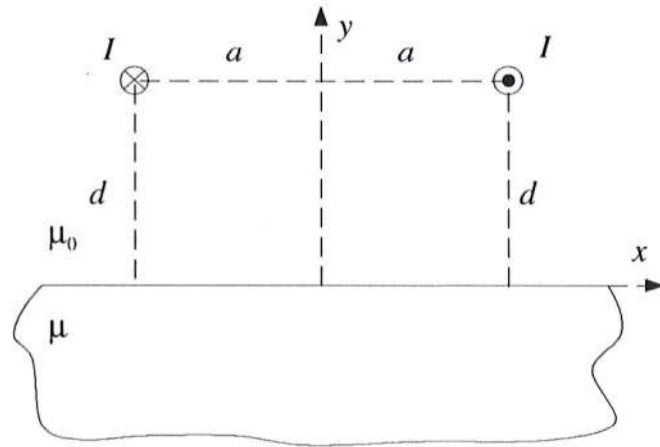


Figure 6.8.1. Two-wire line in front of a magnetic half space

This problem is analogous to the problem of a two-wire line in front of a dielectric half space that we solved in Example 4.8.2.1. Using substitutions (6.8.1) and equations (4.8.23 and 24) we can immediately write a solution for the magnetic vector potential in this problem as

$$A_{z1} = \frac{\mu_0 I}{4\pi} \left\{ \ln \frac{(x-a)^2 + (y-d)^2}{(x-a)^2 + (y+d)^2} + \frac{\mu - \mu_0}{\mu + \mu_0} \ln \frac{(x+a)^2 + (y-d)^2}{(x+a)^2 + (y+d)^2} \right\}, \quad (6.8.3a,b)$$

$$A_{z2} = \frac{\mu_0 I}{4\pi} \frac{2\mu_0}{\mu + \mu_0} \ln \frac{(x-a)^2 + (y-d)^2}{(x-a)^2 + (y+d)^2}.$$

Application of expression (6.8.2) gives us the magnetic induction in both regions of this problem.

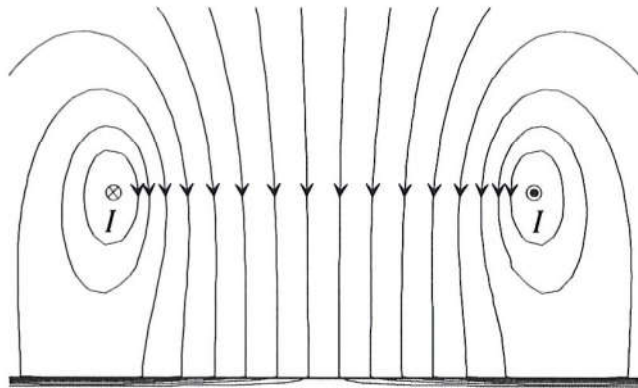


Figure 6.8.2. The static magnetic field of a two-wire line in front of a magnetic half space ($\mu_r = 100$)

PROBLEMS

- 6.1. Discuss the question of appropriateness of introducing the magnetic vector potential in static magnetic field problems !
- 6.2. Determine both the internal and external inductance of an infinitely long coaxial cable where the current I flows through an internal conductor of radius R_1 , and current $-I$ flows through the external conductor of radii R_2 and R_3 ! Assume that the current is uniformly distributed over the cross-section of the internal and external conductor !
- 6.3. Prove the relationship between self-inductances and mutual inductance in a system of two current loops $L_{12}^2 \leq L_{11}L_{22}$! Hint: Analyse the energy stored in such a system!
- 6.4. Determine an approximate expression for the ratio η (6.6.23) in the case of a thin shell of thickness $\delta = (R_2 - R_1) \ll R_1$ and for large μ_r !
- 6.5. Derive expressions for the magnetic induction of a circular current loop at a point P ! Hint: use the identities

$$\frac{\partial K(k)}{\partial k} = \frac{1}{k} \left[\frac{E(k)}{1-k^2} - K(k) \right] \quad ; \quad \frac{\partial E(k)}{\partial k} = \frac{1}{k} [E(k) - K(k)] .$$

- 6.6. Prove the expression

$$\oint_C (\mathbf{r} \cdot \mathbf{r}') d\mathbf{r}' = \oint_C \frac{1}{2} (\mathbf{r}' \times d\mathbf{r}') \times \mathbf{r} ,$$

using the identity for a triple cross product.

- 6.7. Derive the expression for the magnetic vector potential of a circular current loop at a distant point P using a dipole representation!
- 6.8. Derive the expression (6.7.37) for components of a current-carrying cylinder using the Biot-Savart law!
- 6.9. Determine the magnetic induction in both regions of Example 6.8.1.

7. A QUASI-STATIC ELECTROMAGNETIC FIELD

In Chapters 4 – 6, we analysed static problems in electromagnetism. These are the simplest problems and are caused by stationary charge or charge moving with constant velocity. In the case of charge in non-uniform motion, the nature of electromagnetic phenomena becomes very complicated. Electric and magnetic fields cannot be analysed independently because a time dependent magnetic field is necessarily coupled with an electric field. Thus, a dynamic electric field possesses an irrotational component caused by charge and a rotational component, which is usually called the *induced electric field*.

But one can come across a lot of problems in which we can neglect displacement currents and thus reduce the system of Maxwell's equations. Such problems are called *problems of quasi-static electromagnetic fields*. The neglect of displacement currents results in the neglect of the field retardation. Accordingly, the mechanism of energy transfer is diffusion rather than a wave, which is analogous to heat transfer.

In this chapter, we shall introduce diffusion equations for field quantities in quasi-static electromagnetic fields, as well as a phasor notation, which greatly simplifies the solution of sinusoidal fields.

The solution of quasi-static field problems will be arranged into two classes: problems with conductors of large thickness and unknown current distribution, and problems involving thin conductors and uniformly distributed currents.

7.1. EQUATIONS OF A QUASI-STATIC FIELD

The main characteristic of a quasi-static electromagnetic field is the ability to neglect displacement currents, which means that at any moment we can neglect the term $\partial \mathbf{D} / \partial t$ in equation (1.5.15c). The system of Maxwell's equations is then reduced to

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad , \quad \nabla \times \mathbf{H} = \mathbf{J} \quad , \quad (7.1.1a - d)$$

$$\nabla \cdot \mathbf{D} = \rho \quad , \quad \nabla \cdot \mathbf{B} = 0 \quad ,$$

together with the constitutive relationships for a conductive material ($\kappa \neq 0$) $\mathbf{B} = \mu \mathbf{H}$, $\mathbf{D} = \epsilon \mathbf{E}$ and $\mathbf{J} = \kappa \mathbf{E}$. If we apply the curl operator to equation (7.1.1b), we get

$$\nabla \times \nabla \times \mathbf{H} = \nabla(\nabla \cdot \mathbf{H}) - \Delta \mathbf{H} = -\Delta \mathbf{H} = \nabla \times \mathbf{J} = \nabla \times (\kappa \mathbf{E}) = \kappa \nabla \times \mathbf{E} \quad . \quad (7.1.2)$$

Taking into account equation (7.1.1.a) we obtain an equation which the vector $\mathbf{H}$ of a quasi-static electromagnetic field satisfies in linear conductive materials:

$$\Delta H = \mu\kappa \frac{\partial H}{\partial t} . \quad (7.1.3)$$

By a similar procedure for vectors $\mathbf{E}$ and $\mathbf{J}$ in quasi-static electromagnetic fields for problems without free charge ($\rho = 0$), we get

$$\Delta \mathbf{E} = \mu\kappa \frac{\partial \mathbf{E}}{\partial t} \quad ; \quad \Delta \mathbf{J} = \mu\kappa \frac{\partial \mathbf{J}}{\partial t} . \quad (7.1.4a,b)$$

These equations, as well as equation (7.1.3), are *in their form* equal to the equations which govern heat diffusion, and we call them diffusion equations.

The introduction of the magnetic vector potential $\mathbf{A}$ and electric scalar potential φ into the system of equations (7.1.1a – d) in the same manner as they were introduced into the full system of Maxwell's equation results in

$$\mathbf{E} = -\nabla\varphi - \frac{\partial \mathbf{A}}{\partial t} \quad ; \quad \mathbf{B} = \nabla \times \mathbf{A} . \quad (7.1.5a,b)$$

Taking into account the Lorentz gauge and applying a procedure equal to the one described in Chapter 2, we get Poisson's equation for vector magnetic potential and scalar electric potential of a quasi-static electromagnetic field in a lossless material ($\kappa = 0$)

$$\Delta \mathbf{A} = -\mu \mathbf{J} \quad , \quad \Delta \varphi = -\frac{\rho}{\epsilon} . \quad (7.1.6a,b)$$

Solutions of these equations for the potentials of a quasi-static electromagnetic field in a space filled with a linear, non-conductive material, are *in their form* equal to those for static fields

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu}{4\pi} \int_{V'} \frac{\mathbf{J}(\mathbf{r}',t)}{|\mathbf{r} - \mathbf{r}'|} dV' \quad , \quad \varphi(\mathbf{r},t) = \frac{1}{4\pi\epsilon} \int_{V'} \frac{\rho(\mathbf{r}',t)}{|\mathbf{r} - \mathbf{r}'|} dV' . \quad (7.1.7a,b)$$

These are not retarded potentials, meaning that all variations in a field appear instantaneously at all points in space. In conductive materials, the potentials of quasi-static electromagnetic fields satisfy the diffusion equations

$$\Delta \mathbf{A} - \mu\kappa \frac{\partial \mathbf{A}}{\partial t} = -\mu \mathbf{J} \quad , \quad \Delta \varphi - \mu\kappa \frac{\partial \varphi}{\partial t} = -\frac{\rho}{\epsilon} . \quad (7.1.8a,b)$$

The integral forms of equations (7.1.1a – d) are

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = - \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} dS = - \frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{n} dS \quad ; \quad \oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot \mathbf{n} dS , \quad (7.1.9a - d)$$

$$\oint_S \mathbf{D} \cdot \mathbf{n} dS = \int_V \rho dV \quad ; \quad \oint_S \mathbf{B} \cdot \mathbf{n} dS = 0 .$$

Conditions on the boundary between different materials were derived in Section 1.5 and they are valid in quasi-static fields as well.

7.1.1. Validity of a quasi-static approximation

The first question which naturally arises is: when may we analyse an electromagnetic field as a quasi-static one? The answer can be given only by a comparison with the solution of the full system of Maxwell's equations given in Chapter 2. By the introduction of the Lorentz gauge, we showed that such a system has a solution which is in fact the retarded vector magnetic potential

$$A(\mathbf{r}, t) = \frac{\mu}{4\pi} \int_{V'} \frac{J\left(\mathbf{r}', t - \frac{|\mathbf{r} - \mathbf{r}'|}{c}\right)}{|\mathbf{r} - \mathbf{r}'|} dV' . \quad (7.1.10)$$

If we apply Taylor's expansion around the point $\mathbf{r} = \mathbf{r}'$ to the numerator of the integrand in this equation, we get

$$J\left(\mathbf{r}', t - \frac{|\mathbf{r} - \mathbf{r}'|}{c}\right) = J(\mathbf{r}', t) + \sum_{i=1}^{\infty} \frac{|\mathbf{r} - \mathbf{r}'|^i}{i!} \left[\frac{\partial^i}{\partial |\mathbf{r} - \mathbf{r}'|^i} J\left(\mathbf{r}', t - \frac{|\mathbf{r} - \mathbf{r}'|}{c}\right) \right]_{\mathbf{r}=\mathbf{r}'} . \quad (7.1.11)$$

Thus, we may write the solution for the retarded magnetic vector potential as

$$A(\mathbf{r}, t) = \frac{\mu}{4\pi} \int_{V'} \frac{J(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} dV' + \sum_{i=1}^{\infty} \frac{\mu}{4\pi i!} \int_{V'} \frac{(-\tau)^i}{|\mathbf{r} - \mathbf{r}'|} \frac{\partial^i}{\partial t^i} J(\mathbf{r}', t) dV' . \quad (7.1.12)$$

The constant

$$\tau = \frac{|\mathbf{r} - \mathbf{r}'|}{c} , \quad (7.1.13)$$

is the time needed to register any variation of a source positioned at $\mathbf{r}'$ at a point $\mathbf{r}$. We can write the same expansion for a retarded scalar electric potential. The first term in the expansion (7.1.12) is equal to the solution for a quasi-static electromagnetic field. Accordingly, a quasi-static approximation is valid when the contributions of the other terms in expansion (7.1.12) are much smaller than the contribution of the first term. The relationship between these contributions is defined by τ and differentials of a source with respect to time. It is not simple to determine when the sum of other terms will be much smaller than the first term. In any case, the condition

$$J(\mathbf{r}, t) \gg \tau_{\max} \frac{\partial J(\mathbf{r}, t)}{\partial t} , \quad (7.1.14)$$

has to be satisfied and higher-order differentials of the source multiplied by $(\tau_{\max})^i$ have to have a much smaller value than the source. Time $\tau_{\max}$ is the time needed to pass through a distance between two points that are most distant from each other in the problem.

7.2. FIELD EQUATIONS IN A PHASOR DOMAIN AND QUASI-STATIC SINUSOIDAL FIELDS

7.2.1. Sinusoidal time dependent fields and phasor notation

We frequently need to analyse sinusoidal time dependent dynamic electromagnetic fields. In these cases we prescribe that all field quantities vary sinusoidally with the same angular frequency ω_0 . For example, the electric field strength in such fields can be written as $\mathbf{E}(\mathbf{r}, t) = E_x \cos[\omega_0 t + \psi_x(\mathbf{r})] \mathbf{a}_x + E_y \cos[\omega_0 t + \psi_y(\mathbf{r})] \mathbf{a}_y + E_z \cos[\omega_0 t + \psi_z(\mathbf{r})] \mathbf{a}_z$. Other field quantities can be expressed in the same way. Sinusoidal functions can be expressed as the real part of a complex function

$$A \cos(\omega_0 t + \psi) = \operatorname{Re}(A e^{j\psi} e^{j\omega t}) = \operatorname{Re}(\underline{A} e^{j\omega t}) , \quad (7.2.1)$$

and this leads us to the representation of such a function by the complex number $\underline{A}$, which is called the *phasor*. A is amplitude and ψ is phase angle of $\underline{A}$. By application of this representation to scalar and vector quantities of sinusoidally dependent fields, we get their phasor representation. Thus, a phasor representation of a sinusoidal electric field is

$$\begin{aligned} \mathbf{E}(\mathbf{r}, t) &= E_x(\mathbf{r}) \cos[\omega_0 t + \psi_x(\mathbf{r})] \mathbf{a}_x + E_y(\mathbf{r}) \cos(\omega_0 t + \psi_y(\mathbf{r})) \mathbf{a}_y + \\ &+ E_z(\mathbf{r}) \cos[\omega_0 t + \psi_z(\mathbf{r})] \mathbf{a}_z = \\ &= \operatorname{Re} \left\{ \left[E_x(\mathbf{r}) e^{j\psi_x(\mathbf{r})} \mathbf{a}_x + E_y(\mathbf{r}) e^{j\psi_y(\mathbf{r})} \mathbf{a}_y + E_z(\mathbf{r}) e^{j\psi_z(\mathbf{r})} \mathbf{a}_z \right] e^{j\omega t} \right\} = \operatorname{Re}(\underline{\mathbf{E}}(\mathbf{r}) e^{j\omega t}) \end{aligned} \quad (7.2.2)$$

By the introduction of phasor notation, we have *transformed electromagnetic field quantities from a time domain to a phasor domain*. This eradicates time dependence from the electromagnetic field quantities in a phasor domain and they become functions of space variables only. The only difficulty is the visual presentation of such phasors as vectors in three-dimensional space, as every component of the phasor is a complex number. Therefore, a phasor at a point is defined by six numbers. Most frequently, we visually represent amplitudes and phase angles of such fields separately.

In the system of Maxwell's equations, one can find partial differentials with respect to time. In the phasor domain, they are represented by multiples of the complex number $j\omega$

$$\frac{\partial}{\partial t} \mathbf{E}(\mathbf{r}, t) = \operatorname{Re}(j\omega \underline{\mathbf{E}}(\mathbf{r}) e^{j\omega t}) . \quad (7.2.3)$$

Therefore, the system of Maxwell's equations in a phasor domain becomes

$$\begin{aligned} \nabla \times \underline{\mathbf{E}} &= -j\omega \underline{\mathbf{B}} \quad , \quad \nabla \times \underline{\mathbf{H}} = \underline{\mathbf{J}} + j\omega \underline{\mathbf{D}} \quad , \\ \nabla \cdot \underline{\mathbf{D}} &= \rho \quad , \quad \nabla \cdot \underline{\mathbf{B}} = 0 \quad . \end{aligned} \quad (7.2.4a - d)$$

The continuity equation in a phasor domain is $\nabla \cdot \underline{\mathbf{J}} = -j\omega \rho$. The transformation from a phasor to a time domain is done by usage of equation (7.2.1) using the multiplication of

the absolute value of a vector in a phasor domain by the term $\cos(\omega_0 t + \psi)$.

7.2.2. Electromagnetic potentials in a phasor domain and a quasi-static approximation

The Lorentz gauge in a phasor domain is $\nabla \cdot \underline{\mathbf{A}} = -j\omega\mu\epsilon\varphi$ and the phasor form of equation (7.1.5a) is $\underline{\mathbf{E}} = -\nabla\varphi - j\omega\underline{\mathbf{A}}$. If a problem involves a lossless material ($\kappa = 0$), we obtain equations for electromagnetic potential in a phasor domain as being

$$\Delta \underline{\mathbf{A}} + \omega^2 \mu \epsilon \underline{\mathbf{A}} = -\mu \underline{\mathbf{J}} \quad , \quad \Delta \varphi + \omega^2 \mu \epsilon \varphi = -\frac{1}{\epsilon} \rho \quad . \quad (7.2.5a,b)$$

The solutions of these equations are

$$\underline{\mathbf{A}}(\mathbf{r}) = \frac{\mu}{4\pi} \int_{V'} \frac{\underline{\mathbf{J}}(\mathbf{r}') e^{j\omega \frac{|\mathbf{r}-\mathbf{r}'|}{c}}}{|\mathbf{r}-\mathbf{r}'|} dV' \quad , \quad \varphi(\mathbf{r}) = \frac{1}{4\pi\epsilon} \int_{V'} \frac{\rho(\mathbf{r}') e^{j\omega \frac{|\mathbf{r}-\mathbf{r}'|}{c}}}{|\mathbf{r}-\mathbf{r}'|} dV' \quad . \quad (7.2.6a,b)$$

A comparison of these solutions with solutions (7.1.7) leads us to a practically applicable criteria for the validity of a quasi-static approximation. These two classes of solutions are approximately equal when $\omega |\mathbf{r} - \mathbf{r}'|_{\max} / c \ll 1$. The exponential part of the integrands in (7.2.6a,b) is then approximately equal to one and we can solve a field as a quasi-static one. Accordingly, if the greatest distance between points in an electromagnetic field in a lossless material satisfies the condition

$$|\mathbf{r} - \mathbf{r}'| \ll \frac{c}{2\pi f} \quad \text{or} \quad |\mathbf{r} - \mathbf{r}'| \ll \frac{\lambda}{2\pi} \quad , \quad (7.2.7)$$

we can then solve this problem as a quasi-static one.

If we are solving a problem involving conductive materials, meaning that the current density is unknown and that it is prescribed as $\underline{\mathbf{J}} = \underline{\mathbf{J}}_s + \kappa \underline{\mathbf{E}}$, equation (7.2.4b) becomes $\nabla \times \underline{\mathbf{H}} = \underline{\mathbf{J}}_s + \kappa \underline{\mathbf{E}} + j\omega\epsilon \underline{\mathbf{E}}$. The Lorentz gauge for this problem is $\nabla \cdot \underline{\mathbf{A}} = -j\omega\mu\epsilon\varphi + \mu\kappa\varphi$. Utilization of these equations and the system (7.2.4a - d) results in equations for electromagnetic potentials for this problem

$$\Delta \underline{\mathbf{A}} + (\omega^2 \mu \epsilon - j\omega\mu\kappa) \underline{\mathbf{A}} = -\mu \underline{\mathbf{J}}_s \quad , \quad \Delta \varphi + (\omega^2 \mu \epsilon - j\omega\mu\kappa) \varphi = -\frac{\rho_s}{\epsilon} \quad . \quad (7.2.8a,b)$$

The solutions of these equations are

$$\underline{\mathbf{A}}(\mathbf{r}) = \frac{\mu}{4\pi} \int_{V'} \frac{\underline{\mathbf{J}}(\mathbf{r}') e^{|\mathbf{r}-\mathbf{r}'| \sqrt{-\omega^2 \mu \epsilon + j\omega\mu\kappa}}}{|\mathbf{r}-\mathbf{r}'|} dV' \quad , \quad (7.2.9a,b)$$

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\epsilon} \int_{V'} \frac{\rho(\mathbf{r}') e^{|\mathbf{r}-\mathbf{r}'| \sqrt{-\omega^2 \mu \epsilon + j\omega\mu\kappa}}}{|\mathbf{r}-\mathbf{r}'|} dV' \quad .$$

In Unit 7.1, we showed that the potentials in the quasi-static approximation of such problems satisfy the diffusion equations. The phasor forms of these equations are

$$\Delta \underline{A} - j\omega\mu\kappa \underline{A} = -\mu \underline{J}_s, \quad \Delta \varphi - j\omega\mu\kappa \varphi = -\frac{\rho_s}{\epsilon}. \quad (7.2.10a,b)$$

The solutions of these equations in an unbounded space are

$$\begin{aligned} \underline{A}(\mathbf{r}) &= \frac{\mu}{4\pi} \int_{V'} \frac{\underline{J}(\mathbf{r}') e^{-|\mathbf{r}-\mathbf{r}'| \sqrt{j\omega\mu\kappa}}}{|\mathbf{r}-\mathbf{r}'|} dV', \\ \varphi(\mathbf{r}) &= \frac{1}{4\pi\epsilon} \int_{V'} \frac{\rho(\mathbf{r}') e^{-|\mathbf{r}-\mathbf{r}'| \sqrt{j\omega\mu\kappa}}}{|\mathbf{r}-\mathbf{r}'|} dV'. \end{aligned} \quad (7.2.11a,b)$$

A comparison of solutions (7.2.11a,b) with solutions (7.2.9a,b) leads to the criterion for the validity of a quasi-static approximation for such problems

$$\omega\mu\kappa \gg \omega^2\mu\epsilon \quad \Rightarrow \quad \frac{\kappa}{\omega\epsilon} \gg 1. \quad (7.2.12)$$

If we want to apply solutions of Poisson's equation given by (7.1.6) to such problems, the contribution of the exponential part of the integrands in (7.2.9a,b) should be approximately equal to unity. The indices of these integrands can be written as

$$\frac{\omega |\mathbf{r}-\mathbf{r}'|}{c} \sqrt[4]{1 + \frac{\kappa^2}{\omega^2\epsilon^2}} e^{-j\frac{\gamma}{2}}; \quad \gamma = \arctan \frac{\kappa}{\omega\epsilon}. \quad (7.2.13)$$

The exponential parts of the integrands in (7.2.9a,b) are

$$-\frac{\omega |\mathbf{r}-\mathbf{r}'|}{c} \sqrt[4]{1 + \frac{\kappa^2}{\omega^2\epsilon^2}} \sin \frac{\gamma}{2} + j \frac{\omega |\mathbf{r}-\mathbf{r}'|}{c} \sqrt[4]{1 + \frac{\kappa^2}{\omega^2\epsilon^2}} \cos \frac{\gamma}{2}. \quad (7.2.14)$$

From equation (7.2.13), it follows that $0 \leq \gamma \leq \pi/2$ and $\sin(\gamma/2) \leq \cos(\gamma/2)$. Accordingly, the criterion for application of such solutions is that

$$|\mathbf{r}-\mathbf{r}'|_{\max} \ll \frac{1}{\frac{\omega}{c} \sqrt[4]{1 + \frac{\kappa^2}{\omega^2\epsilon^2}} \cos \frac{\gamma}{2}}. \quad (7.2.15)$$

For good conductors one can simplify this criterion to

$$|\mathbf{r}-\mathbf{r}'|_{\max} \ll \sqrt{\frac{2}{\omega\mu\kappa}}, \quad (7.2.16)$$

because for them $\kappa \gg \omega\epsilon$ is valid and from (7.2.13) it follows that $\gamma \approx \pi/2$.

7.3. ENERGY IN A QUASI-STATIC FIELD

The law of conservation of energy in electromagnetic fields is described by Poynting's theorem. For linear materials its integral form is

$$-\oint_S (\mathbf{E} \times \mathbf{H}) \cdot \mathbf{n} dS - \int_V \mathbf{E} \cdot \mathbf{J}_i dV = \int_V \kappa |\mathbf{E}|^2 dV + \frac{d}{dt} \int_V \left(\frac{\epsilon}{2} |\mathbf{E}|^2 + \frac{\mu}{2} |\mathbf{H}|^2 \right) dV. \quad (7.3.1)$$

Problems involving quasi-static electromagnetic fields which most frequently include good conductors and dielectrics. In good conductors, we can neglect displacement currents. This reduces Poynting's theorem to

$$-\oint_S (\mathbf{E} \times \mathbf{H}) \cdot \mathbf{n} dS - \int_V \mathbf{E} \cdot \mathbf{J}_i dV = \frac{1}{\kappa} \int_V |\mathbf{J}|^2 dV + \frac{d}{dt} \int_V \frac{\mu}{2} |\mathbf{H}|^2 dV. \quad (7.3.2)$$

There are no losses in dielectrics ($\kappa = 0$), and Poynting's theorem takes the form

$$-\oint_S (\mathbf{E} \times \mathbf{H}) \cdot \mathbf{n} dS = \frac{d}{dt} \int_V \left(\frac{\epsilon}{2} |\mathbf{E}|^2 + \frac{\mu}{2} |\mathbf{H}|^2 \right) dV. \quad (7.3.3)$$

From equations (7.3.2,3), it follows that a quasi-static electromagnetic field in good conductors and dielectrics can be substituted for by a network of resistors, inductances and capacitances in such a way that the energy in the mesh is equal to the energy in the field.

We can see from equation (7.3.1) that losses and stored energy are functions of the rate of change of time. In keeping with that, the substitute elements are dependent upon the rate of change of time (frequency dependent). In some cases, we may neglect this dependence and so we can differentiate between two basic classes of quasi-static electromagnetic field problems:

- problems involving conductors of large thickness carrying an unknown distribution of current which is dependent upon rate of change of time,
- problems involving thin conductors carrying a prescribed uniformly distributed current which is independent of rate of change of time.

In the next Units 7.4. and 7.5, we shall analyse these two classes of problems and we shall show when we may prescribe a uniform distribution of current in the conductors involved.

7.3.1. Energy and power in a sinusoidal field

In sinusoidal electromagnetic fields, we are interested in time-averaged values, while the instantaneous values of energy flow are not of interest. If we prescribe the field vectors $\mathbf{E}$ and $\mathbf{H}$ with components ($E_x, E_y, E_z, H_x, H_y, H_z$ are maximum values) as

$$\begin{aligned}\mathbf{E} &= E_x \cos(\omega t + \psi_{E_x}) \mathbf{a}_x + E_y \cos(\omega t + \psi_{E_y}) \mathbf{a}_y + E_z \cos(\omega t + \psi_{E_z}) \mathbf{a}_z, \\ \mathbf{H} &= H_x \cos(\omega t + \psi_{H_x}) \mathbf{a}_x + H_y \cos(\omega t + \psi_{H_y}) \mathbf{a}_y + H_z \cos(\omega t + \psi_{H_z}) \mathbf{a}_z,\end{aligned}\quad (7.3.4)$$

then the time-averaged value of Poynting's vector for a sinusoidal field with a period T is

$$N_{av} = \frac{1}{T} \int_0^T (\mathbf{E} \times \mathbf{H}) dt. \quad (7.3.5)$$

Accordingly, the expression for the time-averaged value of Poynting's vector is

$$\begin{aligned}N_{av} &= \frac{1}{2} [E_y H_z \cos(\psi_{E_y} - \psi_{H_z}) - E_z H_y \cos(\psi_{E_z} - \psi_{H_y})] \mathbf{a}_x + \\ &+ \frac{1}{2} [E_z H_x \cos(\psi_{E_z} - \psi_{H_x}) - E_x H_z \cos(\psi_{E_x} - \psi_{H_z})] \mathbf{a}_y + \\ &+ \frac{1}{2} [E_x H_y \cos(\psi_{E_x} - \psi_{H_y}) - E_y H_x \cos(\psi_{E_y} - \psi_{H_x})] \mathbf{a}_z,\end{aligned}\quad (7.3.6)$$

which is equal to the real part of the cross product of the phasors

$$N_{av} = \operatorname{Re} \left\{ \frac{1}{2} (\underline{\mathbf{E}} \times \underline{\mathbf{H}}^*) \right\}, \quad (7.3.7)$$

where $\underline{\mathbf{H}}^*$ is the complex conjugate phasor of $\underline{\mathbf{H}}$

$$\underline{\mathbf{H}}^* = H_x e^{-j\psi_{H_x}} \mathbf{a}_x + H_y e^{-j\psi_{H_y}} \mathbf{a}_y + H_z e^{-j\psi_{H_z}} \mathbf{a}_z. \quad (7.3.8)$$

The time-averaged losses in sinusoidal fields expressed in terms of the phasor $\underline{\mathbf{J}}$ (J_x, J_y, J_z are the maximum values) are

$$\begin{aligned}P_{l,av} &= \frac{1}{\kappa} \frac{1}{T} \int_0^T dt \int_V |J_x \cos(\omega t + \psi_{J_x}) \mathbf{a}_x + J_y \cos(\omega t + \psi_{J_y}) \mathbf{a}_y + \\ &+ J_z \cos(\omega t + \psi_{J_z}) \mathbf{a}_z|^2 dV = \frac{1}{2\kappa} \int_V |\underline{\mathbf{J}}|^2 dV.\end{aligned}\quad (7.3.9)$$

The time-averaged energy stored in the magnetic field expressed in terms of the phasor $\underline{\mathbf{H}}$ and the time-averaged energy stored in the electric field expressed in terms of the phasor $\underline{\mathbf{E}}$ are

$$W_{m,av} = \frac{\mu}{4} \int_V |\underline{\mathbf{H}}|^2 dV, \quad W_{e,av} = \frac{\epsilon}{4} \int_V |\underline{\mathbf{E}}|^2 dV. \quad (7.3.10, 11)$$

Substitution of complex conjugate phasors $\underline{\mathbf{H}}^*$, $\underline{\mathbf{E}}^*$ and $\underline{\mathbf{J}}^*$ into equation (7.2.4b) together with equation (7.2.4a) results in $\nabla \times \underline{\mathbf{H}}^* = \underline{\mathbf{J}}^* - j\omega\epsilon \underline{\mathbf{E}}^*$ and $\nabla \times \underline{\mathbf{E}} = -j\omega\mu \underline{\mathbf{H}}$. If we multiply the

first of these equations by the phasor $\underline{E}$ and the second by the complex conjugate phasor $\underline{H}^*$ and subtract the first equation from the second, we obtain $\underline{H}^* \cdot (\nabla \times \underline{E}) - \underline{E} \cdot (\nabla \times \underline{H}^*) = -\underline{E} \cdot \underline{J}^* - j\omega(\mu \underline{H} \cdot \underline{H}^* - \epsilon \underline{E} \cdot \underline{E}^*)$. After application of the vector identity $-\nabla \cdot (\underline{E} \times \underline{H}^*) = \underline{E} \cdot (\nabla \times \underline{H}^*) - \underline{H}^* \cdot (\nabla \times \underline{E})$, followed by integration over the volume V bounded by the surface S , and then utilization of Gauss' divergence theorem, we obtain

$$\oint_S (\underline{E} \times \underline{H}^*) \cdot \underline{n} dS = - \int_V (\underline{E} \cdot \underline{J}^*) dV + j\omega \int_V (\epsilon \underline{E} \cdot \underline{E}^* - \mu \underline{H} \cdot \underline{H}^*) dV . \quad (7.3.12)$$

Multiplication of this equation by 1/2 followed by a comparison with expressions (7.3.9 – 11) yields the *complex expression for Poynting's theorem*, which states the principle of the conservation of energy in a sinusoidal electromagnetic field

$$P_N = \oint_S \underline{N} \cdot \underline{n} dS = -P_{l,av} + j2\omega(W_{e,av} - W_{m,av}) . \quad (7.3.13)$$

We call P_N the *apparent time-averaged power* which flows in unit time through a closed surface S . Equation (7.3.13) states that in sinusoidal electromagnetic fields, the negative real part of the apparent time-averaged power is equal to the time-averaged losses in volume V , while the imaginary part of the apparent time-averaged power is equal to the product of 2ω and the difference between the time-averaged values of the total electric and magnetic energy stored in the volume V . Therefore, in a sinusoidal electromagnetic field, we can determine the components of an equivalent circuit from the real and imaginary part of the apparent time-averaged power.

7.4. THE SOLUTION OF A QUASI-STATIC FIELD IN CONDUCTORS

In Unit 7.1. we showed that quantities for quasi-static electromagnetic fields in conductors satisfy the diffusion equation. For the solution of that equation, we shall use a separation of variables, as was the case in static fields. We shall assume that the solution for vector $\underline{H}$ can be written as $\underline{H}(\underline{r}, t) = \underline{P}(\underline{r})T(t)$, where $\underline{P}(\underline{r})$ is a vector function independent of time and $T(t)$ is a scalar function dependent only on time. If we include this assumption into equation (7.1.3) it follows that

$$T(t) \Delta \underline{P}(\underline{r}) = \mu \kappa \underline{P}(\underline{r}) \frac{dT(t)}{dt} , \quad (7.4.1)$$

and since $\underline{P}(\underline{r})$ is independent of time, it follows that $T(t)$ must satisfy

$$\mu \kappa \frac{d}{dt} \left(\frac{1}{T(t)} \frac{dT(t)}{dt} \right) = 0 \quad \Rightarrow \quad \frac{1}{T(t)} \frac{dT(t)}{dt} = -\frac{k^2}{\mu \kappa} . \quad (7.4.2)$$

Here, k^2 is an arbitrary constant. $\underline{P}(\underline{r})$ has to satisfy the equation $\Delta \underline{P}(\underline{r}) + k^2 \underline{P}(\underline{r}) = 0$. This equation and equations (7.4.2) are mathematically independent but they are physically connected by constant k . Equation (7.4.2) is a simple second-order differential equation and we can solve it easily. On the other hand, the equation for $\underline{P}(\underline{r})$ can be solved

analytically only in a few geometrically very simple problems by a separation of spatial variables. We obtain a sinusoidal field by a separation in which $T(t) = e^{j\omega t}$, and the constant k^2 is

$$k^2 = -j\omega\mu\kappa \quad \Rightarrow \quad k = (1-j)\sqrt{\frac{\omega\mu\kappa}{2}} = \frac{1-j}{d} . \quad (7.4.3)$$

The quantity

$$d = \sqrt{\frac{2}{\omega\mu\kappa}} , \quad (7.4.4)$$

possesses the dimension of distance and we call it the *depth of penetration*. Through the following examples, we shall show that the ratio between the dimensions of a particular conductor and its depth of penetration, which is frequency dependent, is the most influential factor on the distribution of current in a conductor.

Example 7.4.1. A magnetic tin sheet in a time-dependent magnetic field

The problem is illustrated in Figure 7.4.1. To simplify the problem, we prescribe that the thickness of the sheet is much smaller than its width and height and so we shall assume that the field changes only in the direction of the x -axis

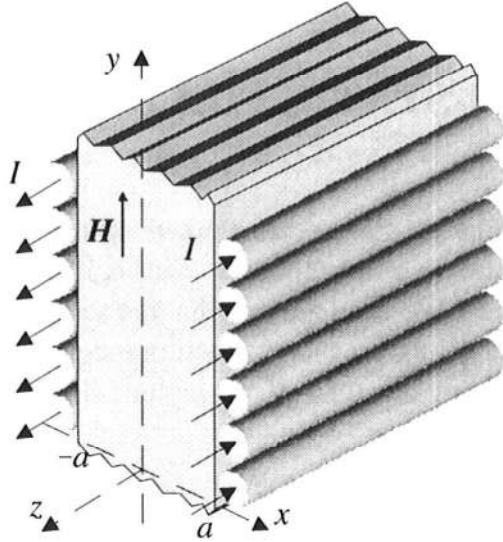


Figure 7.4.1. A magnetic thin sheet in a time-dependent magnetic field

The sheet is placed in a magnetic field which possesses only a y component $\mathbf{H} = H(x,t)\mathbf{a}_y = P(x)T(t)\mathbf{a}_y$.

1. SWITCHING OFF THE CURRENT

In the first step, we shall solve the quasi-static electromagnetic field in the sheet if at a moment in time $t = 0$, the current in the winding that causes $\mathbf{H}_0$ is switched off. The problem is reduced to the solution of the diffusion equation with the following initial conditions

$$\begin{aligned}
H(\pm a, t \geq 0^+) &= 0 \Rightarrow P(\pm a) = 0, \\
H(x, t = 0^+) &= H_0 \Rightarrow T(0^+) = 1, \\
\lim_{t \rightarrow \infty} T &= 0.
\end{aligned} \tag{7.4.5a - c}$$

We determine the functions T and P from the equations

$$\mu\kappa \frac{dT}{dt} + k^2 T = 0, \quad \frac{d^2 P(x)}{dx^2} = -k^2 P(x). \tag{7.4.6a,b}$$

Solutions of these equations are

$$T(t) = e^{-\frac{k^2}{\mu\kappa}t}; \quad P(x) = A \cos(kx) + B \sin(kx). \tag{7.4.7a,b}$$

Because of the condition $H(x) = H(-x)$ the coefficient B must be equal to zero. From the condition that for every moment where $t \geq 0^+$, the field on the boundaries of the sheet must be equal to zero, it follows that $A \cos(kx)$ must be equal to zero for $x = \pm a$ and so $k = n\pi/(2a)$, where n is an odd integer. As the field must be homogeneous over the whole sheet at a moment in time $t = 0$ and as (7.4.7b) does not satisfy this, we assume a solution in the form of a linear combination

$$P(x) = \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{2a}. \tag{7.4.8}$$

This is a Fourier series which has to converge to a constant H_0 within the interval $-a \leq x \leq a$. Coefficients a_n are determined as

$$a_n = \frac{1}{a} \int_{-a}^a H_0 \cos \frac{n\pi x}{2a} dx = \frac{4H_0}{\pi n} \sin \frac{n}{2}\pi. \tag{7.4.9}$$

These coefficients are equal to zero when n is an even number and when n is an odd number they change their sign. From equation $\mathbf{H} = P(x)T(t)\mathbf{a}_y$, it follows that every member of the Fourier series has to be multiplied by a corresponding time function T_n given by equation (7.4.7a)

$$T_n = e^{-\frac{n^2\pi^2}{4a^2\mu\kappa}t}. \tag{7.4.10}$$

The final solution for vector $\mathbf{H}$ is

$$\mathbf{H}(x, t) = \mathbf{a}_y \sum_{n=1}^{\infty} \frac{4H_0}{n\pi} \sin\left(\frac{n\pi}{2}\right) \cos\left(\frac{n\pi x}{2a}\right) e^{-\frac{n^2\pi^2}{4a^2\mu\kappa}t}. \tag{7.4.11}$$

On the basis of this solution, we can see that every space harmonic of the series (7.4.11) is attenuated at a different rate, with the higher harmonics attenuated faster. Figure 7.4.2.

shows the time dependence of the magnetic field in a sheet when the current is switched off. The time constant of the fundamental harmonic ($n = 1$) is defined as $\tau = 4a^2\mu\kappa/\pi^2$. It is the biggest time constant of this series as well, and we call it *the diffusion time constant* of the system.

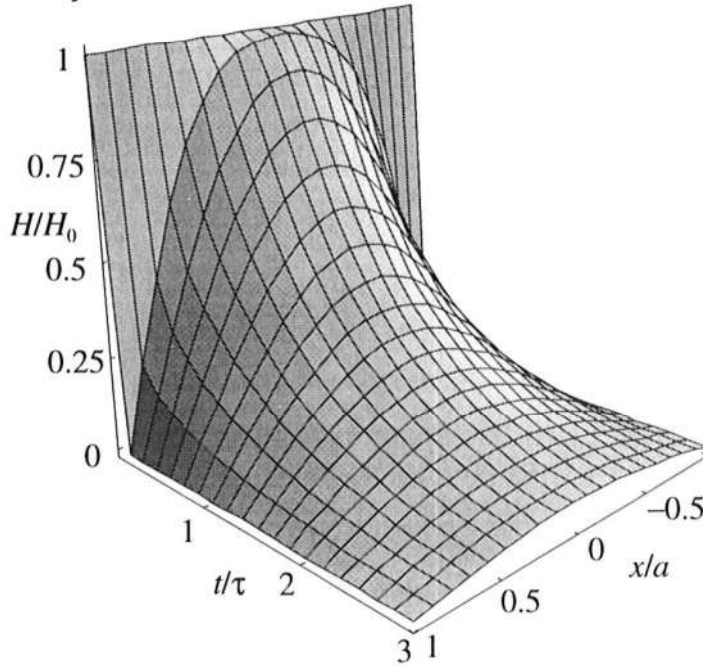


Figure 7.4.2. Time dependence of the magnetic field in a sheet when the current is switched off, $\tau = 4a^2\mu\kappa/\pi^2$.

We determine currents flowing in the sheet by equation (7.1.1b)

$$\mathbf{J} = \mathbf{a}_z \frac{\partial H_y}{\partial x} = \mathbf{a}_z \sum_{n=1}^{\infty} \frac{-2H_0}{a} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi x}{2a}\right) e^{-\frac{n^2\pi^2}{4a^2\mu\kappa}t}. \quad (7.4.12)$$

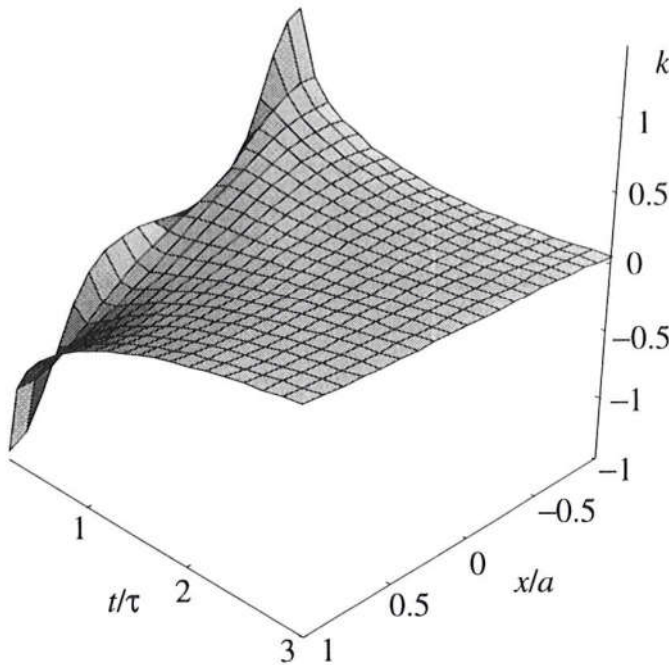


Figure 7.4.3. The time dependence of a normalized current density $k = Ja/2H_0$ in a sheet during switching off $\tau = 4a^2\mu\kappa/\pi^2$, $0.1\tau \leq t \leq 3\tau$.

In Figure 7.4.3, we have shown the time dependence of a normalized volume current density in a sheet when currents are switched off within the time span $0.1\tau \leq t \leq 3\tau$. As

time passes, currents enter into the sheet and decrease.

From equation (7.4.12), we see that the current distribution in the sheet is an odd function of x coordinates. This means that the currents are flowing in one direction on one side of the sheet and that they are flowing in the opposite direction on the other side, while the total current throughout the sheet is equal to zero. These currents cause losses in the material and we call them *eddy currents*.

At $t = 0$ the series (7.4.12) is divergent for $x = \pm a$. This is in accordance with the assumption that the field at this moment outside the sheet is equal to zero and inside the sheet equal to H_0 . To make this possible, the surface current should be established at this moment. The volume current density J is then a singularity.

2. SINUSOIDAL FIELD

We suppose that the thin sheet is in a sinusoidal external magnetic field $H_0 \cos(\omega t)$. We are trying to find the solution in a phasor domain. The equation for the phasor $\underline{H}$ is

$$\frac{d^2 \underline{H}}{dx^2} - j\omega\mu\kappa \underline{H} = 0 \quad (7.4.13)$$

The general form of the solution of this equation is

$$\underline{H} = C_1 e^{x\sqrt{j\omega\mu\kappa}} + C_2 e^{-x\sqrt{j\omega\mu\kappa}} \quad (7.4.14)$$

Due to symmetry, we get $\underline{H}(x) = \underline{H}(-x)$ and this leads to $C_1 = C_2 = C$. We determine the constant C from the condition at the surface of the sheet $\underline{H}(\pm a) = \underline{H}_0$. Finally the phasor of magnetic field strength in the sheet is

$$\underline{H} = H_0 \frac{\cosh(x\sqrt{j\omega\mu\kappa})}{\cosh(a\sqrt{j\omega\mu\kappa})} = H_0 \frac{\cosh\left[(1+j)\frac{x}{d}\right]}{\cosh\left[(1+j)\frac{a}{d}\right]} = H_{\max} e^{j\psi_H} \quad (7.4.15)$$

where the modulus of the phasor of magnetic field strength is

$$H_{\max}(x) = H_0 \sqrt{\frac{\cosh\left(\frac{2x}{d}\right) + \cos\left(\frac{2x}{d}\right)}{\cosh\left(\frac{2a}{d}\right) + \cos\left(\frac{2a}{d}\right)}} \quad (7.4.16)$$

and the phase angle is

$$\psi_H = \arctan\left[\tanh\left(\frac{x}{d}\right) \tan\left(\frac{x}{d}\right)\right] - \arctan\left[\tanh\left(\frac{a}{d}\right) \tan\left(\frac{a}{d}\right)\right] \quad (7.4.17)$$

We determine the phasor of eddy currents in the sheet by

$$\mathbf{J} = \nabla \times \mathbf{H} = -\mathbf{a}_z \frac{\partial H}{\partial x} = H_0 \frac{(1+j)}{d} \frac{\sinh\left((1+j)\frac{x}{d}\right)}{\cosh\left((1+j)\frac{a}{d}\right)} = J_{\max} e^{j\omega t} \mathbf{a}_z. \quad (7.4.18)$$

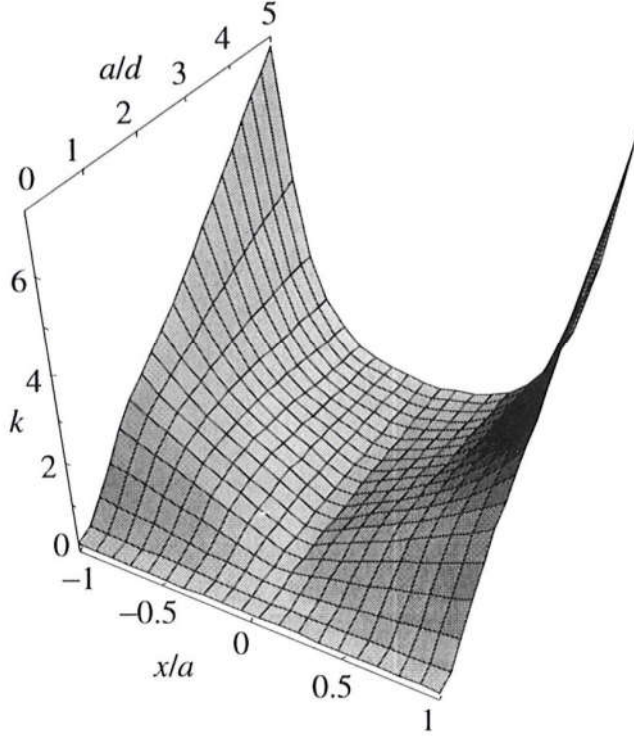


Figure 7.4.4. The distribution of normalized eddy current density $k = J_{\max} a / H_0$ in the sheet dependent on a/d .

Now we can determine the losses due to the eddy currents in the sheet per unit area

$$P_{g,sr} = \frac{1}{2\kappa} \int_{-a}^a J_{\max}^2 dx = \frac{H_0^2}{\kappa d} \frac{\sinh \frac{2a}{d} - \sin \frac{2a}{d}}{\cosh \frac{2a}{d} + \cos \frac{2a}{d}}. \quad (7.4.19)$$

In Figure 7.4.5. we have shown the normalized eddy current losses, which are dependent on the ratio of half of the sheet thickness a and the depth of penetration d .

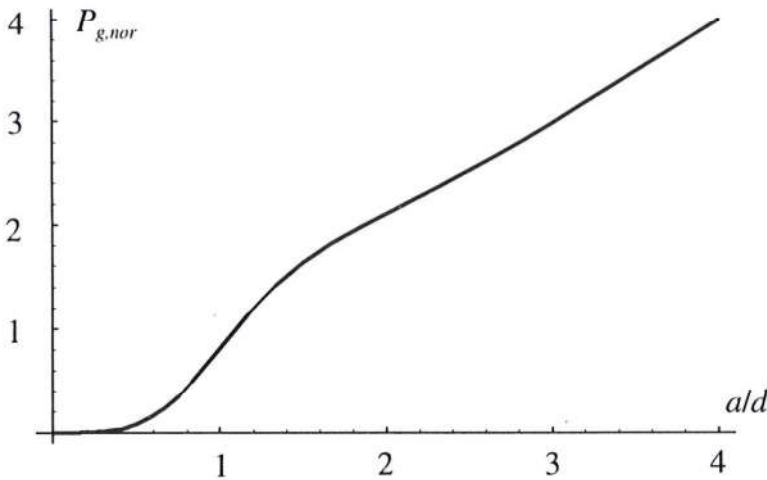


Figure 7.4.5. Dependence of normalized eddy current losses $P_{g,nor} = P_{g,sr} a \kappa / H_0^2$ for the ratio a/d .

From this diagram and equation (7.4.19), we can conclude that losses in the sheet rise sharply with increase of the ratio a/d . When $a \ll d$, the eddy currents are limited by the lack of space, or in other words by a greater resistance, and then we can approximate expression (7.4.19) as

$$P_{g, sr} \approx \frac{1}{3} \omega^2 \kappa \mu^2 H_0^2 a^3 . \quad (7.4.20)$$

Because of these facts, in transformers and electric machines, we use cores consisting of packages of thin magnetic laminae.

Example 7.4.2. Sinusoidal current flowing through a metallic sheet

We shall analyse the current density distribution throughout a sheet that carries a sinusoidal current of phasor $\underline{I}$ flowing in the direction of the z -axis.

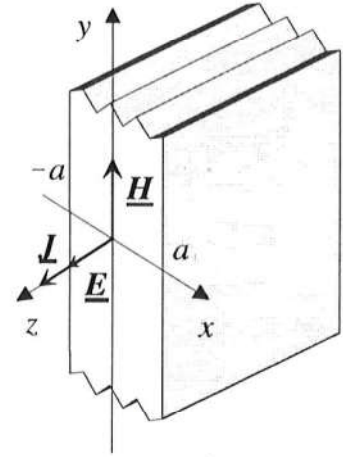


Figure 7.4.6. Sinusoidal current flowing through a magnetic sheet

We shall solve this problem in a phasor domain. The phasor $\underline{H}$ possesses only a y component and phasor $\underline{E}$ has only a z component, thus $\underline{J} = \underline{J}(x)\mathbf{a}_z$, $\underline{E} = \underline{E}(x)\mathbf{a}_z$ and $\underline{H} = \underline{H}(x)\mathbf{a}_y$. The phasors $\underline{E}$ and $\underline{H}$ have to satisfy the diffusion equation in a phasor domain. The general forms of the solutions are:

$$\underline{H} = H_1 e^{x\sqrt{j\omega\mu\kappa}} + H_2 e^{-x\sqrt{j\omega\mu\kappa}} , \quad \underline{E} = E_1 e^{x\sqrt{j\omega\mu\kappa}} + E_2 e^{-x\sqrt{j\omega\mu\kappa}} . \quad (7.4.21a,b)$$

Due to symmetry, we get that $-\underline{H}(-x) = \underline{H}(x)$ and $\underline{E}(-x) = \underline{E}(x)$ and so the constants from equations (7.4.21a,b) have to satisfy the following conditions:

$$H_1 = -H_2 = \frac{H_0}{2} \quad ; \quad E_1 = E_2 = \frac{E_0}{2} . \quad (7.4.22)$$

Now we can write the solutions in a simplified form

$$\begin{aligned} \underline{H} &= H_0 \sinh(x\sqrt{j\omega\mu\kappa}) , \\ \underline{E} &= E_0 \cosh(x\sqrt{j\omega\mu\kappa}) , \\ \underline{J} &= \kappa \underline{E} = \kappa E_0 \cosh(x\sqrt{j\omega\mu\kappa}) . \end{aligned} \quad (7.4.23a - c)$$

The constant E_0 is determined from the total current $\underline{I}$ through the sheet. If the height of

the sheet is $2h$, then $\underline{I}$ is

$$\underline{I} = 2h \int_{-a}^a \kappa E_0 \cosh(x\sqrt{j\omega\mu\kappa}) dx = \frac{4h}{\sqrt{j\omega\mu\kappa}} \kappa E_0 \sinh(a\sqrt{j\omega\mu\kappa}) , \quad (7.4.24)$$

and the expression for the phasor $\underline{E}$ is

$$\underline{E} = \frac{\underline{I}\sqrt{j\omega\mu\kappa}}{4h\kappa} \frac{\cosh(x\sqrt{j\omega\mu\kappa})}{\sinh(a\sqrt{j\omega\mu\kappa})} . \quad (7.4.25)$$

To determine the constant H_0 , we use Ampere's law

$$\oint \underline{H} \cdot d\mathbf{l} = 2h\underline{H}(a) + 2h\underline{H}(-a) = \underline{I} , \quad (7.4.26)$$

and we get

$$H_0 = \frac{\underline{I}}{4h \sinh(b\sqrt{j\omega\mu\kappa})} \Rightarrow \underline{H} = \frac{\underline{I}}{4h} \frac{\sinh(x\sqrt{j\omega\mu\kappa})}{\sinh(a\sqrt{j\omega\mu\kappa})} . \quad (7.4.27a,b)$$

Figure 7.4.7. shows the distribution of normalized current density throughout a sheet as a function of the ratio a/d . In this diagram, we can see that for small values of the ratio, the distribution of the current density is uniform, and that with any rise of this ratio the current increasingly tends to concentrate itself near the surface of the conductor. This effect is called *the skin effect*.

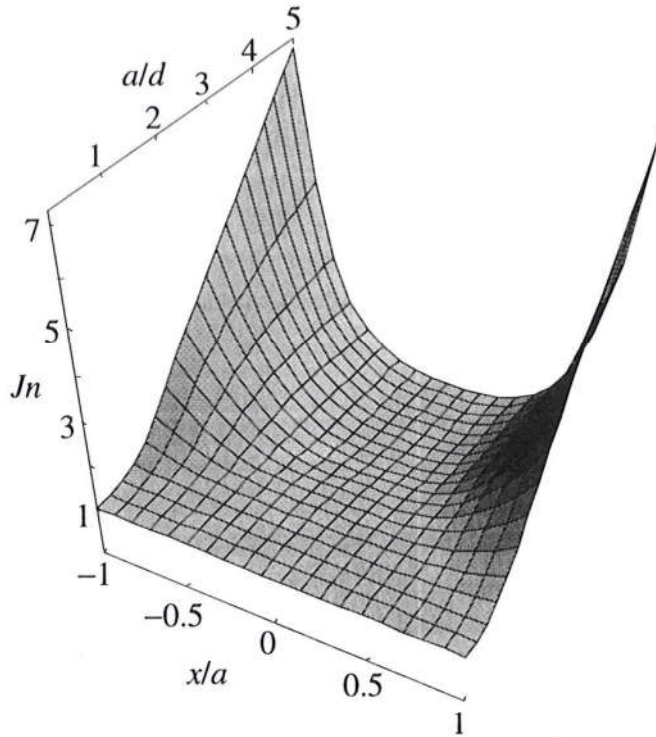


Figure 7.4.7. The distribution of normalized current density $Jn = J_{\max}/J_0$, $J_0 = I/(4ha)$ throughout a sheet as a function of a/d .

Application of expression (7.3.9) for time-averaged losses in a sinusoidal field results in

losses per unit length:

$$P'_{l,av} = \frac{I^2}{8h\kappa d} \frac{\sinh\left(\frac{2a}{d}\right) + \sin\left(\frac{2a}{d}\right)}{\cosh\left(\frac{2a}{d}\right) - \cos\left(\frac{2a}{d}\right)}. \quad (7.4.28)$$

By equating the time-averaged losses in an equivalent resistance and the time-averaged losses in the field, we get an expression for calculating the *equivalent resistance* per unit length:

$$R' = \frac{P'_{l,av}}{\left(\frac{I}{\sqrt{2}}\right)^2}. \quad (7.4.29)$$

It is interesting to analyse the ratio of such a resistance and the resistance $R'_0 = 1/(4h\kappa)$ determined from the uniformly distributed current

$$k_R = \frac{R'}{R'_0} = \frac{a}{d} \frac{\sinh\left(\frac{2a}{d}\right) + \sin\left(\frac{2a}{d}\right)}{\cosh\left(\frac{2a}{d}\right) - \cos\left(\frac{2a}{d}\right)}. \quad (7.4.30)$$

The dependence of k_R on the ratio a/d is shown in Figure 7.4.8. We see that in cases of a small ratio a/d , we can calculate the equivalent resistance by methods described in Chapter 5.

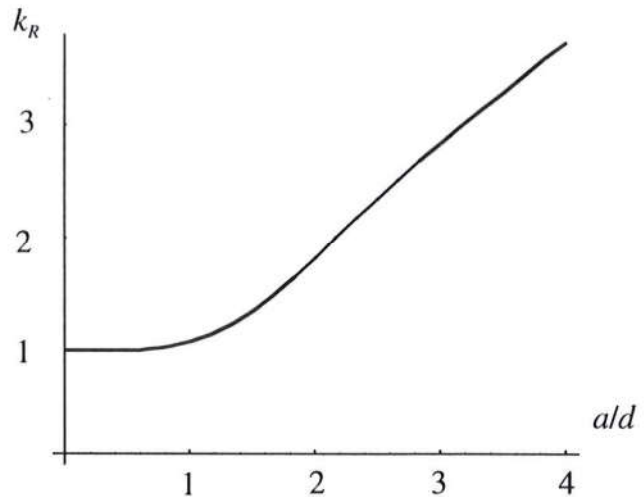


Figure 7.4.8. Dependence of the factor k_R on the ratio a/d

By application of expression (7.3.10), we obtain the time-averaged stored magnetic energy per unit length

$$W'_{m,av} = \frac{\mu I^2 d}{32h} \frac{\sinh\left(\frac{2a}{d}\right) - \sin\left(\frac{2a}{d}\right)}{\cosh\left(\frac{2a}{d}\right) - \cos\left(\frac{2a}{d}\right)}. \quad (7.4.31)$$

Equating the time-averaged energy stored in an equivalent internal inductance and the time-averaged magnetic energy stored in the field results in an expression for calculating the equivalent internal inductance per unit length

$$L'_u = \frac{2W'_{m,av}}{\left(\frac{I}{\sqrt{2}}\right)^2}. \quad (7.4.32)$$

The ratio of such internal inductance and inductance $L'_{u0} = \mu a/(12h)$ determined from a uniformly distributed current is

$$k_L = \frac{L'_u}{L'_{u0}} = \frac{3d}{2a} \frac{\sinh\left(\frac{2a}{d}\right) - \sin\left(\frac{2a}{d}\right)}{\cosh\left(\frac{2a}{d}\right) - \cos\left(\frac{2a}{d}\right)}. \quad (7.4.33)$$

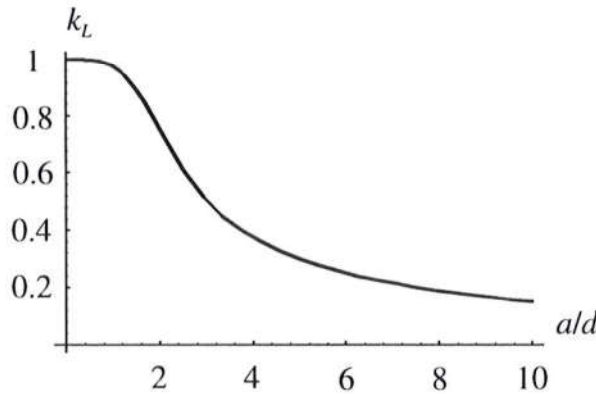


Figure 7.4.9. The dependence of the factor k_L on the ratio a/d .

Example 7.4.3. The skin effect in a cylinder

We shall analyse a cylindrical conductor with constitutive parameters κ and μ , which carries a sinusoidal current in an axial direction. When the length of the conductor is much greater than the diameter $2R$, we can solve this problem as an axially symmetrical one in a cylindrical coordinate system. The current density possesses only a z component $\mathbf{J} = J(r)\mathbf{a}_z$, and so the diffusion equation in a cylindrical coordinate system takes the form

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial J}{\partial r} \right) = \frac{\partial^2 J}{\partial r^2} + \frac{1}{r} \frac{\partial J}{\partial r} = \mu \kappa \frac{\partial J}{\partial t}. \quad (7.4.34)$$

The transformation of this equation into a phasor domain results in

$$\frac{d^2 \underline{J}}{dr^2} + \frac{1}{r} \frac{d\underline{J}}{dr} + (\sqrt{-j\omega\mu\kappa})^2 \underline{J} = 0 . \quad (7.4.35)$$

This is a zero-order Bessel's differential equation, and its solutions are zero-order Bessel functions of the first kind, denoted by J_0 , and zero-order Bessel functions of the second kind, denoted by K_0 . The current density possesses no singularities inside the conductor and therefore the solution has the form

$$\underline{J} = A J_0(r\sqrt{-j\omega\mu\kappa}) . \quad (7.4.36)$$

To determine the constant A we use Ampere's law. Note that the argument of the Bessel function here is a complex number, and therefore its values are complex numbers as well. The real and imaginary part of Bessel functions are determined by Kelvin functions ber and bei which are defined as

$$\text{ber}(u) + j\text{bei}(u) = J_0(ue^{j\frac{3\pi}{4}}) . \quad (7.4.37)$$

The final expression for the current density phasor is

$$\underline{J} = \frac{jL\sqrt{\omega\mu\kappa}}{2R\pi} \frac{\text{ber}(r\sqrt{\omega\mu\kappa}) + j\text{bei}(r\sqrt{j\omega\mu\kappa})}{\text{ber}'(R\sqrt{\omega\mu\kappa}) + j\text{bei}'(R\sqrt{j\omega\mu\kappa})} = J_{\max} e^{j\psi_J} . \quad (7.4.38)$$

The distribution of normalized current density as a function of the ratio R/d is shown in Figure 7.4.10. When the ratio R/d is small, the current is distributed uniformly, but when this ratio rises, the skin effect rises significantly. *At very high frequencies, the current flows in a very narrow zone close to the surface of the conductor.*

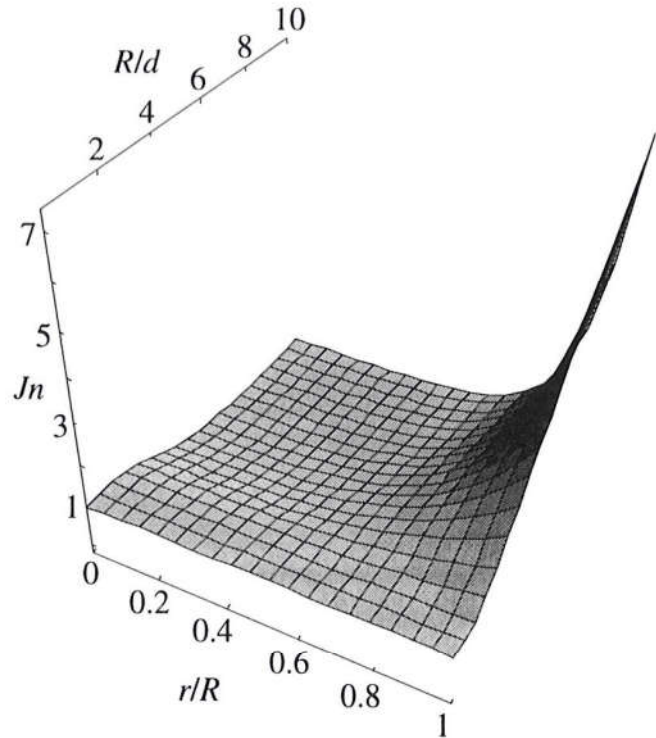


Figure 7.4.10. The distribution of the normalized current density $J_n = J_{\max} S/I$ in a conductor as a function of the ratio R/d , $S = R^2\pi$.

The final expression for the phasor of magnetic field strength is

$$\underline{H} = \frac{I}{2R\pi} \frac{\text{ber}'(r\sqrt{\omega\mu\kappa}) + j\text{bei}'(r\sqrt{j\omega\mu\kappa})}{\text{ber}'(R\sqrt{\omega\mu\kappa}) + j\text{bei}'(R\sqrt{\omega\mu\kappa})} = H_{\max} e^{\Psi_H}, \quad (7.4.39)$$

We can determine the equivalent resistance and inductance from the losses and stored magnetic energy in the same way as in the previous example. However, this procedure involves a complicated integration of Kelvin functions, and so we are going to use Poynting's theorem. The complex Poynting's vector on the surface of the conductor is:

$$\begin{aligned} N &= \frac{1}{2} [\underline{E}(R) \times \underline{H}^*(R)] = \\ &= -\underline{a}_r \frac{1}{2} \frac{jI\sqrt{\omega\mu\kappa}}{2\kappa R\pi} \frac{\text{ber}(R\sqrt{\omega\mu\kappa}) + j\text{bei}(R\sqrt{\omega\mu\kappa})}{\text{ber}'(R\sqrt{\omega\mu\kappa}) + j\text{bei}'(R\sqrt{\omega\mu\kappa})} \frac{I}{2R\pi}. \end{aligned} \quad (7.4.40)$$

The integration surface used in the application of the complex form of Poynting's theorem is the surface of the cylinder (conductor) of length l . At the base of the cylinder N and $\underline{n}$ are perpendicular and on the cylindrical surface the angle between them is π . The apparent power transferred to unit length of the cylinder is

$$P_N = \oint_S N \cdot \underline{n} dS = -\frac{1}{2} \frac{jI^2 \sqrt{\omega\mu\kappa}}{2\kappa R\pi} \frac{\text{ber}(R\sqrt{\omega\mu\kappa}) + j\text{bei}(R\sqrt{\omega\mu\kappa})}{\text{ber}'(R\sqrt{\omega\mu\kappa}) + j\text{bei}'(R\sqrt{\omega\mu\kappa})}. \quad (7.4.41)$$

The real part of the apparent power is the time-averaged power consumed within the conductor of length l . We can determine the equivalent resistance from this power as

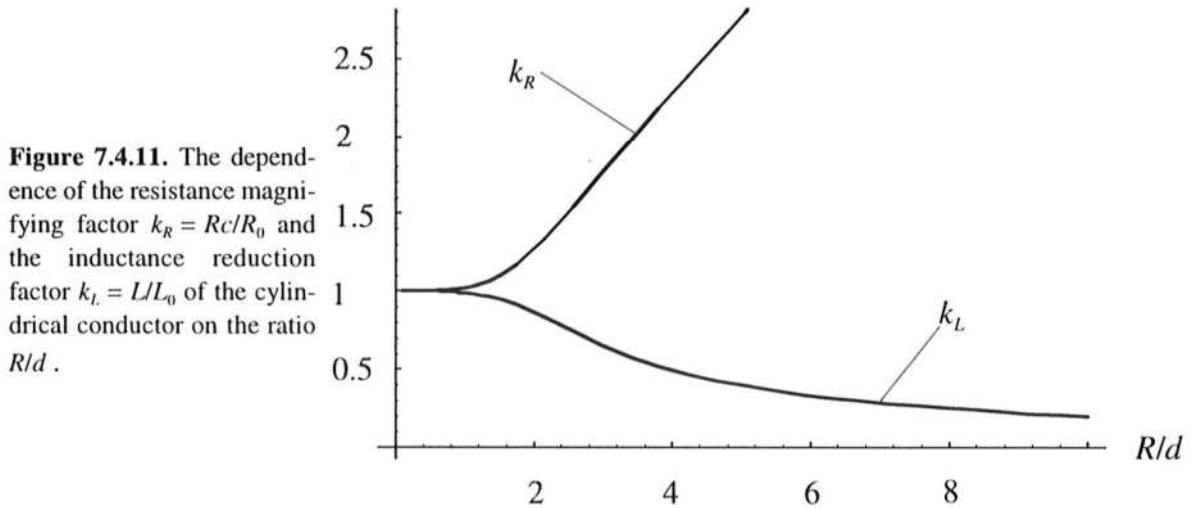
$$R_c = \frac{P_{l,av}}{\left(\frac{I}{\sqrt{2}}\right)^2} = \frac{1}{\sqrt{2}\pi R\kappa d} \frac{\text{ber}\left(\frac{R\sqrt{2}}{d}\right)\text{bei}'\left(\frac{R\sqrt{2}}{d}\right) - \text{bei}\left(\frac{R\sqrt{2}}{d}\right)\text{ber}'\left(\frac{R\sqrt{2}}{d}\right)}{\left[\text{ber}'\left(\frac{R\sqrt{2}}{d}\right)\right]^2 + \left[\text{bei}'\left(\frac{R\sqrt{2}}{d}\right)\right]^2}. \quad (7.4.42)$$

The time-averaged stored magnetic energy can be found from the imaginary part of the apparent power, and the equivalent inductance is therefore:

$$L = \frac{1}{\sqrt{2}\pi R\kappa\omega d} \frac{\text{ber}\left(\frac{R\sqrt{2}}{d}\right)\text{ber}'\left(\frac{R\sqrt{2}}{d}\right) - \text{bei}\left(\frac{R\sqrt{2}}{d}\right)\text{bei}'\left(\frac{R\sqrt{2}}{d}\right)}{\left[\text{ber}'\left(\frac{R\sqrt{2}}{d}\right)\right]^2 + \left[\text{bei}'\left(\frac{R\sqrt{2}}{d}\right)\right]^2}. \quad (7.4.43)$$

In Figure 7.4.10, we can see that in the case of a small ratio of R/d , one can calculate the equivalent resistance $R_0 = 1/(\kappa R^2\pi)$ and inductance $L_0 = \mu/(8\pi)$ with methods valid for static fields. It can be proved by the analysis of equations (7.4.41,42) if we take into account only the first few terms of the expansion of Kelvin functions. The dependence of the resistance magnifying factor $k_R = R_c/R_0$ and the inductance reduction factor $k_L = L/L_0$

on the ratio R/d is shown in Figure 7.4.11.



Example 7.4.4. A sphere in a sinusoidal magnetic field

A sphere of constitutive parameters μ and κ is placed in an external magnetic field of phasor $\underline{B}_0$. We are solving this problem in a spherical coordinate systems using the magnetic vector potential. We thus suppose that

$$\underline{B}_0 = B_0 \underline{a}_z = B_0 \cos \vartheta \underline{a}_r - B_0 \sin \vartheta \underline{a}_\vartheta. \quad (7.4.44)$$

From equation (7.1.5b) it follows that the magnetic vector potential of such a field is

$$\underline{A}_0 = \frac{1}{2} B_0 r \sin \vartheta \underline{a}_\alpha. \quad (7.4.45)$$

Due to the presence of the sphere, the field in its vicinity is changed. We shall determine the magnetic vector potential in the whole domain of this problem by a separation of variables.

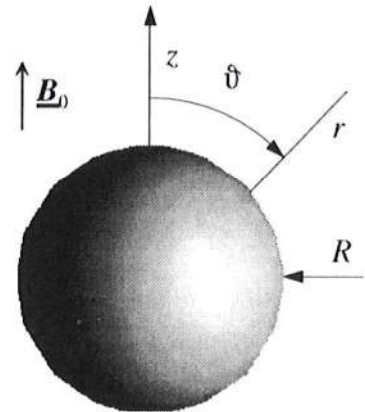


Figure 7.4.12. A sphere in a sinusoidal magnetic field

We suppose the solution takes the form:

$$\underline{A} = \frac{1}{2} B_0 \sin \vartheta F(r) \underline{a}_\alpha. \quad (7.4.46)$$

Outside the sphere, the magnetic vector potential satisfies Laplace's equation in a spheri-

cal coordinate system $\Delta \underline{A}_o = 0$. By substitution of the prescribed solution (7.4.46) into this equation, we get a differential equation for the unknown function F

$$\frac{1}{r^2} \frac{d}{dr} r^2 \frac{dF}{dr} - \frac{2}{r^2} F = 0 . \quad (7.4.47)$$

Solutions of this differential equation were shown in Chapter 4. If $r \rightarrow \infty$, then $A_o \rightarrow A_0$ and so we get

$$\underline{A}_v = \frac{1}{2} B_0 \sin \vartheta \left(r + \frac{a}{r^2} \right) \underline{a}_\alpha . \quad (7.4.48)$$

Inside the sphere, the magnetic vector potential satisfies the diffusion equation $\Delta \underline{A}_i + j\omega\mu\kappa \underline{A}_i = 0$. Substitution of the supposed solution (7.4.46) into this equation results in the final form of the solution inside the sphere being

$$\underline{A}_i = \frac{1}{2} B_0 b \sin \vartheta \sqrt{\frac{\pi}{2r\sqrt{j\omega\mu\kappa}}} I_{3/2}(r\sqrt{j\omega\mu\kappa}) \underline{a}_\alpha . \quad (7.4.49)$$

The modified Bessel function $I_{3/2}$ can be expressed using hyperbolic functions and by substitution of the depth of penetration d we obtain

$$\underline{A}_i = \frac{1}{2} \frac{b B_0 \sin \vartheta}{\frac{r(1+j)}{d}} \left[\cosh \frac{r(1+j)}{d} - \frac{1-j}{2} \frac{d}{r} \sinh \frac{r(1+j)}{d} \right] \underline{a}_\alpha . \quad (7.4.50)$$

We determine the magnetic flux density inside and outside the sphere from the magnetic vector potential using:

$$\underline{B} = \nabla \times \underline{A} = \frac{1}{r \sin \vartheta} \frac{\partial (\sin \vartheta \underline{A})}{\partial \vartheta} \underline{a}_r - \frac{1}{r} \frac{\partial (r \underline{A})}{\partial r} \underline{a}_\vartheta = \underline{B}_r \underline{a}_r + \underline{B}_\vartheta \underline{a}_\vartheta . \quad (7.4.51)$$

Outside the sphere, the magnetic flux density is

$$\underline{B}_o = \underline{B}_0 \cos \vartheta \left(1 + \frac{a}{r^3} \right) \underline{a}_r - \underline{B}_0 \sin \vartheta \left(1 - \frac{a}{2r^3} \right) \underline{a}_\vartheta , \quad (7.4.52)$$

and inside the sphere the magnetic flux density is

$$\begin{aligned} \underline{B}_{ri} &= \frac{b B_0 d \cos \vartheta}{r^2} \left[(1-j) \cosh \frac{r(1+j)}{d} + \frac{jd}{r} \sinh \frac{r(1+j)}{d} \right] , \\ \underline{B}_{\vartheta i} &= \frac{b B_0 d \sin \vartheta}{2r^2} \left[(1-j) \cosh \frac{r(1+j)}{d} + \frac{jd}{r} \sinh \frac{r(1+j)}{d} - \frac{2r}{d} \sinh \frac{r(1+j)}{d} \right] . \end{aligned} \quad (7.4.53a,b)$$

The unknown constants a and b are determined from the boundary conditions at $r = R$

$$\begin{aligned} B_{ri}(R, \vartheta) &= B_{ro}(R, \vartheta) , \\ B_{\vartheta i}(R, \vartheta) &= \mu_r B_{\vartheta o}(R, \vartheta) . \end{aligned} \quad (7.4.54a,b)$$

The eddy current density inside the sphere is determined as

$$\mathbf{J} = \frac{1}{\mu} \nabla \times \mathbf{B}_i = \frac{1}{\mu} \nabla \times \nabla \times \mathbf{A}_i = \frac{1}{\mu} (-\Delta \mathbf{A}_i) = -j\omega\kappa \mathbf{A}_i , \quad (7.4.55)$$

and its value is

$$\mathbf{J}(r, \vartheta) = \frac{-3jH_0 \left(\frac{R}{d} \right)^2 \frac{\sin \vartheta}{r} \left[(1-j) \cosh \frac{r(1+j)}{d} + \frac{jd}{r} \sinh \frac{r(1+j)}{d} \right]}{(\mu_r - 1) \left[(1-j) \cosh \frac{R(1+j)}{d} + \frac{jd}{R} \sinh \frac{R(1+j)}{d} \right] + \frac{2R}{d} \sinh \frac{R(1+j)}{d}} \mathbf{a}_\vartheta . \quad (7.4.56)$$

Eddy currents flow in a circumferential direction. The distribution of the normalized maximum value of eddy current density as a function of ratios r/R and R/d for angle $\vartheta = \pi/2$ is shown in Figure 7.4.13.

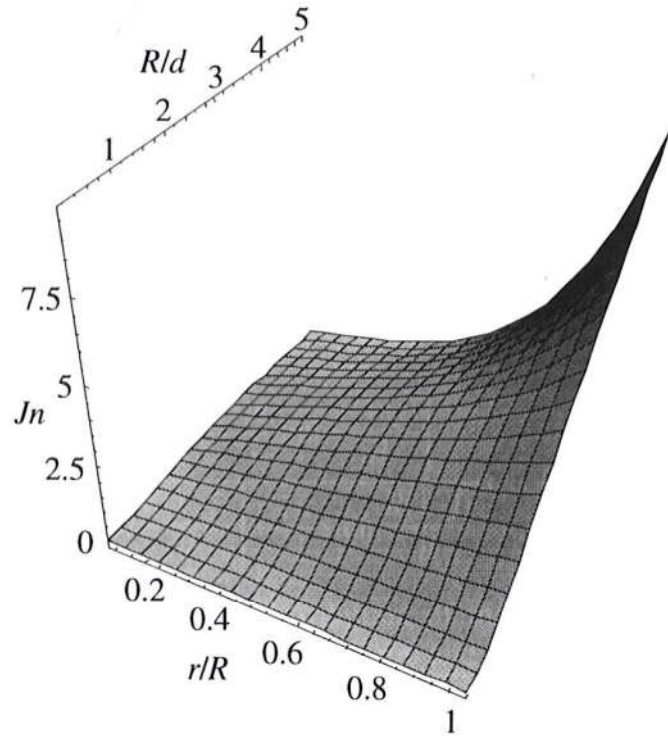


Figure 7.4.13. The distribution of the normalized maximum value of eddy current density $Jn = J_{\max}R/H_0$ in a sphere as a function of the ratios r/R and R/d for angle $\vartheta = \pi/2$ and $\mu_r = 1$.

7.5. CURRENT LOOPS IN A QUASI-STATIC FIELD

A current loop is a loop that consists of conductors of cross-sections that are so small that the cross-sectional distribution of current density can be regarded as a uniform one even in a quasi-static electromagnetic field. In this case, we may calculate the total current I through a conductor of cross-section S as

$$I = \int_S \mathbf{J} \cdot \mathbf{n} dS = JS . \quad (7.5.1)$$

Results from the previous unit showed that the ratio between the cross-sectional dimensions and the depth of penetration is the fundamental criterion guiding current distribution. Since the depth of penetration is frequency dependent, the validity of the approximation of a conductor to a current loop is frequency dependent too.

The solutions of such problems are based on the integral form of the quasi-static electromagnetic field equations. Equation (7.1.9b) for current loops becomes equal to equation (6.1.3)

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot \mathbf{n} dS = JS = I . \quad (7.5.2)$$

Therefore we can calculate quantities in a quasi-static magnetic field in problems involving current loops by methods that are valid for a static magnetic field described in Chapter 6. The magnetic field is time dependent in such a case, and it causes an induced electric field $\mathbf{E}_i$ which we determine using equation (7.1.9a).

If the distribution of free charge ρ is prescribed, then the electric field $\mathbf{E}$ caused by it can be determined by those methods valid for static electric fields described in Chapter 4. In a quasi-static electromagnetic field, there is no interaction between this electric field and the magnetic field. The total electric field is the sum $\mathbf{E}_i + \mathbf{E}$.

7.5.1. Steady current loops in a quasi-static field

We shall analyse a system of N steady current loops. Energy is transferred to them via an external electric field $\mathbf{E}_s$. The current densities in the loops are

$$\mathbf{J} = \kappa(\mathbf{E}_s + \mathbf{E}) = \kappa \left(\mathbf{E}_s - \frac{\partial \mathbf{A}}{\partial t} - \nabla \varphi \right) . \quad (7.5.3)$$

The current density distribution is uniform in every current loop, therefore the current density in the k -th loop $\mathbf{J}_k$ is equal to

$$\mathbf{J}_k = \frac{i_k}{S_k} \frac{d\mathbf{l}_k}{dl_k} , \quad (7.5.4)$$

where i_k is the total current in the k -th loop, S_k is the cross-section of the k -th loop and $d\mathbf{l}_k$ is a differential line element directed in the same direction as the current in the k -th loop. As a first step, we substitute equation (7.5.4) into (7.5.3), and then apply a dot product with $d\mathbf{l}_k$ to the resulting equation. Finally we perform a line integration over the closed contour C_k , which lies in the axis of the k -th loop and get

$$i_k \oint_{C_k} \frac{dl_k}{\kappa S_k} = \oint_{C_k} \mathbf{E}_S \cdot d\mathbf{l}_k - \oint_{C_k} \frac{\partial \mathbf{A}}{\partial t} \cdot d\mathbf{l}_k - \oint_{C_k} \nabla \varphi \cdot d\mathbf{l}_k . \quad (7.5.5)$$

The line integral on the left-hand side of this equation is the resistance R_k of the k -th loop. The first term on the right-hand side of this equation is the total electromotive force of external sources in the k -th loop. The third term is equal to zero when the contour C_k is closed, and the second term can be written as

$$- \oint_{C_k} \frac{\partial \mathbf{A}}{\partial t} \cdot d\mathbf{l}_k = - \frac{d\Phi}{dt} . \quad (7.5.6)$$

This term represents the total electromotive force induced by a time variation in magnetic flux. Therefore for the k -th loop, we get

$$R_k i_k + \frac{d\Phi}{dt} = \mathcal{E}_k . \quad (7.5.7)$$

In the case of steady loops, a time variation of magnetic flux may only be caused by a time variation in current. In simple materials, the total magnetic flux linked by the k -th loop is a linear function of the currents in all loops

$$\Phi_k = \sum_{i=1}^N L_{ik} i_i \quad ; \quad i = 1, 2, \dots, N . \quad (7.5.8)$$

For steady loops, the inductances L_{ik} are constant and equation (7.5.5) is transformed to

$$R_k i_k + \sum_{i=1}^N L_{ik} \frac{di_i}{dt} = \mathcal{E}_k . \quad (7.5.9)$$

We can write similar equations for all other contours, and so we get a system of N linear differential equations for unknown currents i_k , $k = 1 \dots N$. If the time dependence of the external electromotive forces $\mathcal{E}_S(t)$ is prescribed together with the initial conditions for currents, $i_k(t=0)$, we can evaluate the current i_k at any moment in time t by solving this system. Equation (7.5.9) states that we can substitute for a system of current loops by using a mutually coupled series $R - L$ circuit.

Example 7.5.1. Two circular loops at rest

Two circular loops of radii r_1 and r_2 and resistance R_1 and R_2 placed in air are shown in Figure 7.5.1. If the loop C_1 carries a current $i_1 = I_{\max} \cos(\omega t)$, we need to determine the current i_2 in loop C_2 taking into account the initial condition $i_2(t=0) = 0$. To solve this problem, we shall apply equation (7.5.9) to loop C_2

$$R_2 i_2 + L_{12} \frac{di_1}{dt} + L_{22} \frac{di_2}{dt} = 0 . \quad (7.5.10)$$

The solution of this linear differential equation is

$$i_2(t) = B e^{-\frac{R_2}{L_{22}}t} - \frac{\omega L_{12} [\omega L_{22} \cos(\omega t) - R_2 \sin(\omega t)]}{\omega^2 L_{22}^2 + R_2^2} I_{\max} . \quad (7.5.11)$$

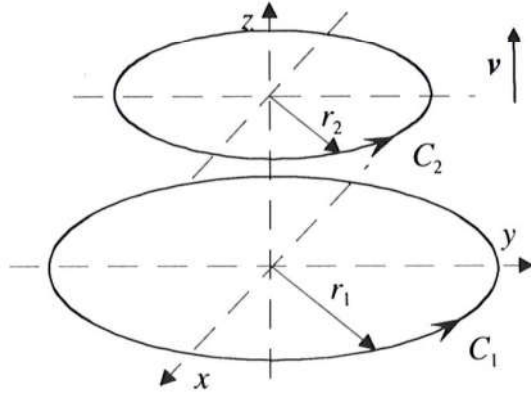


Figure 7.5.1. Two circular current loops

The unknown constant B is calculated from the initial condition

$$i_2(0) = 0 \quad \Rightarrow \quad B = \frac{\omega^2 L_{12} L_{22}}{\omega^2 L_{22}^2 + R_2^2} I_{\max} . \quad (7.5.12)$$

To calculate the current, we have to determine the mutual inductance L_{12} and self inductance L_{22} . In Example 6.8.1.1. we derived the expression (6.7.16) for the magnetic vector potential of a circular current loop. The mutual inductance between two given loops is

$$L_{12} = \frac{\oint_{C_2} \mathbf{A}_1 \cdot d\mathbf{l}_2}{I_1} = \mu_0 \sqrt{r_1 r_2} \left[\left(\frac{2}{k} - k \right) K(k) - \frac{2}{k} E(k) \right] . \quad (7.5.13)$$

In order to determine the self inductance of the loop C_2 we have to take into consideration the dimensions of its cross-section. We assume a circular cross-section of the loop C_2 with radius r_0 . We calculate the external inductance L_{22e} of this loop assuming that the total current is compressed into its axis and neglecting any magnetic flux in the loop. The linked external magnetic flux can then be determined by integration of the vector magnetic potential, and the external inductance results from (7.5.13) if we substitute r_2 and $r_2 - r_0$ for the radii of circular current loops together with $z = 0$

$$L_{22e} = \mu_0 \sqrt{r_1(r_1 - r_0)} \left[\left(\frac{2}{k} - k \right) K(k) - \frac{2}{k} E(k) \right] ; \quad k^2 = \frac{4r_1(r_1 - r_0)}{(2r_1 - r_0)^2} . \quad (7.5.14)$$

We are currently analysing problems with $r_0 \ll r_2$, and therefore we can approximate the modulus of the elliptic integral k as

$$k = 2 \frac{\sqrt{r_1(r_1 - r_0)}}{2r_1 - r_0} \quad \Rightarrow \quad k^2 \approx 1 - \left(\frac{r_0}{2r_1} \right)^2 \approx 1 . \quad (7.5.15)$$

The calculation of the elliptic integral in (7.5.14) is essentially simpler in this case. We

have that $E(k) \approx E(1) = 1$, and $K[1 - (r_0/2r_1)^2] \approx \ln(8r_1/r_0)$, and finally we get

$$L_{22e} \approx \mu_0 r_1 \left(\ln \frac{8r_1}{r_0} - 2 \right). \quad (7.5.16)$$

The magnetic field inside the conductor is approximately equal to that which would be inside an imaginary infinite line conductor carrying the same current. Therefore, we can determine the internal inductance L_{22i} by using the result of example 6.5.1. for a conductor of length $2r_1\pi$

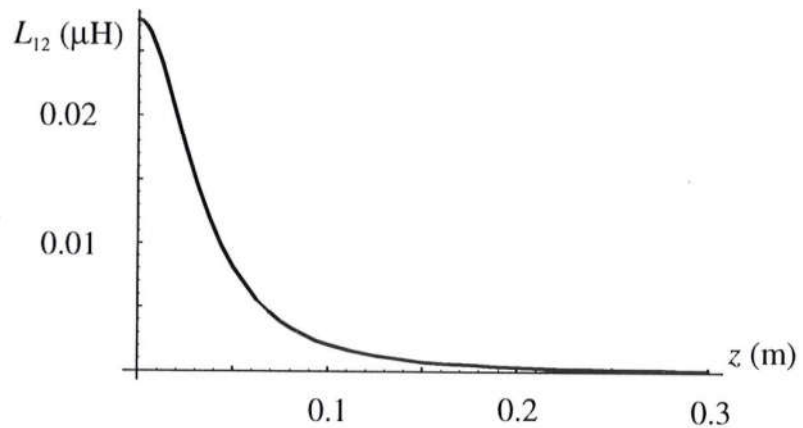
$$L_{22i} = \frac{\mu_0}{8\pi} 2r_1\pi = \frac{\mu_0 r_1}{4}. \quad (7.5.17)$$

Total self inductance of the loop C_2 is

$$L_{22} = L_{22i} + L_{22e} \approx \mu_0 r_1 \left(\ln \frac{8r_1}{r_0} - \frac{7}{4} \right). \quad (7.5.18)$$

For example, we have chosen loops with radii $r_1 = 0.05$ m, $r_2 = 0.025$ m and $r_0 = 0.0003$ m. The dependence of the mutual inductance L_{12} on the z coordinate is shown in Figure 7.5.2.

Figure 7.5.2. The dependence of the mutual inductance between two circular loops on distance apart



The time dependence of current i_2 in loop C_2 for frequency $f = 50$ kHz, $R_2 = 0.019 \Omega$, $L_{22} = 0.149 \mu\text{H}$, and $z = 0.01$ m is shown in Figure 7.5.3.

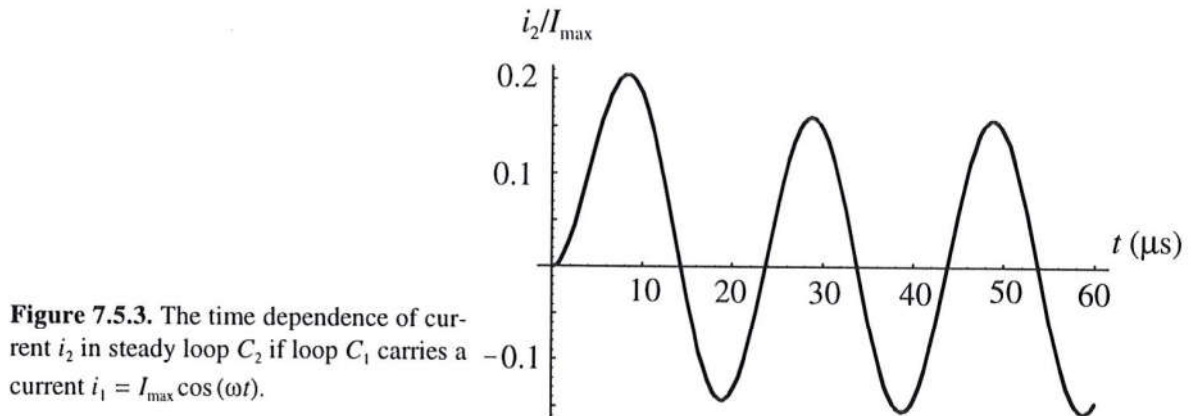


Figure 7.5.3. The time dependence of current i_2 in steady loop C_2 if loop C_1 carries a current $i_1 = I_{\max} \cos(\omega t)$.

7.5.2. Movement of a current loop in a quasi-static field: electromagnetic induction

If we analyse a current loop moving in a magnetic field that is dependent on time and space coordinates, we have to consider the form of equation (7.1.9a) that includes *time and space variations of the magnetic flux*:

$$\oint_C \mathbf{E}_i \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{n} dS . \quad (7.5.19)$$

Here, $\mathbf{E}_i$ is the induced electric field, and the right-hand side is the *total differential of magnetic flux with respect to time*. The negative sign indicates *Lenz's law* (of cussedness) (H.F.E. Lenz, 1804 – 1865): *the induced electric field causes a current that tries to annihilate its cause* (change of magnetic flux). If we mark the position of the loop at a moment in time $t - dt$ with S_1 , and the position of the loop at a moment in time t with S_2 , the total differential of magnetic flux with respect to time is

$$\frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{n} dS = \frac{d}{dt} \left[\int_{S_2} \mathbf{B}(t) \cdot \mathbf{n} dS_2 - \int_{S_1} \mathbf{B}(t-dt) \cdot \mathbf{n} dS_1 \right] . \quad (7.5.20)$$

While moving, the loop forms a cylinder bounded by S_1 , S_2 and a surface of width vdt . The area of the surface is

$$\oint_C d\mathbf{l} \times \mathbf{v} dt . \quad (7.5.21)$$

Let us apply Gauss' divergence theorem to the volume V of the cylinder

$$\int_V \nabla \cdot \mathbf{B}(t) dV = 0 = \int_{S_2} \mathbf{B}(t) \cdot \mathbf{n} dS - \int_{S_1} \mathbf{B}(t) \cdot \mathbf{n} dS + dt \oint_C \mathbf{B}(t) \cdot d\mathbf{l} \times \mathbf{v} . \quad (7.5.22)$$

The negative sign in front of the integral over S_1 is there because the normal on S_1 is directed into volume V . Since

$$\mathbf{B}(t-dt) = \mathbf{B}(t) - \frac{\partial \mathbf{B}}{\partial t} dt , \quad (7.5.23)$$

the integral over S_1 in (7.5.22) can be written as

$$\int_{S_1} \mathbf{B}(t) \cdot \mathbf{n} dS_1 = \int_{S_1} \mathbf{B}(t-dt) \cdot \mathbf{n} dS_1 + dt \int_{S_1} \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} dS_1 . \quad (7.5.24)$$

From equations (7.5.23,24) and (7.5.20) we get that

$$\int_{S_2} \mathbf{B}(t) \cdot \mathbf{n} dS_2 - \int_{S_1} \mathbf{B}(t-dt) \cdot \mathbf{n} dS_1 = dt \left[\int_{S_1} \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} dS_1 - \oint_C \mathbf{B} \cdot (d\mathbf{l} \times \mathbf{v}) \right]. \quad (7.5.25)$$

Substitution of (7.5.25) into (7.5.19) and usage of the identity for a triple cross product results in

$$\oint_C \mathbf{E}_i \cdot d\mathbf{l} = - \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} dS + \oint_C (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l}. \quad (7.5.26)$$

This equation is known as Faraday's law of electromagnetic induction, which was experimentally proved by English physicist Michael Faraday in the year 1831. The first term on the right-hand side of this equation represents the *induction of an electric field due to changes in the magnetic field* and the second represents the *induction due to the actual movement in the magnetic field*.

Differential equations for currents in a system of moving current loops can be derived from equation (7.5.8). If the k -th loop is moving in the direction of the s -coordinate with velocity $\mathbf{v}$, the total differential of magnetic flux with respect to time is

$$\begin{aligned} \frac{d\Phi_k}{dt} &= \sum_{i=1}^N \left\{ L_{ik} \frac{di_i}{dt} + i_i \frac{dL_{ik}}{dt} \right\} = \sum_{i=1}^N \left\{ L_{ik} \frac{di_i}{dt} + i_i \frac{dL_{ik}}{ds} \frac{ds}{dt} \right\} = \\ &= \sum_{i=1}^N \left\{ L_{ik} \frac{di_i}{dt} + i_i v \frac{dL_{ik}}{ds} \right\}, \end{aligned} \quad (7.5.27)$$

and so the equation

$$R_k i_k + \sum_{i=1}^N \left\{ L_{ik} \frac{di_i}{dt} + i_i v \frac{dL_{ik}}{ds} \right\} = \mathcal{E}_k, \quad (7.5.28)$$

is valid for the current in the k -th loop moving in a direction s with velocity $\mathbf{v}$.

Example 7.5.2.1. Movement of one current loop in the field of another

The current loop C_2 is placed in the field of the loop C_1 that carries the current $i_1(t) = I_{\max} \cos(\omega t)$ as shown in Figure 7.5.1. At a moment $t = 0$, the loop C_2 starts to move with constant velocity $\mathbf{v}$ in the direction of the z coordinate. In order to determine the current in the loop C_2 we shall apply equation (7.5.28) to the loop C_2

$$R_2 i_2 + L_{12} \frac{di_1}{dt} + i_1 v \frac{dL_{12}}{dz} + L_{22} \frac{di_2}{dt} = 0, \quad (7.5.29)$$

together with the fact that $z = vt$ and the initial condition $z(t = 0) = d$. The new term in the equation is the differentiation of the mutual inductance L_{12} with respect to z . We determine this term from equation (7.5.13) as being

$$\frac{dL_{12}}{dz} = -\mu_0 \frac{zk}{4\sqrt{r_1 r_2}} \left[\frac{2-k^2}{1-k^2} E(k) - 2K(k) \right]. \quad (7.5.30)$$

The dependence of the differential of mutual inductance on the z coordinate for loops with dimensions given in Example 7.5.1.1. is shown in Figure 7.5.4.

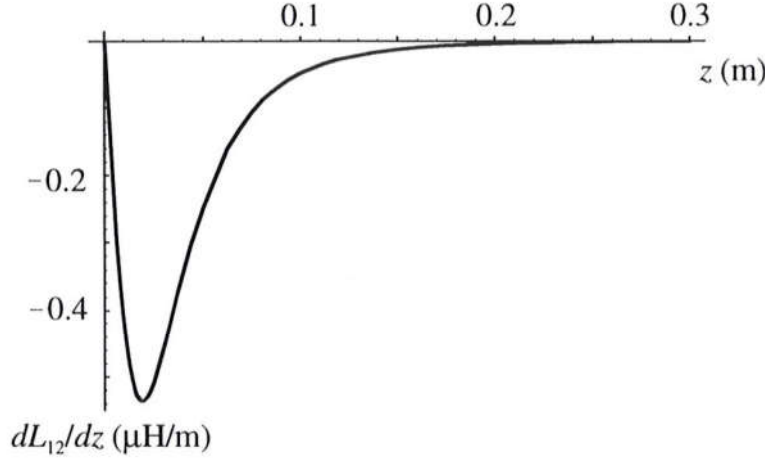


Figure 7.5.4. The dependence of the differential of mutual inductance between two circular loops on the z coordinate

Equation (7.5.30) is a non-linear differential equation and we can only solve it numerically. Here, we have applied a fourth-order Runge-Kutta method. The solution for current i_2 with dimensions of loops given in Example 7.5.1.1. and $z(t=0) = 0.01$ m and $v = 1000$ m/s is shown in Figure 7.5.5.

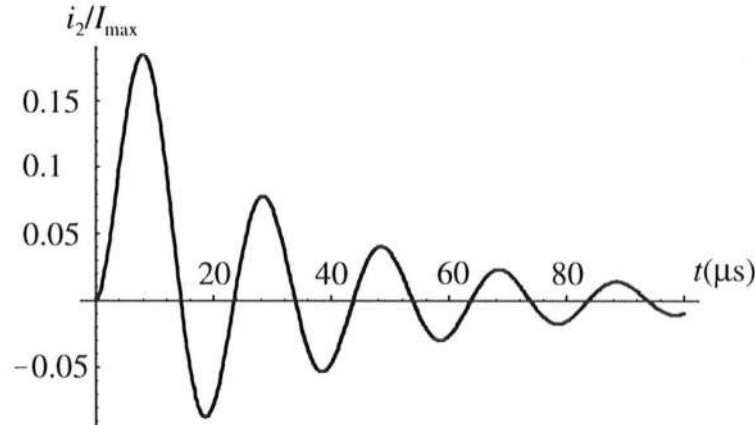


Figure 7.5.5. Time dependence of current i_2 if loop C_2 starts to move at a moment $t = 0$ from a point $z = 0.01$ m with velocity $v = 1000$ m/s while the current in loop C_1 is $i_1 = I_{\max} \cos(\omega t)$

7.5.3. Electromechanical energy conversion

Electromechanical energy converters convert mechanical energy into electrical energy and vice versa. They utilize electrical or mechanical forces, and have at least two connectors for connection of the source or the receiver. In practice, the most usual converters utilize magnetic forces because they need a smaller volume than ones based on electrical forces using the same power.

In Chapter 3. we showed that the surface stress on a conductor immersed in an electric field in air was $t^e = \epsilon_0 E^2/2$, while the surface stress on a conductor immersed in a

magnetic field in air was $t^m = B^2/2\mu_0$. The maximum electric field strength in air is around 3 MV/m, and the maximum magnetic field density on the surface of ferromagnetic materials in air is around 2 T. The ratio of the greatest possible surface stresses in a magnetic and electric field in air is

$$\frac{t^m}{t^e} \approx \frac{B_{\max}^2}{\mu_0 \epsilon_0 E_{\max}^2} \approx 10^4, \quad (7.5.31)$$

and so we need a considerably smaller volume of electromechanical converter to utilize magnetic forces. Therefore, we shall restrict our analysis to such "magneto-mechanical" converters.

On the basis of the general principle of energy conservation, one can find that the total energy taken in by a converter must be equal to the sum of the change in kinetic energy, the energy stored in the magnetic field and losses within the quasi-static electromagnetic field. If the total force $\mathbf{F}$ acting on the moveable part of a converter moving with velocity $\mathbf{v}$ from a position $\mathbf{r}_1$ to a position $\mathbf{r}_2$ is the sum of the magnetic $\mathbf{F}_{mag}$ and the mechanical force $\mathbf{F}_{mech}$, then the change in kinetic energy is equal to the work done by these forces:

$$\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^t (\mathbf{F}_{mag} + \mathbf{F}_{mech}) \cdot \mathbf{v} dt = \int_0^t dW_{kin}. \quad (7.5.32)$$

The energetic balance of the electromagnetic part of the system, which is in fact a current loop enclosing the moveable part of a converter, can be found if we multiply equation (7.5.5) by the current and integrate over the same time interval 0 – t

$$\int_0^t Ri^2 dt + \int_0^t i d\Phi = \int_0^t i \mathcal{E} dt. \quad (7.5.33)$$

The energy that we are taking into the system is the sum of all the electromagnetic energy taken by way of the external field and the work done by mechanical forces. From the principle of energy conservation it follows that

$$\int_0^t i \mathcal{E} dt + \int_0^t \mathbf{F}_{mech} \cdot \mathbf{v} dt = \int_0^W (dW_{mag} + dW_{kin}) + \int_0^t i^2 R dt. \quad (7.5.34)$$

Substitution of equations (7.5.32,33) into (7.5.34) results in the law of electromagnetic energy conversion in magneto-mechanical converters

$$\int_0^\Phi i d\Phi = \int_0^t \mathbf{F}_{mag} \cdot \mathbf{v} dt + \int_0^W dW_{mag} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}_{mag} \cdot d\mathbf{r} + \int_0^W dW_{mag}. \quad (7.5.35)$$

If we assume that the stored magnetic energy is a single-valued and differentiable function of both the magnetic flux Φ and displacement r , we can write

$$dW_{mag} = \frac{\partial W_{mag}}{\partial \Phi} d\Phi + \frac{\partial W_{mag}}{\partial r} dr . \quad (7.5.36)$$

It follows that

$$i = \frac{\partial W_{mag}(\Phi, r)}{\partial \Phi} \quad ; \quad F_{mag} = - \frac{\partial W_{mag}(\Phi, r)}{\partial r} . \quad (7.5.37)$$

and so we can determine the force on a moving loop as well as the current through the loop. If magnetic energy is given as a function of current and displacement r , we can introduce magnetic coenergy $W_{co, mag}(i, r) = -W_{mag}(\Phi, r) + i\Phi$. Usage of this definition results in

$$\Phi = \frac{\partial W_{co, mag}(i, r)}{\partial i} \quad ; \quad F_{mag} = \frac{\partial W_{co, mag}(i, r)}{\partial r} . \quad (7.5.38)$$

Magnetic coenergy as the area under the curve $\Phi(i)$, and magnetic energy as the area under the curve $i(\Phi)$ are shown in Figure 7.5.6. For linear materials these areas are equal.

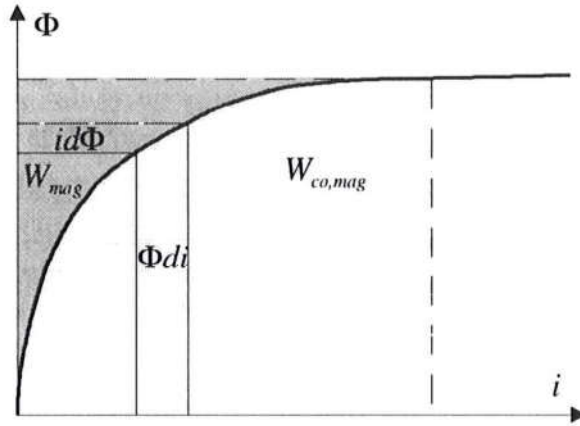


Figure 7.5.6. Magnetic energy and coenergy

We can derive equations of motion of the moveable part of a converter by direct application of Newton's laws and vector kinematics, or with the help of a scalar energetic formulation by using Lagrange's equations.

Example 7.5.3.1. Two circular current loops acting as a magnetic accelerator

The purpose of magnetic accelerators is to accelerate massive objects. As an example of an analysis in the design of such apparatus, we shall consider a simple geometry as defined in Figure 7.5.1. We assume that the loop C_1 is fixed and that this loop carries a prescribed current, while the loop C_2 can move in the direction of the z coordinate. We prescribe an impulse current in the loop C_1 :

$$i_1(t) = \begin{cases} I_{\max} \sin\left(\frac{\pi t}{\tau_0}\right) & ; \quad 0 \leq t \leq \tau_0 , \\ 0 & ; \quad \tau_0 < t . \end{cases} \quad (7.5.39)$$

We need to establish three differential equations in order to determine the current i_2 in loop C_2 , the velocity v of the loop C_2 and the distance through which C_2 moves. Loop C_2 can move only in the z direction and its velocity is given by:

$$v = \frac{dz}{dt} . \quad (7.5.40)$$

We calculate the force with which the magnetic part of the quasi-static electromagnetic field acts on the loop C_2 using expression (7.5.37). Usage of a balance of forces results in the second differential equation

$$m \frac{dv}{dt} = -F_{mag} = \frac{\partial W_{mag}}{\partial r} = i_1(t) i_2(t) \frac{dL_{12}(z)}{dz} . \quad (7.5.41)$$

The third differential equation follows from (7.5.29)

$$R_2 i_2(t) + L_{12} \frac{di_1(t)}{dt} + i_1(t) v \frac{dL_{12}(z)}{dz} + L_{22} \frac{di_2(t)}{dt} = 0 . \quad (7.5.42)$$

We can only solve this set of non-linear differential equations numerically. The time dependence of velocity v and the current i_2 of loop C_2 for the data given in Example 7.5.1.1. with $\tau_0 = 20 \mu s$ and $m = 0.4 \text{ g}$ are shown in Figure 7.5.7a,b.

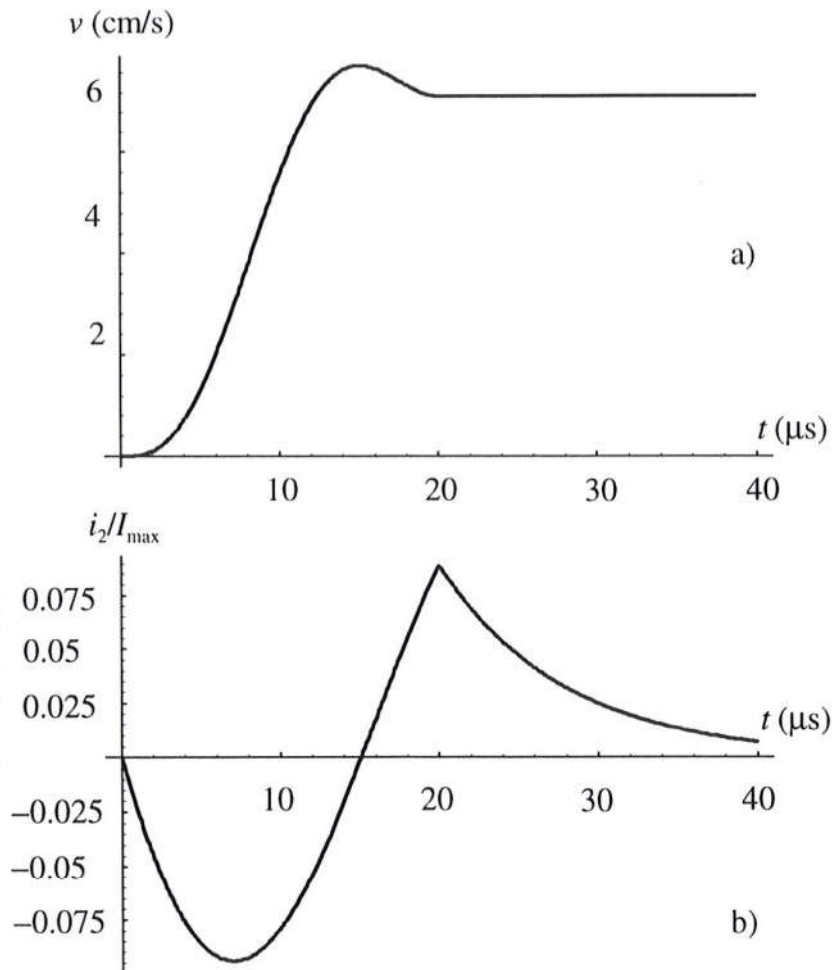


Figure 7.5.7. a) the time dependence of the velocity v of loop C_2 and b) time dependence of the normalized current i_2/I_{max} through loop C_2 , while loop C_1 carries an impulse current given by expression (7.5.41).

PROBLEMS

- 7.1. Analyse the relationship between the Fourier transform and phasor notation!
- 7.2. Determine criteria for the validity of a quasi-static approximation in air ($\mu = \mu_0$, $\epsilon = \epsilon_0$, $\kappa = 0$), sea water ($\mu = \mu_0$, $\epsilon = 81\epsilon_0$, $\kappa = 5$ S/m) and copper ($\mu = \mu_0$, $\epsilon = \epsilon_0$, $\kappa = 58$ MS/m) for a frequency of 50 Hz!
- 7.3. Calculate the eddy currents in the magnetic tin sheet from Example 7.4.1. during the switching on of a current!
- 7.4. Prove expressions for the modulus and phase angle of the phasor of magnetic field strength (7.4.16,17) of a magnetic tin sheet in a sinusoidal field!
Hint: use the identities

$$\cosh(a + jb) = \cosh(a) \cos(b) + j \sinh(a) \sin(b) ,$$

$$\sinh(a + jb) = \sinh(a) \cos(b) + j \cosh(a) \sin(b) ,$$

$$\cosh^2(a) = \frac{1}{2} [\cosh(2a) + 1] ,$$

$$\sinh^2(a) = \frac{1}{2} [\cosh(2a) - 1] .$$

- 7.5. Determine the constant A in expression (7.4.26) for the current density in a cylindrical conductor!
- 7.6. Calculate the losses in a sphere in a sinusoidal magnetic field (Example 7.4.4) using expression (7.4.46) for eddy current density inside the sphere!

8. ELECTROMAGNETIC WAVES

Maxwell's discovery of electromagnetic waves two decades before Hertz experimentally proved them, along with the explanation of the electromagnetic nature of light, is one of the greatest scientific contributions of the nineteenth century. A fundamental property of Maxwell's equations for an electromagnetic field is the existence of a solution for *travelling waves* that represents the transfer of energy through a space. Therefore, the analysis of plane waves is of fundamental importance in electromagnetic theory.

The simplest solutions of the system of equations for dynamic fields, except for static fields that were analysed in Chapters 4, 5, and 6, are ones that depend on time and on only one space coordinate. These one-dimensional solutions in a Cartesian coordinate system represent *plane waves*.

In this chapter, we shall study the propagation of homogeneous transverse plane waves through conductive materials and dielectrics. A lot of properties of the propagation of electromagnetic waves will be explained by the study of the characteristics of plane waves. We shall examine the response of a highly-conductive material to an excitation by a one-dimensional electromagnetic field with sinusoidal time dependence and emphasise the occurrence of *the skin effect*. The incidence of electromagnetic waves on a plane boundary between two materials will be described afterwards. The propagation of guided plane waves will be applied in the analysis of transmission lines, waveguides and resonant cavities. Finally, we shall study the Hertz dipole and a linear antenna in an unbounded space as sources of radiation.

8.1. WAVE EQUATIONS OF A FIELD IN AN UNBOUNDED SOURCE-FREE SPACE

Before we proceed with the derivation of the wave equations and the obtaining of their one-dimensional solutions for a plane wave, we shall focus our attention on some facts that are essential for our choice of solving procedure. As was shown in Chapter 2, where we introduced the electromagnetic potentials φ and $\mathbf{A}$ instead of the fields $\mathbf{E}$ and $\mathbf{B}$, there are two possible approaches to the solution of fields that result in the application of two different types of equations.

The first type are *partial differential equations* in wave form that can be attenuated or not, homogeneous or non-homogeneous. The second type are *integral equations* which we usually use in cases where the *sources of the field are given*. Equations that arise in solving such problems are usually Fredholm's equations of the first and second kind. We shall use the integral form of these equations later in the calculation of radiation of dipoles and antennae.

8.1.1. Wave equations

We assume that an electromagnetic field propagates in an unbounded space filled with a linear material at rest. In the inside of such a material with constitutive parameters ϵ , μ and κ , the density of the free charge is equal to zero $\rho_s = 0$, and free current can be represented as $\mathbf{J}_s = \kappa \mathbf{E}$.

The behaviour of an electromagnetic field in such an unbounded material is defined by the set of Maxwell's equations (1.5.15a – d) together with the constitutive relationships (1.5.17a – c). If we take into the consideration the fact that $\rho_s = 0$, we get an initial set of Maxwell's equations for an electromagnetic field in terms of the vectors $\mathbf{E}$, $\mathbf{H}$

$$\begin{aligned} \nabla \times \mathbf{E} + \mu \frac{\partial \mathbf{H}}{\partial t} &= 0, & \nabla \times \mathbf{H} - \epsilon \frac{\partial \mathbf{E}}{\partial t} - \kappa \mathbf{E} &= 0, \\ \nabla \cdot \mathbf{E} &= 0, & \nabla \cdot \mathbf{H} &= 0. \end{aligned} \quad (8.1.1a - d)$$

It is worth mentioning that the set of equations (8.1.1a – d) defines the vectors $\mathbf{E}$ and $\mathbf{H}$ completely as it prescribes the curl and divergence of both of these vectors, which is necessary if vectors are indirectly prescribed.

We shall apply the curl operator to each of the main Maxwell's equations (8.1.1a,b), to obtain

$$\nabla \times \nabla \times \mathbf{E} + \mu \frac{\partial}{\partial t} (\nabla \times \mathbf{H}) = 0, \quad \nabla \times \nabla \times \mathbf{H} - \epsilon \frac{\partial}{\partial t} (\nabla \times \mathbf{E}) - \kappa (\nabla \times \mathbf{E}) = 0.$$

If we replace the double curl of $\mathbf{E}$ as $\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \Delta \mathbf{E}$ and analogously for $\mathbf{H}$, and substitute Maxwell's equations (8.1.1a – d) we obtain

$$\Delta \mathbf{E} - \mu \epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} - \mu \kappa \frac{\partial \mathbf{E}}{\partial t} = 0, \quad \Delta \mathbf{H} - \mu \epsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} - \mu \kappa \frac{\partial \mathbf{H}}{\partial t} = 0, \quad (8.1.2a,b)$$

which are *attenuated wave equations for an electromagnetic field*. These are two vector, homogeneous second-order partial differential equations for the vectors $\mathbf{E}$ and $\mathbf{H}$.

The third term in equations (8.1.2a,b) represents the attenuation of the wave if the material it is propagated in is conductive ($\kappa \neq 0$). The first two terms of the wave equations indicate the wave nature of the field. If a material is non-conductive ($\kappa = 0$), then from equations (8.1.2a,b) it follows that the undamped wave field equations are

$$\Delta \mathbf{E} - \mu \epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0, \quad \Delta \mathbf{H} - \mu \epsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0. \quad (8.1.3a,b)$$

8.1.2. A separation of variables of the wave equation

We shall solve the wave equations derived in the previous section by the method of a separation of variables. To begin with, we shall assume that the solution of any field vector $\mathbf{W}(\mathbf{r}, t)$ of a vector wave equation is of the form

$$\Delta \mathbf{W}(\mathbf{r},t) - \mu\epsilon \frac{\partial^2 \mathbf{W}(\mathbf{r},t)}{\partial t^2} - \mu\kappa \frac{\partial \mathbf{W}(\mathbf{r},t)}{\partial t} = 0 , \quad (8.1.4)$$

and can be separated into two parts, one that depends only on time, and another that depends on spatial coordinates. Due to the linearity of equation (8.1.4) and the fact that the coefficients in this equation are real, we may consider the complex solutions, that is, those vector functions of time that have complex components. If we want a real solution we take the real part of the complex solution that is, along with the imaginary part, the solution of equation (8.1.4).

Let us express the vector $\mathbf{W}$ as the product $\mathbf{W}(\mathbf{r},t) = \mathbf{V}(\mathbf{r})T(t)$, where $\mathbf{V}(\mathbf{r})$ is a field vector independent of time, and $T(t)$ is a function independent of space. The vector function $\mathbf{V}$ and the scalar function T are complex functions. Instantaneous real values of the field $\mathbf{W}$ are in fact the *real parts of the product* of these two functions $\mathbf{V}$ and T . Substitution of the assumed solution into the wave equation (8.1.4) results in the differential equation $\Delta \mathbf{V} - f(t)\mathbf{V} = 0$, where

$$f(t) = \frac{\mu\epsilon \frac{\partial^2 T}{\partial t^2} + \mu\kappa \frac{\partial T}{\partial t}}{T} . \quad (8.1.5)$$

Differentiation of equation $\Delta \mathbf{V} - f(t)\mathbf{V} = 0$ with respect to t results in

$$\frac{d}{dt}(\Delta \mathbf{V}) = \frac{d\mathbf{V}}{dt}f(t) + \frac{df(t)}{dt}\mathbf{V} = 0 , \quad (8.1.6)$$

since $\mathbf{V}(\mathbf{r})$, and consequently $\Delta \mathbf{V}$, are independent of time. From equation (8.1.6) it follows that

$$\frac{df(t)}{dt} = 0 \quad \Rightarrow \quad f(t) = -k^2 . \quad (8.1.7)$$

By $(-k^2)$ we have denoted a constant that can be any complex number. The negative sign is used in order to transform the equation for $\mathbf{V}(\mathbf{r})$ into the standard form of a Helmholtz complex differential equation $\Delta \mathbf{V}(\mathbf{r}) + k^2 \mathbf{V}(\mathbf{r}) = 0$. This is the first separated differential equation. It is dependent only on spatial variables.

The second separated differential equation from the wave equation (8.1.4) is obtained from expressions (8.1.6,7) as

$$\mu\epsilon \frac{d^2 T(t)}{dt^2} + \mu\kappa \frac{dT(t)}{dt} + k^2 T(t) = 0 . \quad (8.1.8)$$

In this way, we have separated the vector wave equation (8.1.4) into a system of two differential equations. The first equation for $\mathbf{V}(\mathbf{r})$ is a “static” Helmholtz equation and depends only on the geometry of the field, thus determining the *space shape of the wave* in an unbounded space filled with a linear material. The second equation (8.1.8) is an ordinary second-order differential equation and is only time-dependent. It determines the form and extent of the time dependence of the field $\mathbf{W}$ at any point in space.

Although these two differential equations are mathematically independent of each other, they are physically dependent because the same constant k^2 is used in both equations. This is to be expected since both equations describe one unique electromagnetic wave.

Analytically, we solve Helmholtz equation for $V(\mathbf{r})$ by a separation of variables, as described in Unit 4.6. It is necessary to ensure a separability of this equation in a specified coordinate system in order to enable a separation of the variables. The simplest geometrical forms of waves are ones for which the vector function of space

$$\mathbf{V} = \sum_i V_i(u_1, u_2, u_3) \mathbf{a}_i = V(u_1, u_2, u_3) , \quad (8.1.9)$$

with three spatial components V_i dependent on three spatial variables u_1 , u_2 and u_3 can be expressed by only one spatial vector that depends on only one spatial variable. If this is the case, we may assume that

$$\mathbf{V} = \sum_i V_i(u_1) \mathbf{a}_i = V(u_1) . \quad (8.1.10)$$

These are *one-dimensional forms of waves*, and include such examples as plane waves, circularly-cylindrical waves, spherical waves and others. The simplest form is obtained in a Cartesian coordinate system when we assume that the field quantities are functions of only one spatial variable. We call such a wave a *plane electromagnetic wave*.

8.2. A PLANE WAVE IN AN UNBOUNDED SPACE

We shall apply a Cartesian coordinate system when separating variables in the Helmholtz equation assuming an arbitrary time dependence of fields $\mathbf{E}$ and $\mathbf{H}$.

8.2.1. The separation of the Helmholtz equation

Due to the simple relationship $\Delta V = \mathbf{a}_x \Delta V_x + \mathbf{a}_y \Delta V_y + \mathbf{a}_z \Delta V_z$ in a Cartesian coordinate system, one can separate the complex Helmholtz vector equation for $V(\mathbf{r})$ into three scalar complex Helmholtz equations with respect to the components of field vectors of the form $\Delta V + k^2 V = 0$. We shall again apply a separation of variables by assuming that one can express the solutions of this equation in the form $V(x, y, z) = X(x)Y(y)Z(z)$. After substitution of this assumption into the scalar Helmholtz equation we obtain a system of ordinary differential equations

$$\frac{1}{X} \frac{d^2 X}{dx^2} = -k_x^2 , \quad \frac{1}{Y} \frac{d^2 Y}{dy^2} = -k_y^2 , \quad \frac{1}{Z} \frac{d^2 Z}{dz^2} = -k_z^2 . \quad (8.2.1a - c)$$

Here $-k_x^2$, $-k_y^2$, $-k_z^2$ are *separation constants* for each of these three differential equations. They are generally complex and we can describe them as *eigenvalues* of the operator d^2/dx^2 in the space defined by twice differentiable functions of x .

Each eigenvalue $-k_x^2$ corresponds to a two-dimensional space defined by the *eigenfunctions* of the d^2/dx^2 operator. These functions are obtained by a linear combination of the functions $h(k_x x) = e^{\pm j k_x x}$ with complex constants. When k_x^2 is a non-zero value, k_x is one of its roots. Conversely, if $k_x^2 = 0$, these eigenfunctions are obtained by a linear combination of 1 and x . These linear combinations are the general solutions of equations (8.2.1a – c).

By multiplying solutions $h(k_x x)$, $h(k_y y)$ and $h(k_z z)$ together, and at the same time satisfying the condition

$$k^2 = k_x^2 + k_y^2 + k_z^2, \quad (8.2.2)$$

we obtain solutions of the scalar Helmholtz equation as

$$V = \sum_{k_x} \sum_{k_y} V_{k_x k_y} h(k_x x) h(k_y y) h(k_z z). \quad (8.2.3)$$

The products of individual eigenfunctions spanning spaces from solutions of equations (8.2.1a – c) give solutions to the scalar Helmholtz equation and we call them *elementary wave functions*. These are, for example, solutions of the form

$$e^{\pm j(k_x x + k_y y + k_z z)}.$$

Linear combinations of these elementary wave functions are the solutions of the scalar Helmholtz equation as well. From all the types of solutions (8.2.3) we shall extract the simplest one: a *plane wave that propagates in one direction* in space.

Let us suppose that k_x^2 , k_y^2 and k_z^2 are real positive constants. We shall use this notation for complex constants as well, but here k_x^2 , k_y^2 and k_z^2 are real and strictly positive. The solution of the Helmholtz equation associated to them is

$$V = K e^{+j(k_x x + k_y y + k_z z)} + L e^{-j(k_x x + k_y y + k_z z)}, \quad (8.2.4)$$

where K and L are complex constants too. According to (8.2.2) $k^2 = k_x^2 + k_y^2 + k_z^2$, and it is convenient to introduce the vector $\mathbf{k} = k_x \mathbf{a}_x + k_y \mathbf{a}_y + k_z \mathbf{a}_z$, as a vector that determines the *direction of wave propagation*. If the position vector is $\mathbf{r} = a_x x + a_y y + a_z z$, then the dot product can be expressed as $\mathbf{k} \cdot \mathbf{r} = k_x x + k_y y + k_z z$. Finally, we can write the chosen elementary solution (8.2.4) of the scalar complex Helmholtz equation in a Cartesian system as

$$V = K e^{+j\mathbf{k} \cdot \mathbf{r}} + L e^{-j\mathbf{k} \cdot \mathbf{r}}. \quad (8.2.5)$$

We call it the three-dimensional Cartesian zonal harmonic.

8.2.2. Equations of a plane wave

In continuation, we need to determine the relationship between the individual components of fields $\mathbf{E}$ and $\mathbf{H}$ that follow from the physical nature of a dynamic electromagnetic field before we can determine equations of the plane wave. Therefore, it is convenient to introduce a new Cartesian system (ξ, η, ζ) with the same origin as the (x, y, z) system. Let us suppose that the ζ -axis of the new system coincides with the direction of the vector $\mathbf{k}$,

meaning the direction of wave propagation. If we introduce a new unit vector oriented in the direction of the ζ -axis

$$\mathbf{a}_\zeta = \cos\alpha \mathbf{a}_x + \cos\beta \mathbf{a}_y + \cos\gamma \mathbf{a}_z, \quad (8.2.6)$$

then the vector in the direction of wave propagation can be simply expressed as $\mathbf{k} = k\mathbf{a}_\zeta$, where we determine the quantity k using equation (8.2.2). Since a position vector in the new system can be expressed as $\mathbf{r} = \mathbf{a}_\xi \xi + \mathbf{a}_\eta \eta + \mathbf{a}_\zeta \zeta$, the dot product is $\mathbf{k} \cdot \mathbf{r} = k\zeta$. A consequence of our assumption is that for all points of the plane $\mathbf{r} \cdot \mathbf{a}_\zeta = \zeta = \text{const.}$ representing a plane perpendicular to the ζ axis as shown in Figure 8.2.1, all components V_i determined by equation (8.2.5) are constant, which means that they are independent of ξ and η coordinates.

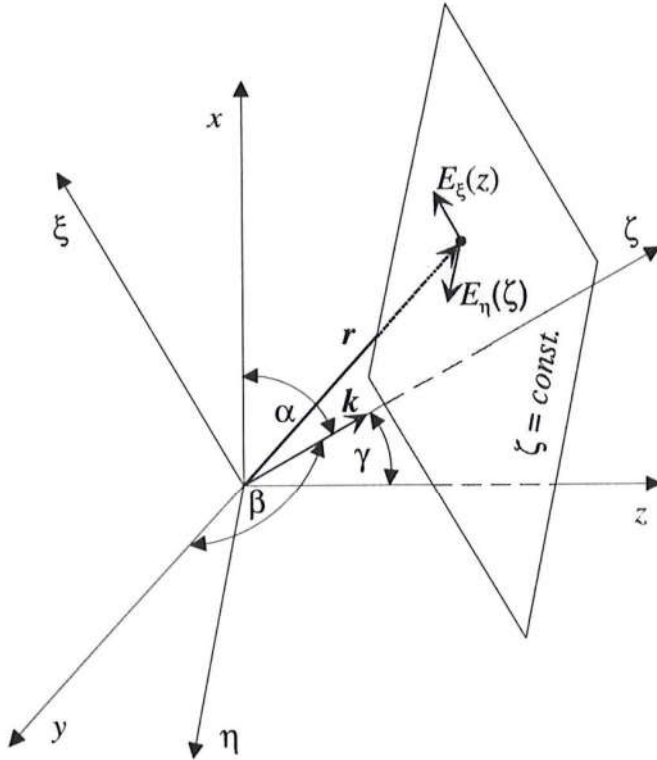


Figure 8.2.1. Propagation of a transverse homogeneous plane wave in the direction of the ζ -axis

It is obvious that if we substitute $\mathbf{k} \cdot \mathbf{r} = k\zeta$ into solutions (8.2.5), we then obtain equations of the form

$$V = K e^{+jk\zeta} + L e^{-jk\zeta}. \quad (8.2.7)$$

We call such a wave a *homogeneous wave*. The previously presented example of separation with positive constants shows that the field vectors $\mathbf{E}$ and $\mathbf{H}$ determined by Maxwell's system of equations (8.1.1a – d) under the specified conditions in a Cartesian coordinate system can be expressed dependent on only one spatial coordinate ζ and time t as $\mathbf{E} = \mathbf{E}(\zeta, t)$ and $\mathbf{H} = \mathbf{H}(\zeta, t)$.

The vectors $\mathbf{E}$ and $\mathbf{H}$ simultaneously satisfy the wave equations (8.1.2a,b) can be expressed in the form of separated complex solutions as $\mathbf{E} = \mathbf{E}_0(\zeta)T(t)$ and $\mathbf{H} = \mathbf{H}_0(\zeta)T(t)$. If we take into consideration that these solutions have to satisfy Maxwell's equations (8.1.1a,b), we obtain

$$\nabla \times \mathbf{E}(\zeta, t) + \mu \frac{\partial}{\partial t} \mathbf{H}(\zeta, t) = 0 \quad , \quad \nabla \times \mathbf{H}(\zeta, t) - \epsilon \frac{\partial}{\partial t} \mathbf{E}(\zeta, t) - \kappa \mathbf{E}(\zeta, t) = 0 \quad .$$

Substitution of the corresponding operators into these equations and subsequent rearrangement results in

$$\begin{aligned} & \mathbf{a}_\zeta \times [\mathbf{a}_\xi E_{0\xi} + \mathbf{a}_\eta E_{0\eta} + \mathbf{a}_\zeta E_{0\zeta}] (\pm jk) e^{\pm jk\zeta} T(t) + \\ & + \mu [\mathbf{a}_\xi H_{0\xi} + \mathbf{a}_\eta H_{0\eta} + \mathbf{a}_\zeta H_{0\zeta}] e^{\pm jk\zeta} \frac{\partial T(t)}{\partial t} = 0 \quad , \\ & \mathbf{a}_\zeta \times [\mathbf{a}_\xi H_{0\xi} + \mathbf{a}_\eta H_{0\eta} + \mathbf{a}_\zeta H_{0\zeta}] (\pm jk) e^{\pm jk\zeta} T(t) - \\ & - (\epsilon \frac{\partial T}{\partial t} + \kappa T) [\mathbf{a}_\xi E_{0\xi} + \mathbf{a}_\eta E_{0\eta} + \mathbf{a}_\zeta E_{0\zeta}] e^{\pm jk\zeta} = 0 \quad , \end{aligned}$$

and from this expression it follows that

$$\begin{aligned} H_\zeta &= 0 \quad , \\ E_\zeta &= E_{0\zeta} e^{\pm jk\zeta} T_0 e^{-\frac{t}{\tau}} \quad ; \quad \tau = \frac{\epsilon}{\kappa} \quad . \end{aligned} \tag{8.2.8a,b}$$

From this analysis, we may conclude that there is no longitudinal component of magnetic field H_ζ . This is the component in the direction of propagation of the wave. If the conductivity of the material is finite, then there is no longitudinal component of electric field either, i.e. $E_\zeta = 0$. The static electric field vanishes exponentially with time, which is to be expected because inside a conductive material a static electric field cannot exist. Therefore, the vectors $\mathbf{E}$ and $\mathbf{H}$ possess only two components and are in the plane $\zeta = \text{const}$, which is perpendicular to the direction of wave propagation

$$\begin{aligned} \mathbf{E}(\zeta, t) &= \mathbf{a}_\xi E_\xi(\zeta, t) + \mathbf{a}_\eta E_\eta(\zeta, t) = \mathbf{E}_0(\zeta) T(t) \quad , \\ \mathbf{H}(\zeta, t) &= \mathbf{a}_\xi H_\xi(\zeta, t) + \mathbf{a}_\eta H_\eta(\zeta, t) = \mathbf{H}_0(\zeta) T(t) \quad . \end{aligned} \tag{8.2.9a,b}$$

Such a plane wave, whose vector of electric and magnetic field are in a plane perpendicular to the direction of wave propagation is called a *transverse electromagnetic wave* or TEM wave.

Finally, we need to determine how the electromagnetic wave field vector's time dependence influences the behaviour of the wave. Therefore, we must analyse solutions of the differential equation (8.1.8). It is convenient to write this equation in the form

$$\frac{d^2 T}{dt^2} + \frac{\kappa}{\epsilon} \frac{dT}{dt} + \frac{k^2}{\mu\epsilon} T = 0 \quad . \tag{8.2.10}$$

Its characteristic equation is

$$s^2 + \frac{\kappa}{\epsilon}s + \frac{k^2}{\mu\epsilon} = 0, \quad (8.2.11)$$

and the roots of this quadratic equation are generally complex

$$s_{1,2} = -\frac{\kappa}{2\epsilon} \pm \sqrt{\left(\frac{\kappa}{2\epsilon}\right)^2 - \frac{k^2}{\mu\epsilon}}. \quad (8.2.12)$$

Therefore, the solution of the differential equation (8.2.10) is

$$T(t) = M e^{s_1 t} + N e^{s_2 t},$$

where M and N are general complex constants. Solutions of the attenuated wave equations for field vectors $\mathbf{E}$ and $\mathbf{H}$ satisfying the previously introduced assumptions about electromagnetic fields, i.e. that in an unbounded source-free space filled with a linear material, a homogeneous plane TEM wave exists that propagates in the direction of the ζ -axis, are obtained in the form of complex expressions as being

$$\mathbf{E}(\zeta, t) = \mathbf{E}_1 e^{st + jk\zeta} + \mathbf{E}_2 e^{st - jk\zeta}, \quad \mathbf{H}(\zeta, t) = \mathbf{H}_1 e^{st + jk\zeta} + \mathbf{H}_2 e^{st - jk\zeta}, \quad (8.2.13a, b)$$

where

$$\begin{aligned} \mathbf{E}_1 &= a_\xi \mathbf{E}_{1\xi} + a_\eta \mathbf{E}_{1\eta}, & \mathbf{E}_2 &= a_\xi \mathbf{E}_{2\xi} + a_\eta \mathbf{E}_{2\eta}, \\ \mathbf{H}_1 &= a_\xi \mathbf{H}_{1\xi} + a_\eta \mathbf{H}_{1\eta}, & \mathbf{H}_2 &= a_\xi \mathbf{H}_{2\xi} + a_\eta \mathbf{H}_{2\eta}, \end{aligned} \quad (8.2.14a - d)$$

and $E_{1\xi}, E_{1\eta}, E_{2\xi}, E_{2\eta}, H_{1\xi}, H_{1\eta}, H_{2\xi}, H_{2\eta}$ are eight complex constants that represent four complex vectors $\mathbf{E}_1, \mathbf{E}_2, \mathbf{H}_1$ and $\mathbf{H}_2$, all of which are in the transverse plane $\zeta = \text{const}$.

Since the two complex vector wave equations (8.1.2a, b) are of second order, their solutions incorporate two pairs of complex vector constants. But, due to the mutual physical connection between vectors $\mathbf{E}$ and $\mathbf{H}$, only two of the complex vector constants are independent for a unique electromagnetic plane wave, while the other two are not. We can find out this mutual relationship between the constants in solutions (8.2.13a, b) by substituting these solutions into the main Maxwell's equation (8.1.1a). In such a way we obtain the complex equation

$$\left(a_\zeta \frac{\partial}{\partial \zeta} \right) \times [\mathbf{E}_1 e^{st + jk\zeta} + \mathbf{E}_2 e^{st - jk\zeta}] + \mu \frac{\partial}{\partial t} [\mathbf{H}_1 e^{st + jk\zeta} + \mathbf{H}_2 e^{st - jk\zeta}] = 0.$$

After differentiation we get

$$[jk(a_\zeta \times \mathbf{E}_1) + \mu s \mathbf{H}_1] e^{st + jk\zeta} - [jk(a_\zeta \times \mathbf{E}_2) - \mu s \mathbf{H}_2] e^{st - jk\zeta} = 0.$$

Equating the coefficients of the corresponding exponential functions to zero results in the following two expressions

$$\mathbf{H}_1 = -\frac{jk}{\mu s}(\mathbf{a}_\zeta \times \mathbf{E}_1) \quad , \quad \mathbf{H}_2 = \frac{jk}{\mu s}(\mathbf{a}_\zeta \times \mathbf{E}_2) \quad , \quad (8.2.15a,b)$$

and the solutions (8.2.13a,b) for the field vectors can be expressed only by two independent complex vector integration constants $\mathbf{E}_1$ and $\mathbf{E}_2$ as

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_1 e^{st+jk\zeta} + \mathbf{E}_2 e^{st-jk\zeta} \quad , \\ \mathbf{H} &= -\frac{jk}{\mu s}(\mathbf{a}_\zeta \times \mathbf{E}_1) e^{st+jk\zeta} + \frac{jk}{\mu s}(\mathbf{a}_\zeta \times \mathbf{E}_2) e^{st-jk\zeta} \quad . \end{aligned} \quad (8.2.16a,b)$$

This system of equations shows that the electric and magnetic fields of a unique electromagnetic wave are *always spatially mutually perpendicular*, and that in a TEM wave they are both perpendicular to the direction of wave propagation. If we separate the vector equations into complex components we obtain the equations

$$(\mathbf{a}_\xi H_{1\xi} + \mathbf{a}_\eta H_{1\eta}) = -\frac{jk}{\mu s}(\mathbf{a}_\eta E_{1\xi} - \mathbf{a}_\xi E_{1\eta}) \quad , \quad (\mathbf{a}_\xi H_{2\xi} + \mathbf{a}_\eta H_{2\eta}) = \frac{jk}{\mu s}(\mathbf{a}_\eta E_{2\xi} - \mathbf{a}_\xi E_{2\eta}) \quad ,$$

and from these equations we get four scalar complex expressions

$$H_{1\xi} = \frac{jk}{\mu s} E_{1\eta} \quad ; \quad H_{1\eta} = -\frac{jk}{\mu s} E_{1\xi} \quad , \quad H_{2\xi} = -\frac{jk}{\mu s} E_{2\eta} \quad ; \quad H_{2\eta} = \frac{jk}{\mu s} E_{2\xi} \quad .$$

It is obvious that for individual complex components of the field, we may write

$$\begin{aligned} E_\xi &= E_{1\xi} e^{st+jk\zeta} + E_{2\xi} e^{st-jk\zeta} \quad , \quad E_\eta = E_{1\eta} e^{st+jk\zeta} + E_{2\eta} e^{st-jk\zeta} \quad , \\ H_\eta &= -\frac{jk}{\mu s} E_{1\xi} e^{st+jk\zeta} + \frac{jk}{\mu s} E_{2\xi} e^{st-jk\zeta} \quad , \\ H_\xi &= \frac{jk}{\mu s} E_{1\eta} e^{st+jk\zeta} - \frac{jk}{\mu s} E_{2\eta} e^{st-jk\zeta} \quad . \end{aligned} \quad (8.2.17a - d)$$

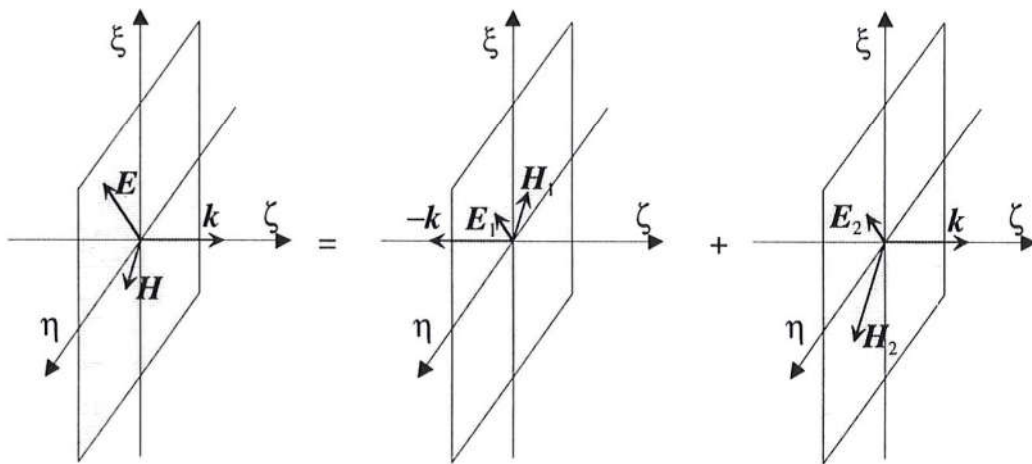


Figure 8.2.2. The separation of the total field of a plane TEM wave ($\mathbf{E}, \mathbf{H}$) into inverse ($\mathbf{E}_1, \mathbf{H}_1$) and direct ($\mathbf{E}_2, \mathbf{H}_2$) waves that are propagating in the direction of the $-\zeta$ - and $+\zeta$ -axes

The mutual relationship between the field vectors and their components are shown in Figure 8.2.2.

8.2.3. Plane waves sinusoidal in time

We suppose that *the field is sinusoidal with time*, which means that the field varies with respect to time as either $\sin \omega t$ or $\cos \omega t$, where ω is constant angular frequency. We shall often use *phasors* as a symbolical representation of sinusoidal functions in a complex domain.

Suppose the time function $T(t)$ defined by the differential equation (8.2.10) is sinusoidally dependent on time, $T = T_0 e^{j\omega t}$. In this case equation (8.2.10) transforms to

$$-\omega^2 T_0 e^{j\omega t} + j\omega \frac{\kappa}{\epsilon} T_0 e^{j\omega t} + \frac{k^2}{\mu\epsilon} T_0 e^{j\omega t} = 0 ,$$

which results in the complex separation constant $k^2 = \omega^2 \mu\epsilon - j\omega\mu\kappa$. We often write the separation constant in a complex domain as

$$k = k' \left(1 + \frac{\kappa}{j\omega\epsilon} \right)^{\frac{1}{2}} ; \quad k' = \omega \sqrt{\mu\epsilon} = \frac{\omega}{c} . \quad (8.2.18a,b)$$

It is convenient to introduce (if we suppose that $\beta > 0$ always)

$$\gamma = jk = \alpha + j\beta , \quad (8.2.19)$$

as *the propagation constant*. This constant replaces jk in such a way that its real part represents an attenuation and its imaginary part represents a phase constant. Obviously, one can easily calculate the propagation constant as $\gamma^2 = -\omega^2 \mu\epsilon + j\omega\mu\kappa$. Now we need to substitute k into the characteristic equation (8.2.11) and we thus obtain

$$s^2 + \frac{\kappa}{\epsilon} s + \omega^2 - j\omega \frac{\kappa}{\epsilon} = 0 \quad \Rightarrow \quad s_{1,2} = -\frac{\kappa}{2\epsilon} \pm \left(\frac{\kappa}{2\epsilon} + j\omega \right) .$$

We shall choose the root $s = j\omega$. The solutions of the wave equations (8.1.2a,b) in a complex domain where the fields are sinusoidal with respect to time are

$$\begin{aligned} E(\zeta, \omega t) &= E_1 e^{\alpha\zeta + j(\omega t + \beta\zeta)} + E_2 e^{-\alpha\zeta + j(\omega t - \beta\zeta)} , \\ H(\zeta, \omega t) &= H_1 e^{\alpha\zeta + j(\omega t + \beta\zeta)} + H_2 e^{-\alpha\zeta + j(\omega t - \beta\zeta)} . \end{aligned} \quad (8.2.20a,b)$$

Most frequently, we write these complex solutions for the fields $E(\zeta, \omega t)$ and $H(\zeta, \omega t)$ in a simpler and shorter form by omitting the time function $e^{j\omega t}$. This means that we introduce phasors of fields $\underline{E}$ and $\underline{H}$ as $E(\zeta, \omega t) = \underline{E} e^{j\omega t}$ and $H(\zeta, \omega t) = \underline{H} e^{j\omega t}$, where *the field phasors* are determined by

$$\underline{E} = E_1 e^{+\gamma\zeta} + E_2 e^{-\gamma\zeta} , \quad \underline{H} = H_1 e^{+\gamma\zeta} + H_2 e^{-\gamma\zeta} , \quad (8.2.20c,d)$$

which are solutions of the phasor equations $\Delta \underline{E} = \gamma^2 \underline{E}$ and $\Delta \underline{H} = \gamma^2 \underline{H}$. This is a simple way

of writing wave equations, thus simplifying any further calculation as well. The symbols $\underline{E}(\zeta, j\omega)$ and $\underline{H}(\zeta, j\omega)$ are often used to denote the field phasors. Real (instantaneous) values of the fields are simply determined from the field phasors using the expressions

$$\mathbf{E}_{Re}(\zeta, \omega t) = \text{Re}\{\underline{E}e^{j\omega t}\} \quad , \quad \mathbf{H}_{Re}(\zeta, \omega t) = \text{Re}\{\underline{H}e^{j\omega t}\} . \quad (8.2.21a,b)$$

If we expand the solutions (8.2.20a,b) with respect to components from expressions (8.2.17a – d), we obtain four scalar equations in a complex domain. We express the complex constants in these relations as

$$E_{1\xi} = A_1 e^{j\delta_1} \quad , \quad E_{2\xi} = A_2 e^{j\delta_2} \quad , \quad E_{1\eta} = B_1 e^{j\nu_1} \quad , \quad E_{2\eta} = B_2 e^{j\nu_2} \quad , \quad (8.2.22a - d)$$

where all new constants, the moduli A_1, A_2, B_1, B_2 and the arguments $\delta_1, \delta_2, \nu_1, \nu_2$, are real. Thus, the components of the electric field of a plane wave (8.2.25a – d) can be written as

$$\begin{aligned} E_{\xi} &= A_1 e^{+a\zeta + j(\omega t + \beta\zeta + \delta_1)} + A_2 e^{-a\zeta + j(\omega t - \beta\zeta + \delta_2)} \quad , \\ E_{\eta} &= B_1 e^{+a\zeta + j(\omega t + \beta\zeta + \nu_1)} + B_2 e^{-a\zeta + j(\omega t - \beta\zeta + \nu_2)} \quad , \end{aligned} \quad (8.2.23a,b)$$

and the components of the magnetic field can be written as

$$\begin{aligned} H_{\xi} &= -\frac{j\gamma}{\mu\omega} B_1 e^{+a\zeta + j(\omega t + \beta\zeta + \nu_1)} + \frac{j\gamma}{\mu\omega} B_2 e^{-a\zeta + j(\omega t - \beta\zeta + \nu_2)} \quad , \\ H_{\eta} &= \frac{j\gamma}{\mu\omega} A_1 e^{-a\zeta + j(\omega t + \beta\zeta + \delta_1)} - \frac{j\gamma}{\mu\omega} A_2 e^{+a\zeta + j(\omega t - \beta\zeta + \delta_2)} \quad . \end{aligned} \quad (8.2.23c,d)$$

If we arrange the coefficients that arise from the moduli of the complex components of the magnetic field we obtain

$$\begin{aligned} \frac{j\gamma}{\mu\omega} &= -\frac{\beta - j\alpha}{\mu\omega} = -\frac{\sqrt{\alpha^2 + \beta^2}}{\mu\omega} e^{j\psi} = \frac{-|\gamma|}{\mu\omega} e^{j\psi} \quad , \\ \frac{-j\gamma}{\mu\omega} &= \frac{\beta - j\alpha}{\mu\omega} = \frac{\sqrt{\alpha^2 + \beta^2}}{\mu\omega} e^{j\psi} = \frac{|\gamma|}{\mu\omega} e^{j\psi} \quad , \end{aligned} \quad (8.2.24a,b)$$

where

$$|\gamma| = \sqrt{\alpha^2 + \beta^2} \quad ; \quad \tan \psi = \frac{-\alpha}{\beta} . \quad (8.2.25a,b)$$

After this, the components of the magnetic field (8.2.23c,d) of a plane TEM wave can be written as

$$\begin{aligned}
 H_{\xi} &= \frac{|\gamma|}{\mu\omega} B_1 e^{+\alpha\zeta + j(\omega t + \beta\zeta + \nu_1 + \psi)} - \frac{|\gamma|}{\mu\omega} B_2 e^{-\alpha\zeta + j(\omega t - \beta\zeta + \nu_2 + \psi)}, \\
 H_{\eta} &= -\frac{|\gamma|}{\mu\omega} A_1 e^{+\alpha\zeta + j(\omega t + \beta\zeta + \delta_1 + \psi)} + \frac{|\gamma|}{\mu\omega} A_2 e^{-\alpha\zeta + j(\omega t - \beta\zeta + \delta_2 + \psi)}.
 \end{aligned}
 \tag{8.2.26a,b}$$

Since equations (8.2.23a,b) and (8.2.26a,b) are solutions of wave equations (8.1.2a,b) in a complex domain, both real and imaginary parts have to be solutions too.

In order to express the actual physical essence of the field, we shall use the real part of equations (8.2.23a,b) and (8.2.26a,b) as solutions for components of a plane TEM wave in a real space-time domain

$$E_{\xi Re} = A_1 e^{+\alpha\zeta} \cos(\omega t + \beta\zeta + \delta_1) + A_2 e^{-\alpha\zeta} \cos(\omega t - \beta\zeta + \delta_2), \tag{8.2.27a - d}$$

$$E_{\eta Re} = B_1 e^{+\alpha\zeta} \cos(\omega t + \beta\zeta + \nu_1) + B_2 e^{-\alpha\zeta} \cos(\omega t - \beta\zeta + \nu_2),$$

$$H_{\xi Re} = \frac{|\gamma|}{\mu\omega} B_1 e^{+\alpha\zeta} \cos(\omega t + \beta\zeta + \nu_1 + \psi) - \frac{|\gamma|}{\mu\omega} B_2 e^{-\alpha\zeta} \cos(\omega t - \beta\zeta + \nu_2 + \psi)$$

$$H_{\eta Re} = -\frac{|\gamma|}{\mu\omega} A_1 e^{+\alpha\zeta} \cos(\omega t + \beta\zeta + \delta_1 + \psi) + \frac{|\gamma|}{\mu\omega} A_2 e^{-\alpha\zeta} \cos(\omega t - \beta\zeta + \delta_2 + \psi)$$

Obviously, the fields $\mathbf{E}$ and $\mathbf{H}$, as well as all their components, are *sinusoidal in time and sinusoidally attenuated in space*.

Waves propagating in two opposite directions, i.e. in direction $+\zeta$ and in direction $-\zeta$, arise in all the previously described solutions for a plane wave as a matter of course. Therefore, we call these waves *travelling waves*. The mutual relationship between the fields of direct and inverse travelling waves are shown in Figure 8.2.2.

We have presented one pair of field components of a direct plane wave ($E_{2\xi}$, $H_{2\eta}$), which are obtained from the system of solutions (8.2.27a - d) as it propagates along the axis $+\zeta$ in Figure 8.2.3.

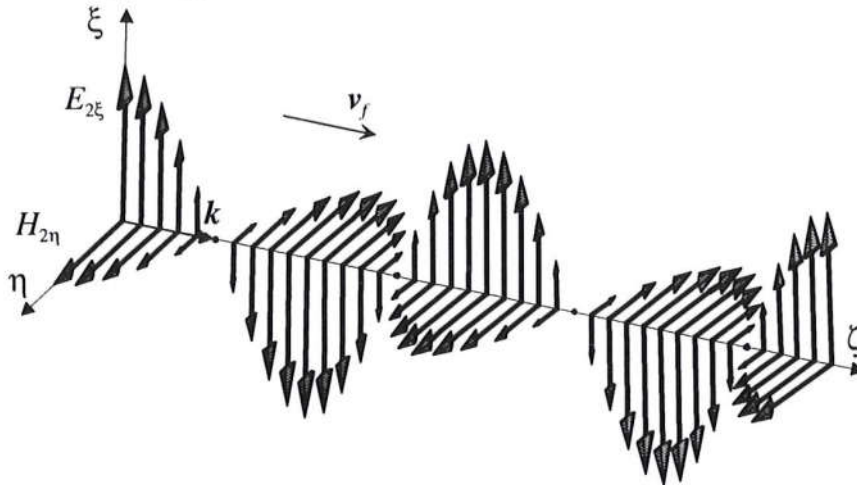


Figure 8.2.3. The electric and magnetic field of a plane sinusoidal direct travelling TEM wave propagating in an unbounded space filled with a lossless material in the direction of the $+\zeta$ -axis

For the sake of simplicity, we have supposed that the material in which the wave is propagated is lossless and therefore the fields are in phase $\psi = 0$, there is no damping $\alpha = 0$, and for the initial condition, we have chosen that the angle $\delta_2 = 0$. The propagation

of the wave at a moment in time $t = 0$ is shown in Figure 8.2.3.

The arguments $(\omega t \pm \beta \zeta + \varphi)$ of the periodical functions in the system (8.2.27a – d) are called *phases*. The angle φ determined by the initial conditions and by various relationships regarding the material, is called the *phase angle*. The vectors $\mathbf{E}$ and $\mathbf{H}$ are constant in the complete plane $\zeta = \text{const.}$ at any moment in time, and we call this plane the *plane of constant phase*.

An observer has to move in the direction of propagation of an individual wave in such a way that he always measures the same constant phase in order to determine the phase velocity. Accordingly, for all components of a plane wave given by equations (8.2.27a – d) we can write that $\omega t \pm \beta \zeta + \varphi = \text{const.}$, which, after differentiation, results in $\omega dt \mp \beta d\zeta = 0$. Therefore, the *phase velocity* is given as

$$v_f = \frac{d\zeta}{dt} = \pm \frac{\omega}{\beta} . \quad (8.2.28)$$

This simply represents the propagation velocity of the phase of the wave and it is not always necessary that it coincides with the velocity of propagation of energy of the wave or signal. In fact, v_f can be greater than the critical speed of light c_0 without any conflict with postulates of the theory of relativity. Obviously, $v_f = +\omega/\beta$ means that the phase of the wave moves in the direction of the $+\zeta$ -axis for a wave that we call a *direct wave*, i.e. with a phase $(\omega t - \beta \zeta + \varphi)$, and $v_f = -\omega/\beta$ is the phase velocity for an *inverse wave* that is moving in the direction of the $-\zeta$ -axis with a phase $(\omega t + \beta \zeta + \varphi)$. The frequency f of the field is related to the angular frequency ω as $f = \omega/(2\pi)$ or $\omega = 2\pi f$. The time period is

$$T = \frac{1}{f} = \frac{2\pi}{\omega} , \quad (8.2.29)$$

and is measured in seconds. The spatial period is called the *wavelength*, it is measured in metres and defined as

$$\lambda = \frac{v_f}{f} , \quad (8.2.30a)$$

which, by usage of (8.2.28), results in

$$\lambda = \frac{2\pi}{\beta} . \quad (8.2.30b)$$

We shall determine two very important constants α and β using equation (8.2.19) and separating this complex equation into two real equations $\alpha^2 - \beta^2 = -\omega^2 \mu \epsilon$ and $2\alpha\beta = \omega \mu \kappa$. By the simultaneous solution of these equations, we obtain the constants

$$\alpha = \frac{\omega}{\sqrt{2}c} \left(\sqrt{1 + \left(\frac{\kappa}{\omega \epsilon} \right)^2} - 1 \right)^{1/2} , \quad \beta = \frac{\omega}{\sqrt{2}c} \left(\sqrt{1 + \left(\frac{\kappa}{\omega \epsilon} \right)^2} + 1 \right)^{1/2} , \quad (8.2.31a,b)$$

where $c = 1/(\mu \epsilon)^{1/2}$. The multiplier $e^{\pm \alpha \zeta}$ represents the attenuation of the wave, and we thus call α the *attenuation constant*. Clearly, a direct wave in the direction $+\zeta$ is attenuated by a factor $e^{-\alpha \zeta}$, and an inverse wave in the direction $-\zeta$ is attenuated by a factor $e^{+\alpha \zeta}$, as

shown in Figure 8.2.4.

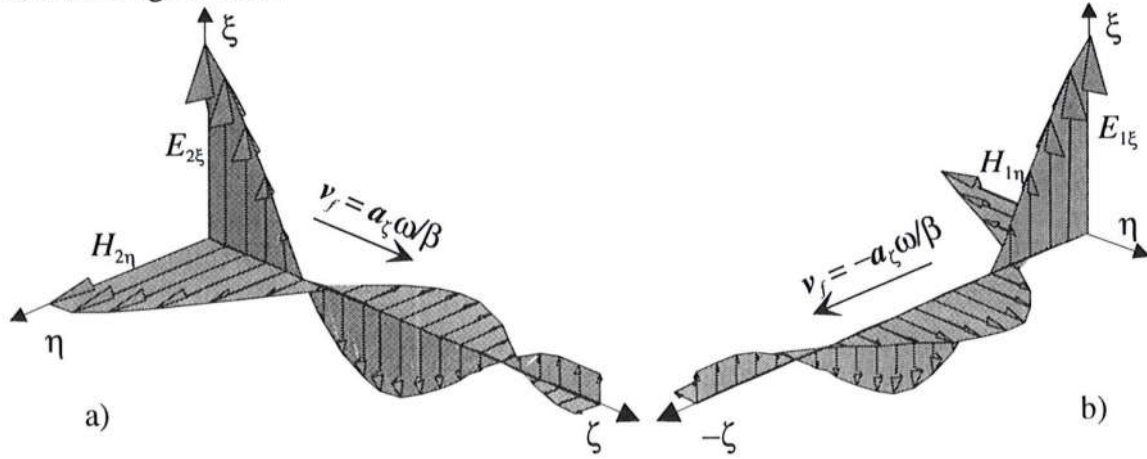


Figure 8.2.4. Attenuation of plane waves, a) direct by a factor $e^{-\alpha\zeta}$ and b) inverse by a factor $e^{+\alpha\zeta}$

The terms $\pm \beta\zeta$ are parts of the phase of both direct and inverse waves, and therefore we call β the *phase constant*. The phase velocity can be calculated from equation (8.2.28) using the expression for phase constant β (8.2.31b), as

$$v_f = \pm c \left[\frac{1}{2} \left(\sqrt{1 + \left(\frac{\kappa}{\omega\epsilon} \right)^2} + 1 \right) \right]^{-1/2}, \quad (8.2.32)$$

where $c = 1/(\mu\epsilon)^{1/2}$. The quantities α , β and v_f determine the behaviour of a plane sinusoidal wave in an unbounded space dependent on angular frequency ω and on the constitutive parameters ϵ , μ , κ of the material in that space.

8.2.4. Plane waves sinusoidal in space

In the previous Section 8.2.3, we assumed that the field is sinusoidally time dependent (as $e^{j\omega t}$, or $\cos \omega t$). This resulted in the complex separation constant k . As a consequence, we obtained an electromagnetic wave whose fields were sinusoidal in time and sinusoidally attenuated in space.

Suppose the separation constant k is real. In this case, the root becomes complex according to (8.2.12)

$$s = -\frac{\kappa}{2\epsilon} \pm j \sqrt{\frac{k^2}{\mu\epsilon} - \left(\frac{\kappa}{2\epsilon} \right)^2} = -p \pm jg. \quad (8.2.33)$$

One solution of the complex wave equations (8.1.2a,b) in this case will be given by the expressions

$$\begin{aligned}
E(\zeta, t) &= E_1 e^{-pt+j(gt+k\zeta)} + E_2 e^{-pt+j(gt-k\zeta)} , \\
H(\zeta, t) &= H_1 e^{-pt+j(gt+k\zeta)} + H_2 e^{-pt+j(gt-k\zeta)} .
\end{aligned}
\tag{8.2.34a,b}$$

These solutions will be equal to those in (8.2.13a,b) for a plane wave in cases where g is real. If $(\kappa/2\epsilon)^2 < k^2/\mu\epsilon$, then the quantity g is an imaginary number and the fields (8.2.34a,b) can be represented as a plane wave that propagates in both directions of the ζ -axis with phase velocity

$$v = \frac{d\zeta}{dt} = \pm \frac{g}{k} . \tag{8.2.35}$$

The amplitude of this wave at any point ζ decreases exponentially with time t , and its attenuation is determined by the time constant $\tau = \epsilon/\kappa$. Accordingly, if the sinusoidal distribution of the field with respect to the ζ coordinate is given at a moment t , then, if $(\kappa/2\epsilon)^2 < k^2/\mu\epsilon$, a sinusoidal plane wave can exist in the given material as demonstrated by equations (8.2.34a,b). This is surely the case in a dielectric where $\kappa = 0$. This condition can be satisfied in a good insulator, whose constitutive parameters are close to the constitutive parameters of an ideal dielectric.

In the second case, when $(\kappa/2\epsilon)^2 > k^2/\mu\epsilon$, the quantity g is imaginary, and g is a real quantity. We introduce a new real quantity $r = p - jg$. In this case, solutions of the equations (8.1.2a,b) using a supposition of the *sinusoidal distribution of the field in space*, are

$$\begin{aligned}
E(\zeta, t) &= E_1 e^{-rt+jk\zeta} + E_2 e^{-rt-jk\zeta} , \\
H(\zeta, t) &= H_1 e^{-rt+jk\zeta} + H_2 e^{-rt-jk\zeta} .
\end{aligned}
\tag{8.2.36a,b}$$

In a real domain, where we suppose that the constant vectors are real, these equations are

$$\begin{aligned}
E_{Re}(\zeta, t) &= E_1 e^{-rt} \cos(k\zeta) - E_2 e^{-rt} \cos(k\zeta) , \\
H_{Re}(\zeta, t) &= H_1 e^{-rt} \cos(k\zeta) - H_2 e^{-rt} \cos(k\zeta) .
\end{aligned}
\tag{8.2.37a,b}$$

Obviously, this represents neither propagation of an electromagnetic field nor of energy, that is, not in any way we have met so far. Electric and magnetic fields are sinusoidally distributed along the ζ axis, but they *decay with time* t . There is no movement of the initial waveform along the ζ -axis. This is a wave phenomenon which has degenerated into a diffusion. The electromagnetic energy in such a field propagates by diffusion, and not as a wave, similar to heat transfer. This is possible only in materials in which the relationship

$$\left(\frac{\kappa}{2\epsilon} \right)^2 > \frac{k^2}{\mu\epsilon} ,$$

is valid. These are materials with $\kappa > 0$, i.e. they are conductors. If a material is a better conductor, the effect of diffusion is more pronounced.

8.2.5. Fundamental characteristics of sinusoidal plane waves

8.2.5.1. Wave impedance

The notion of impedance is well known in the theory of electric circuits and in the theory of lines. It links voltage and current in both amplitude and phase. The impedance of a material in an unbounded space for waves of a given form and type represents the response of the material to excitation by such a wave.

In Section 8.2.3, we showed that the system of equations (8.2.23a – d) written in a complex domain results in the solution for components of the electric and magnetic field of a sinusoidal plane TEM wave. These equations are comprised of two components of electric and magnetic field of a direct wave, and two components of electric and magnetic field of an inverse wave. The quotients of the moduli of the corresponding components of electric and magnetic field are *positive for direct waves* and we denote them as $+Z$

$$Z = \frac{E_{2\xi}}{H_{2\eta}} = \frac{A_2 e^{j\delta_2}}{\frac{-j\gamma}{\mu\omega} A_2 e^{j\delta_2}} = \frac{j\omega\mu}{\gamma} \quad , \quad Z = -\frac{E_{2\eta}}{H_{2\xi}} = -\frac{B_2 e^{j\nu_2}}{\frac{j\gamma}{\mu\omega} B_2 e^{j\nu_2}} = \frac{j\omega\mu}{\gamma} \quad , \quad (8.2.38a,b)$$

while they are negative for inverse waves and we denote them with $-Z$

$$-Z = \frac{E_{1\xi}}{H_{1\eta}} = -\frac{j\omega\mu}{\gamma} \quad , \quad -Z = \frac{-E_{1\eta}}{H_{1\xi}} = -\frac{j\omega\mu}{\gamma} \quad . \quad (8.2.38c,d)$$

Accordingly, the *wave impedance* is defined as the ratio of the corresponding complex components of electric and magnetic field of the corresponding direct sinusoidal electromagnetic wave. For the ratio of the pair of components of an inverse wave, we use the negative wave impedance. Therefore, the wave impedance is

$$Z = \frac{j\omega\mu}{\gamma} \quad . \quad (8.2.39a)$$

Using expression (8.2.19), where α and β are determined by expressions (8.2.31a,b), we can write the complex wave impedance in exponential form as

$$Z = \frac{\omega\mu}{|\gamma|} e^{j\varphi} = |Z| e^{j\varphi} \quad , \quad (8.2.39b)$$

where the modulus and argument of Z are determined as

$$|Z| = \frac{\omega\mu}{\sqrt{\alpha^2 + \beta^2}} \quad ; \quad \tan \varphi = \frac{\alpha}{\beta} \quad . \quad (8.2.39c,d)$$

In free space, the wave impedance reduces to a real number

$$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 376.6 \Omega . \quad (8.2.39e)$$

It is suitable to write the wave impedance as

$$Z = Z' \left(1 + \frac{\kappa}{j\omega\epsilon} \right)^{-1/2} ; \quad Z' = \sqrt{\frac{\mu}{\epsilon}} . \quad (8.2.39f)$$

By introduction of the impedance Z , we can express the components of the fields of a *direct*, linearly polarised sinusoidal plane wave as

$$E_\xi = E_{\xi 0}^+ e^{-\alpha\zeta + j(\omega t - \beta\zeta)} , \quad H_\eta = \frac{E_{\xi 0}^+}{|Z|} e^{-\alpha\zeta + j(\omega t - \beta\zeta - \varphi)} , \quad (8.2.40a,b)$$

and for an *inverse wave* as

$$E_\xi = E_{\xi 0}^- e^{+\alpha\zeta + j(\omega t + \beta\zeta)} , \quad H_\eta = -\frac{E_{\xi 0}^-}{|Z|} e^{+\alpha\zeta + j(\omega t + \beta\zeta - \varphi)} , \quad (8.2.41a,b)$$

where $E_{\xi 0}^+$ and $E_{\xi 0}^-$ are real numbers.

8.2.5.2. Energy flow

In Chapter 3, we defined Poynting's vector that determines the electromagnetic energy transmitted by a plane wave through unit surface of a plane of constant phase in unit time in the direction of wave propagation. In sinusoidal systems, we usually do calculations using time-averaged values of power. *The time-averaged value of energy flow transmitted in unit time through unit surface* is calculated using expression for Poynting's vector, which is written in a complex domain as

$$N_{av} = \frac{1}{2} \mathbf{E} \times \mathbf{H}^* , \quad (8.2.42)$$

where $\mathbf{H}^*$ is the complex conjugate value of the vector of magnetic field $\mathbf{H}$.

Let us, for the sake of simplicity, prescribe a direct wave with a vector $\mathbf{E}$ linearly polarised in the direction of the ξ -axis and with a vector $\mathbf{H}$ in the direction of the η -axis. The wave propagates in the direction of the $+\zeta$ -axis as

$$E_\xi = E_{\xi 0}^+ e^{-\alpha\zeta + j(\omega t - \beta\zeta + \nu)} , \quad H_\eta = \frac{E_{\xi 0}^+}{|Z|} e^{-\alpha\zeta + j(\omega t - \beta\zeta + \nu - \varphi)} , \quad (8.2.43a,b)$$

and its complex conjugate H_η^* is

$$H_{\eta}^* = \frac{E_{\xi 0}^+}{|Z|} e^{-\alpha\zeta - j(\omega t - \beta\zeta + \nu - \varphi)} . \quad (8.2.43c)$$

The time-averaged value of the complex Poynting's vector is, according to (8.2.42),

$$(N_{\zeta})_{av} = \frac{1}{2} E_{\xi} H_{\eta}^* = \frac{E_{\xi 0}^{+2}}{2|Z|} e^{-2\alpha\zeta + j\varphi} . \quad (8.2.44a)$$

The real part of the complex Poynting's vector $N_{\zeta,av}$ represents the time-averaged electromagnetic energy that is transmitted by a plane wave through space in the direction of the $+\zeta$ -axis crossing unit area of the plane of constant phase (ξ, η) in unit time

$$(N_{\zeta})_{Re,av} = \frac{E_{\xi 0}^{+2}}{2|Z|} e^{-2\alpha\zeta} \cos\varphi , \quad (8.2.44b)$$

where $E_{\xi 0}^+$ is a real number and $\cos\varphi$ can be determined as

$$\cos\varphi = \frac{\beta}{\sqrt{\alpha^2 + \beta^2}} . \quad (8.2.44c)$$

The multiplier $\cos\varphi$ arises due to the phase shift between the complex vectors E_{ξ} and H_{η} , or in other words, because the phasor $\underline{H}_{\eta}$ is retarded in phase by angle φ with respect to phasor $\underline{E}_{\xi}$. In ideal dielectrics $\kappa = 0$, and therefore $\alpha = 0$, which results in $\cos\varphi = 1$, or $\varphi = 0$. Thus, there is no phase shift between the phasors of electric and magnetic field in dielectrics. If we substitute (8.2.39) and (8.2.44c) into expression (8.2.44b), we obtain

$$(N_{\zeta})_{Re,av} = \frac{\beta}{2\mu\omega} e^{-2\alpha\zeta} E_{\xi 0}^{+2} . \quad (8.2.44d)$$

It is obvious that with an increase of ζ , or by the propagation of the wave, the specific power that is transmitted by the wave decreases due to the term $e^{-2\alpha\zeta}$. This loss of electromagnetic power is converted into heat in the material in which the wave propagates. We calculate the real losses within unit volume of the material as being

$$(P_g')_{Re,av} = -\nabla \cdot (N_{\zeta})_{Re,av} = -\frac{\partial N_{\zeta}}{\partial \zeta} = \frac{\kappa}{2} e^{-2\alpha\zeta} E_{\xi 0}^{+2} . \quad (8.2.45)$$

This is the part of the energy transmitted by the electromagnetic wave that is converted to heat loss in unit time within unit volume.

8.2.5.3. Polarization

A locus of the tip of the vector $\mathbf{E}$ of a plane wave as a function of time at a given point in space determines, on the whole, the polarization of the wave. In the case of linear polarization, this locus is a line segment, while in the case of circular polarization, this locus is a circle. In a general case, the curve determined by the tip of the vector $\mathbf{E}$ is an

ellipse whose dimensions can be calculated using analytical geometry.

A direct sinusoidal homogeneous plane wave is completely defined by the frequency of the field ω , direction of propagation $+\zeta$, and the magnitude and direction of the vector of the electric field, for example $\mathbf{E}_{\xi 0} = \mathbf{a}_{\xi} E_{\xi 0}$. If the space of propagation is unbounded and if the constitutive parameters ϵ , μ , and κ of a material are given, then vector of the magnetic field $\mathbf{H}_{\eta 0} = \mathbf{a}_{\eta} H_{\eta 0}$ is completely defined. We write the electric field vector in a real domain as

$$\mathbf{E}_{Re}(\zeta, t) = \mathbf{a}_{\xi} \text{Re} [E_{\xi 0} e^{-\alpha \zeta + j(\omega t - \beta \zeta)}] = \mathbf{a}_{\xi} |E_{\xi 0}| e^{-\alpha \zeta} \cos(\omega t - \beta \zeta + \delta) , \quad (8.2.46a)$$

where $E_{\xi 0} = |E_{\xi 0}| e^{j\delta}$. This wave is linearly polarized in the direction of the ξ axis. In a similar way, we can write a linearly polarized wave in the direction of the η axis as

$$\mathbf{E}_{Re}(\zeta, t) = \mathbf{a}_{\eta} \text{Re} [E_{\eta 0} e^{-\alpha \zeta + j(\omega t - \beta \zeta)}] = \mathbf{a}_{\eta} |E_{\eta 0}| e^{-\alpha \zeta} \cos(\omega t - \beta \zeta + \nu) , \quad (8.2.46b)$$

where $E_{\eta 0} = |E_{\eta 0}| e^{j\nu}$. A linear combination of these two fields is the solution of the wave equation as well, and we obtain a new wave

$$\mathbf{E}_{Re}(\zeta, t) = \text{Re} [\mathbf{a}_{\xi} |E_{\xi 0}| E^{-\alpha \zeta + j(\omega t - \beta \zeta + \delta)} + \mathbf{a}_{\eta} |E_{\eta 0}| E^{-\alpha \zeta + j(\omega t - \beta \zeta + \nu)}] . \quad (8.2.47)$$

By an appropriate choice of amplitudes and phase angles of components of the electric field vector, we can influence the polarization of the field and determine if the field will be linearly, circularly or generally elliptically polarized. Suppose that the following expressions are given

$$|E_{\xi 0}| = |E| \cos \varphi \quad ; \quad |E_{\eta 0}| = |E| \sin \varphi \quad ; \quad \delta = \nu = \psi , \quad (8.2.48a - c)$$

where the quantities $|E|$, φ and ψ are also given. Substitution of these expressions into equation (8.2.49) results in the field

$$\mathbf{E}_{Re}(\zeta, t) = [\mathbf{a}_{\xi} \cos \varphi + \mathbf{a}_{\eta} \sin \varphi] |E| e^{-\alpha \zeta} \cos(\omega t - \beta \zeta + \psi) , \quad (8.2.49)$$

that represents a linearly polarized wave in the direction of the vector ρ with angle φ with respect to the ξ -axis as shown in Figure 8.2.5 a).

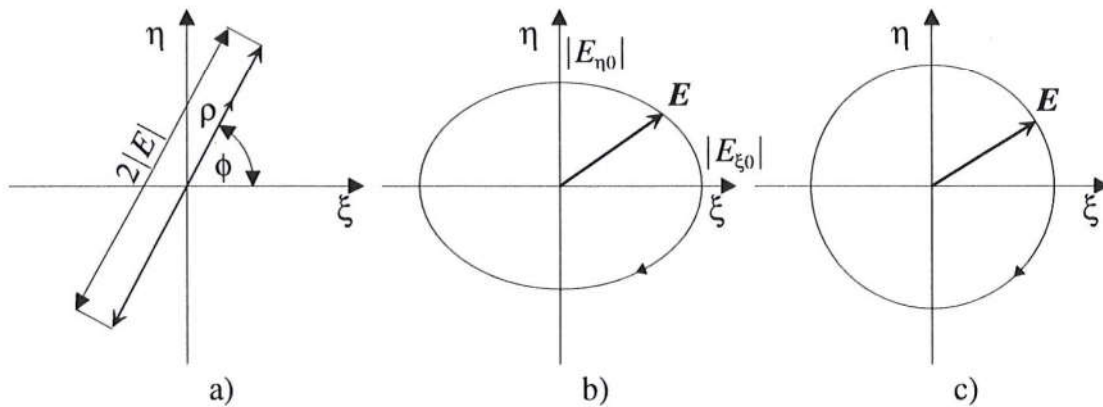


Figure 8.2.5. Linearly a), elliptically b) and circularly c) polarized electric field of a plane wave

Let us suppose in the second case that the following quantities from equation (8.2.47) are

$$|E_{\xi 0}| \neq |E_{\eta 0}| \quad ; \quad \nu = \delta \pm \frac{\pi}{2} . \quad (8.2.50a,b)$$

Substitution of these expressions into equation (8.2.47) and the squaring of the result gives us

$$\left[\frac{E_{\xi}(\zeta, t)}{|E_{\xi 0}|} \right]^2 + \left[\frac{E_{\eta}(\zeta, t)}{|E_{\eta 0}|} \right]^2 = e^{-2\alpha\zeta} . \quad (8.2.51)$$

This equation determines the relationship between components E_{ξ} and E_{η} . It represents an equation of an ellipse at a given $\zeta = \text{const}$. The semi-axes of the ellipse shown in Figure 8.2.5 b) are $|E_{\xi 0}|e^{-\alpha\zeta}$ and $|E_{\eta 0}|e^{-\alpha\zeta}$. We say that such a wave is elliptically polarized. In general, the semi-axes of an ellipse can be at any position.

If we suppose that $|E_{\xi 0}| = |E_{\eta 0}| = |E|$, then equation (8.2.51) is transformed into the equation of a circle of radius $|E|e^{-\alpha\zeta}$. This is a circularly polarized wave. The polarization of waves is used in the processes of wave filtering.

8.3. THE PROPAGATION OF PLANE WAVES

We shall start with a dielectric as, in this case, the wave phenomena and subsequent mathematical analysis are the simplest. In continuation, we shall analyse the more complex case of wave propagation in a real lossy material.

8.3.1. Waves in a dielectric

We shall study the propagation of a sinusoidal homogeneous transverse electromagnetic (TEM) travelling plane wave in a lossless material (dielectric, $\kappa = 0$) that occupies an unbounded space. We shall assume that the frequency of the wave is given, and that it is small enough to be able to consider the constitutive parameters ϵ and μ as constants. With these assumptions we can calculate the fundamental characteristics of the material, α , β , γ , ν_f and Z , that determine the propagation of the wave in a given dielectric. Using equations (8.2.31a,b), we determine the phase and attenuation constant of a dielectric material as

$$\alpha = 0 \quad , \quad \beta = \frac{\omega}{c} = k' , \quad (8.3.1a,b)$$

where

$$c = \frac{1}{\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{\mu_r\epsilon_r}} \frac{1}{\sqrt{\mu_0\epsilon_0}} = \frac{c_0}{n} . \quad (8.3.2)$$

With c we have denoted the speed of light in a dielectric, and with $c_0 = 1/(\mu_0\epsilon_0)^{1/2}$ we have denoted the speed of light in free space. In optics the constant

$$n = \sqrt{\mu_r \epsilon_r} , \quad (8.3.3)$$

is called *the index of refraction*. Since in all but ferromagnetic materials, $\mu_r \approx 1$, the index of refraction is approximately $n \approx \epsilon_r^{1/2}$. This result was first established by Maxwell and it was the foundation of his prediction that light is an electromagnetic phenomenon. We determine the phase velocity and wavelength of a wave in a dielectric as

$$v_f = \frac{\omega}{\beta} = c = \frac{1}{\sqrt{\mu\epsilon}} \quad ; \quad \lambda = \frac{v_f}{f} = \frac{c}{f} = \frac{\lambda_0}{n} , \quad (8.3.4)$$

where $\lambda_0 = c_0/f$ is the wavelength of the same frequency in free space. Finally, we determine the propagation constant and the wave impedance of a material as

$$\gamma = j\beta = j\frac{\omega}{c} \quad , \quad Z = \frac{j\omega\mu}{\gamma} = \frac{j\omega\mu}{j\frac{\omega}{c}} = \sqrt{\frac{\mu}{\epsilon}} = Z' \quad ; \quad \varphi = 0 . \quad (8.3.5a - c)$$

Therefore, there is no power loss during the propagation of a wave in a dielectric and a wave in a dielectric propagates without attenuation. The phase velocity of movement of a plane of constant phase in a dielectric is independent of frequency and is equal to the speed of light in the given material. In all dielectrics, the index of refraction is $n > 1$, and therefore the phase velocity is less than that of a vacuum $v_f = c_0/n$. As a consequence, the wavelength is shorter in a dielectric compared to that of a vacuum $\lambda = \lambda_0/n$.

Since there is no attenuation of the amplitude of a wave, and since the phase velocities of waves with different frequencies are the same, no distortion of a signal composed from a whole spectrum of sinusoidal waves of different frequencies in an ideal dielectric results.

The wave impedance of a dielectric is a real number $Z = Z'$. Therefore, the vectors of electric and magnetic field are spatially perpendicular, while in time they are in phase.

8.3.1.1. Travelling waves

In Section 8.2.3. we derived a few forms of the solution of the wave equations in the case of a travelling sinusoidal wave in a linear material. We shall use here *the phasor form* of the solution (8.2.20c,d) written for a homogeneous TEM wave that contains a direct wave in the +z-direction and an inverse wave in the -z-direction. The field vectors $\underline{E}$ and $\underline{H}$ are in a plane of constant phase $x - y$ and their phasor form is

$$\underline{E} = E_1 e^{\gamma z} + E_2 e^{-\gamma z} \quad , \quad \underline{H} = H_1 e^{\gamma z} + H_2 e^{-\gamma z} . \quad (8.3.6a,b)$$

In a general case, the complex vectors $\underline{E}_1$, $\underline{E}_2$, $\underline{H}_1$ and $\underline{H}_2$ can be resolved according to equation (8.2.14a - d) into four pairs of components in the plane of constant phase

$$\begin{aligned} \mathbf{E}_1 &= \mathbf{a}_x E_{1x} + \mathbf{a}_y E_{1y} \quad , \quad \mathbf{E}_2 = \mathbf{a}_x E_{2x} + \mathbf{a}_y E_{2y} \quad , \\ \mathbf{H}_1 &= \mathbf{a}_x H_{1x} + \mathbf{a}_y H_{1y} \quad , \quad \mathbf{H}_2 = \mathbf{a}_x H_{2x} + \mathbf{a}_y H_{2y} \quad , \end{aligned} \quad (8.3.7a - d)$$

where E_{1x} , E_{1y} , E_{2x} , E_{2y} , H_{1x} , H_{1y} , H_{2x} and H_{2y} are complex scalar quantities. These are complex integration constants and according to (8.2.22a - d) we have to prescribe four of them, and we thus determine the other four using the wave impedance as being

$$\begin{aligned} E_{1x} &= A_1 e^{j\alpha_1} \Rightarrow H_{1x} = \frac{B_1}{Z'} e^{j\beta_1} \quad , \quad E_{2x} = A_2 e^{j\alpha_2} \Rightarrow H_{2x} = -\frac{B_2}{Z'} e^{j\beta_2} \quad , \\ E_{1y} &= B_1 e^{j\beta_1} \Rightarrow H_{1y} = -\frac{A_1}{Z'} e^{j\alpha_1} \quad , \quad E_{2y} = B_2 e^{j\beta_2} \Rightarrow H_{2y} = \frac{A_2}{Z'} e^{j\alpha_2} \quad , \end{aligned} \quad (8.3.8a - d)$$

where A_1 , A_2 , B_1 and B_2 are real constants. Finally, we shall write these general expressions for homogeneous travelling waves in the form of phasors with respect to the x and y components of electric and magnetic field in a dielectric as

$$\begin{aligned} \underline{\mathbf{E}}_x^- &= E_{1x} e^{+\gamma z} = A_1 e^{j(\alpha_1 + k'z)} \quad ; \quad \underline{\mathbf{H}}_y^- = H_{1y} e^{+\gamma z} = -\frac{A_1}{Z'} e^{j(\alpha_1 + k'z)} \quad , \\ \underline{\mathbf{E}}_x^+ &= E_{2x} e^{-\gamma z} = A_2 e^{j(\alpha_2 - k'z)} \quad ; \quad \underline{\mathbf{H}}_y^+ = H_{2y} e^{-\gamma z} = \frac{A_2}{Z'} e^{j(\alpha_2 - k'z)} \quad , \\ \underline{\mathbf{E}}_y^- &= E_{1y} e^{+\gamma z} = B_1 e^{j(\beta_1 + k'z)} \quad ; \quad \underline{\mathbf{H}}_x^- = H_{1x} e^{+\gamma z} = \frac{B_1}{Z'} e^{j(\beta_1 + k'z)} \quad , \\ \underline{\mathbf{E}}_y^+ &= E_{2y} e^{-\gamma z} = B_2 e^{j(\beta_2 - k'z)} \quad ; \quad \underline{\mathbf{H}}_x^+ = H_{2x} e^{-\gamma z} = -\frac{B_2}{Z'} e^{j(\beta_2 - k'z)} \quad , \end{aligned} \quad (8.3.9a - d)$$

where $k' = \omega/c$. The two pairs of components of electric and magnetic field of direct waves propagating in the $+z$ direction are denoted here as $(\underline{\mathbf{E}}_x^+, \underline{\mathbf{H}}_y^+)$ and $(\underline{\mathbf{E}}_y^+, \underline{\mathbf{H}}_x^+)$, while fields of the two other inverse waves that are propagating in the $-z$ direction are denoted as $(\underline{\mathbf{E}}_x^-, \underline{\mathbf{H}}_y^-)$ and $(\underline{\mathbf{E}}_y^-, \underline{\mathbf{H}}_x^-)$. Equations (8.3.9a - d) are general phasor equations of plane travelling waves (in the directions $+z$ and $-z$). Any homogeneous wave in a dielectric can be expressed as the superposition of the corresponding equations from (8.3.9a - d).

For the sake of simplicity, we shall analyse one direct and one inverse travelling wave by a superposition of fields from the first and second equations in expressions (8.3.9a - d). A general x -polarized (electric) field that comprises of two waves with different amplitudes moving in opposite directions can be expressed as

$$\begin{aligned} \underline{\mathbf{E}}_x &= \underline{\mathbf{E}}_x^- + \underline{\mathbf{E}}_x^+ = A_1 e^{j(\alpha_1 + k'z)} + A_2 e^{j(\alpha_2 - k'z)} \quad , \\ \underline{\mathbf{H}}_y &= \underline{\mathbf{H}}_y^- + \underline{\mathbf{H}}_y^+ = -\frac{A_1}{Z'} e^{j(\alpha_1 + k'z)} + \frac{A_2}{Z'} e^{j(\alpha_2 - k'z)} \quad . \end{aligned} \quad (8.3.10a,b)$$

These are the equations of a *general linearly polarized travelling wave*. When $A_1 = 0$, the resulting wave is “purely” *direct* and linearly polarized, and when $A_2 = 0$, the resulting

wave is a “purely” *inverse* travelling wave. The case where $|A_1| = |A_2|$ is of special interest and we shall analyse it in Section 8.3.1.2.

If the real constants $A_1 \neq A_2$, and if, for the sake of simplicity, $\alpha_1 = \alpha_2 = 0$, then the resulting x -polarized electric field that results from the superposition of the two oppositely directed travelling waves at any point of the z -axis can be expressed as

$$\underline{E}_x = (A_1^2 + A_2^2 + 2A_1A_2 \cos 2k'z)^{1/2} e^{-j \arctan \left(\frac{A_1 - A_2}{A_1 + A_2} \tan k'z \right)} = |\underline{E}_x| e^{-j\psi} . \quad (8.3.11)$$

The dependence of the amplitude $|\underline{E}_x|$ of the resultant electric field on the position on the z -axis is shown in Figure 8.3.1.

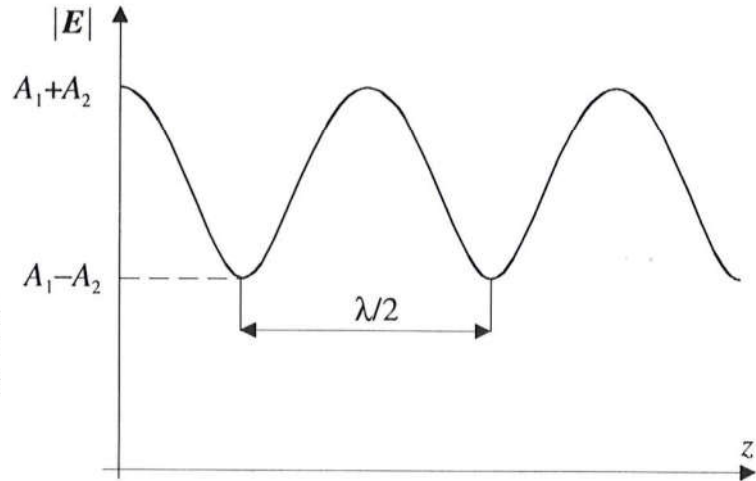


Figure 8.3.1. The dependence of the amplitude of the phasor of the resultant electric field of two waves propagating in opposite directions on distance z .

A small probe (receiving antenna) put in the field of such a wave and connected to a rectifier would show a voltage that essentially follows the pattern shown in Figure 8.3.1. For a pure travelling wave ($A_1 = 0$ or $A_2 = 0$), the diagram in Figure 8.3.1. would be constant, meaning a line parallel to the z -axis.

Now we are going to analyse the special case of a *linearly polarized pure direct travelling wave* that represents an elementary example of a homogeneous plane TEM wave. The phasor equations of such a wave are

$$\underline{E}_x^+ = E_0 e^{-jk'z} , \quad \underline{H}_y^+ = H_0 e^{-jk'z} = \frac{E_0}{Z'} e^{-jk'z} . \quad (8.3.12a,b)$$

The instantaneous values of the field vectors of a linearly polarized pure direct travelling wave in a dielectric are

$$\begin{aligned} E_{x_{Re}}^+ &= E_0 \cos(\omega t - k'z) = E_0 \cos \left(\omega t - 2\pi \frac{z}{\lambda} \right) , \\ H_{y_{Re}}^+ &= \frac{E_0}{Z'} \cos(\omega t - k'z) = \frac{E_0}{Z'} \cos \left(\omega t - 2\pi \frac{z}{\lambda} \right) , \end{aligned} \quad (8.3.13a,b)$$

and are shown in Figure 8.2.3.

The energetic relationships are very interesting. The densities of total stored electric and magnetic energy in a dielectric material can be calculated in a real domain as being

$$\begin{aligned}
 (w_e)_{Re} &= \frac{\epsilon}{2} \{Re(\underline{E}_x^+ e^{j\omega t})\}^2 = \frac{\epsilon}{2} \{Re(E_0 e^{j(\omega t - k'z)})\}^2 = \\
 &= \frac{\epsilon}{2} E_0^2 \cos^2\left(\omega t - \frac{\omega z}{c}\right) = \frac{\epsilon}{4} E_0^2 \left[1 + \cos 2\left(\omega t - \frac{\omega z}{c}\right)\right], \\
 (w_m)_{Re} &= \frac{\mu}{2} \{Re(\underline{H}_y^+ e^{j\omega t})\}^2 = \frac{\mu}{2} \{Re(H_0 e^{j(\omega t - k'z)})\}^2 = \\
 &= \frac{\epsilon}{4} E_0^2 \left[1 + \cos 2\left(\omega t - \frac{\omega z}{c}\right)\right],
 \end{aligned} \tag{8.3.14a,b}$$

where E_0 is the amplitude, or maximum value, of the electric field strength. Note that the instantaneous values of density of stored electric and magnetic energy of the linearly polarized pure travelling wave are equal, $w_e = w_m$, which means that half of the total stored energy is electric energy and the other half is magnetic.

The time-averaged flow of energy in unit time is determined by the time-averaged value of the complex Poynting's as

$$N_{av}^+ = \frac{1}{2} \underline{E} \times \underline{H}^+ = \frac{1}{2} \underline{a}_x \underline{E}_x^+ \times \underline{a}_y \underline{H}_y^+ = \frac{1}{2} (\underline{a}_x \times \underline{a}_y) \frac{E_0^2}{Z'} = \underline{a}_z \frac{E_0^2}{2Z'}. \tag{8.3.15}$$

It is obvious that the density of the time-averaged power transmitted by a linearly polarized travelling wave through a dielectric is *constant*, meaning that it is independent of distance. The instantaneous value of electromagnetic power transmitted by a pure direct travelling wave can be determined as

$$N_{Re}^+ = Re(\underline{a}_x \underline{E}_x^+ e^{j\omega t}) \times Re(\underline{a}_y \underline{H}_y^+ e^{j\omega t}) = \underline{a}_z \frac{E_0^2}{2Z'} \left[1 + \cos 2\left(\omega t - \frac{\omega z}{c}\right)\right]. \tag{8.3.16}$$

It is usual to define the *velocity of propagation of energy* v_e by a wave as the *ratio of the instantaneous values* of power flow density and stored energy density

$$v_e = \frac{N_{Re}}{w_{eRe} + w_{mRe}}. \tag{8.3.17}$$

For a homogeneous travelling wave from equations (8.3.14a,b) and (8.3.16) it follows that

$$v_e = \frac{1}{\epsilon Z'} = \frac{1}{\sqrt{\mu\epsilon}} = c = v_f. \tag{8.3.18}$$

In general, these two velocities do not have to be equal. The phase velocity can be greater than the speed of light, but the velocity of propagation of energy is never greater than the speed of light.

We shall proceed with the analysis of the special case of a travelling wave with a *circularly polarized field*. Let us observe a pure direct travelling wave that has x and y components of its field in a plane of constant phase. This wave is the result of a superposition of both the components of a direct wave and is given by

$$\begin{aligned}\underline{E}^+ &= \underline{a}_x \underline{E}_x^+ + \underline{a}_y \underline{E}_y^+ = (\underline{a}_x E_{2x} + \underline{a}_y E_{2y}) e^{-\gamma z}, \\ \underline{H}^+ &= \underline{a}_y \underline{H}_y^+ + \underline{a}_x \underline{H}_x^+ = (\underline{a}_y H_{2y} + \underline{a}_x H_{2x}) e^{-\gamma z}.\end{aligned}\quad (8.3.19a,b)$$

After rearrangement, we obtain the expressions

$$\underline{E}^+ = (\underline{a}_x A_2 e^{j\alpha_2} + \underline{a}_y B_2 e^{j\beta_2}) e^{-jk'z}, \quad \underline{H}^+ = (\underline{a}_y A_2 e^{j\alpha_2} - \underline{a}_x B_2 e^{j\beta_2}) \frac{1}{Z'} e^{-jk'z}.$$

Suppose that the integration constants are given as $\alpha_2 = 0$ and $\beta_2 = \pi/2$, and that $A_2 = B_2 e^{j\pi/2} = E_0$. This results in the expressions

$$\underline{E}^+ = (\underline{a}_x - j\underline{a}_y) E_0 e^{-jk'z}, \quad \underline{H}^+ = (\underline{a}_x - j\underline{a}_y) j \frac{E_0}{Z'} e^{-jk'z} \quad (8.3.20a,b)$$

representing phasor equations of a circularly polarized pure direct travelling wave in a dielectric.

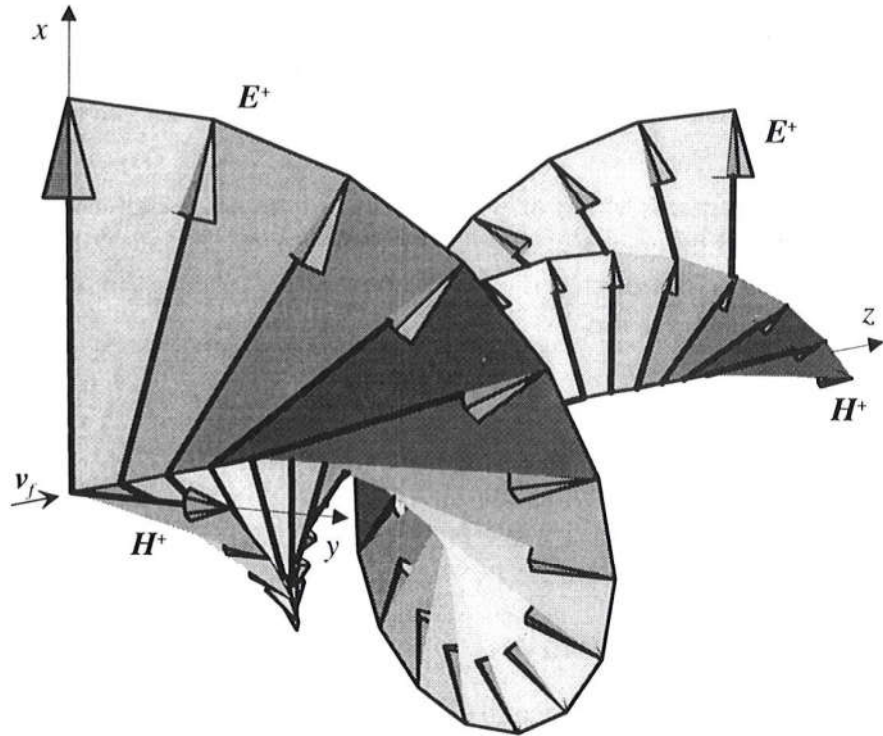


Figure 8.3.2. Propagation of a circularly polarized pure direct (in the $+z$ direction) travelling wave with phase velocity $v_f = c$.

We obtain the instantaneous values of field strength as:

$$\begin{aligned}
 \mathbf{E}_{Re}^+ &= \mathbf{a}_x E_0 \cos\left(\omega t - 2\pi \frac{z}{\lambda}\right) + \mathbf{a}_y E_0 \cos\left(\omega t - 2\pi \frac{z}{\lambda} - \frac{\pi}{2}\right), \\
 \mathbf{H}_{Re}^+ &= \mathbf{a}_y \frac{E_0}{Z'} \cos\left(\omega t - 2\pi \frac{z}{\lambda}\right) - \mathbf{a}_x \frac{E_0}{Z'} \cos\left(\omega t - 2\pi \frac{z}{\lambda} - \frac{\pi}{2}\right),
 \end{aligned} \tag{8.3.20c,d}$$

and these are shown in Figure 8.3.2. Three vectors $\mathbf{E}^+$, $\mathbf{H}^+$ and $\mathbf{v}_f$ in Figure 8.3.2. form a Cartesian coordinate system that moves in the direction of the $+z$ axis and that rotates around this axis following the right-hand screw rule as time passes. The instantaneous values of vectors $\mathbf{E}_{Re}^+$ and $\mathbf{H}_{Re}^+$ are constant and their values are $|\mathbf{E}_{Re}^+| = E_0$ and $|\mathbf{H}_{Re}^+| = E_0/Z'$. The expression

$$\mathbf{N}_{av}^+ = \frac{1}{2} \mathbf{E}^+ \times \mathbf{H}^+ = \frac{1}{2} (\mathbf{a}_x - j\mathbf{a}_y) E_0 e^{-jk'z} \times (-j\mathbf{a}_x + \mathbf{a}_y) \frac{E_0}{Z'} e^{+jk'z} = \mathbf{a}_z \frac{E_0^2}{Z'}. \tag{8.3.21}$$

determines the flow of time-averaged energy transmitted by a circularly polarized pure direct travelling wave in unit time on the basis of the time-averaged value of the complex Poynting's vector. A circularly polarized wave transmits time-averaged electromagnetic power that is twice the power transmitted by a linearly polarized wave.

The instantaneous value of power density that is transmitted by the circularly polarized wave can be calculated as

$$\mathbf{N}_{Re}^+ = \mathbf{E}_{Re}^+ \times \mathbf{H}_{Re}^+ = \mathbf{a}_z \frac{E_0^2}{Z'} \left[\cos^2\left(\omega t - 2\pi \frac{z}{\lambda}\right) + \sin^2\left(\omega t - 2\pi \frac{z}{\lambda}\right) \right] = \mathbf{a}_z \frac{E_0^2}{Z'}. \tag{8.3.22}$$

The instantaneous value of power density transmitted by a circularly polarized wave is constant and independent of time, and is equal to the value of the time-averaged power.

The densities of stored electric and magnetic energy in a dielectric during the propagation of the wave are

$$(w_e)_{Re} = \frac{\epsilon}{2} |\mathbf{E}_{Re}^+|^2 = \frac{\epsilon}{2} E_0^2, \quad (w_m)_{Re} = \frac{\mu}{2} |\mathbf{H}_{Re}^+|^2 = \frac{\mu}{2} \left(\frac{E_0}{Z'} \right)^2 = \frac{\epsilon}{2} E_0^2, \tag{8.3.23}$$

where E_0 is the maximum value (i.e. amplitude) of the electric field strength. Accordingly, the circularly polarized wave possesses a steady flow of energy analogous to a multi-phase symmetric system for power transmission.

8.3.1.2. Standing waves

Other properties of waves can be presented using standing waves. We shall rearrange the equation for a travelling wave given in Section 8.3.1.1. by supposing that in expression (8.3.10a,b) for a linearly polarized travelling wave, the constants are

$$A_1 = \frac{E_0}{2}; \quad \alpha = -\frac{\pi}{2}, \quad A_2 = \frac{E_0}{2}; \quad \alpha_2 = +\frac{\pi}{2}. \tag{8.3.24a - d}$$

In such a way, we shall obtain a linearly polarized wave of the form

$$\begin{aligned}\underline{E}_x &= \frac{E_0}{2} e^{j\left(-\frac{\pi}{2} + k'z\right)} + \frac{E_0}{2} e^{j\left(\frac{\pi}{2} - k'z\right)} = E_0 j \frac{e^{-jk'z} - e^{+jk'z}}{2} = E_0 \sin(k'z) , \\ \underline{H}_y &= -\frac{E_0}{2Z'} e^{j\left(-\frac{\pi}{2} + k'z\right)} + \frac{E_0}{2Z'} e^{j\left(\frac{\pi}{2} - k'z\right)} = j \frac{E_0}{Z'} \frac{e^{-jk'z} + e^{+jk'z}}{2} = j \frac{E_0}{Z'} \cos(k'z) .\end{aligned}\quad (8.3.25a,b)$$

These are phasor equations for a plane wave. The corresponding instantaneous values in a real domain are

$$E_{x_{Re}} = E_0 \sin\left(2\pi \frac{z}{\lambda}\right) \cos(\omega t) , \quad H_{y_{Re}} = -\frac{E_0}{Z'} \cos\left(2\pi \frac{z}{\lambda}\right) \sin(\omega t) . \quad (8.3.26a,b)$$

We may see that the vectors $\underline{E}_x$ and $\underline{H}_y$ are out of phase by 90° . Also, whatever the time of observation, the electric and magnetic field of a linearly polarized plane wave are sinusoidally distributed along the z -axis.

Null-points of E_x and H_y are fixed on the z -axis in space and the null-point of E_x is spatially shifted by a quarter of a wavelength ($\lambda/4$) with respect to the null-point of H_y . Therefore, there is no movement along the z -axis. Such a wave is called a *standing wave*. It is homogeneous and linearly polarized, and it is shown in Figure 8.3.3. at a specified instant in time.

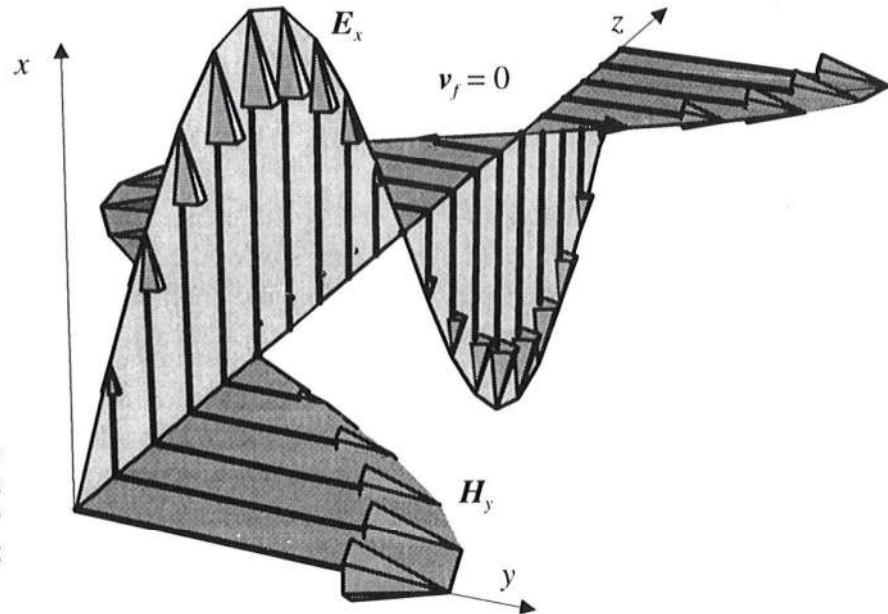


Figure 8.3.3. The electric and magnetic field of a homogeneous linearly polarised plane standing wave

We are going to determine energetic relationships in a linearly polarized pure standing wave and we shall compare them to similar expressions for a pure travelling wave. The density of stored energy is

$$\begin{aligned}
 (w_e)_{Re} &= \frac{\epsilon}{2} E_{x_{Re}}^2 = \frac{\epsilon}{2} E_0^2 \sin^2 \left(2\pi \frac{z}{\lambda} \right) \cos^2(\omega t) , \\
 (w_m)_{Re} &= \frac{\mu}{2} H_{y_{Re}}^2 = \frac{\epsilon}{2} E_0^2 \cos^2 \left(2\pi \frac{z}{\lambda} \right) \sin^2(\omega t) .
 \end{aligned}
 \tag{8.3.27a,b}$$

We shall calculate the time-averaged value of the complex Poynting's vector using expressions (8.3.25a,b) as

$$N_{av} = \frac{1}{2} (\mathbf{a}_x \underline{E}_x \times \mathbf{a}_y \underline{H}_y^*) = -\mathbf{a}_z \frac{1}{4} j \frac{E_0^2}{Z'} \sin 2 \left(2\pi \frac{z}{\lambda} \right) . \tag{8.3.28a}$$

In a real domain, the instantaneous value of Poynting's vector for a pure standing wave is, according to (8.3.26a,b),

$$N_{Re} = (\mathbf{a}_x E_{x_{Re}} \times \mathbf{a}_y H_{y_{Re}}) = -\mathbf{a}_z \frac{1}{4} \frac{E_0^2}{Z'} \sin^2 \left(2\pi \frac{z}{\lambda} \right) \sin(2\omega t) . \tag{8.3.28b}$$

Using this expression, we see that the time-averaged value of Poynting's vector in a real domain is equal to zero since the time-averaged value of the function $\sin(2\omega t)$ is always equal to zero.

8.3.2. Waves in a lossy conductive material

We shall start our analysis by observing the propagation of a direct sinusoidal TEM wave of given frequency ω in an unbounded space filled with a linear conductive material of known constitutive parameters ϵ , μ and κ . We shall use the phasor form of the equations for a plane wave given by expressions (8.2.20c,d). These equations written for a direct wave in the $+z$ direction are

$$\underline{\mathbf{E}}_x^+ = E_0 e^{-\gamma z} , \quad \underline{\mathbf{H}}_y^+ = H_0 e^{-\gamma z} . \tag{8.3.29a,b}$$

One should note that the only difference between these solutions for lossy materials and those for ideal lossless dielectrics is that *the propagation constant γ is complex* for lossy materials and purely imaginary for lossless materials. This fact results from expressions for the characteristics α , β , γ , Z and v_p , defining the behaviour of a wave in a lossy material. They are given as

$$\alpha = \frac{\omega}{\sqrt{2}c} \left(\sqrt{1 + \left(\frac{\kappa}{\omega\epsilon} \right)^2} - 1 \right)^{1/2}, \quad \beta = \frac{\omega}{\sqrt{2}c} \left(\sqrt{1 + \left(\frac{\kappa}{\omega\epsilon} \right)^2} + 1 \right)^{1/2}$$

$$\gamma = \alpha + j\beta, \quad Z = \frac{j\omega\mu}{\gamma} = \frac{\omega\mu}{\sqrt{\alpha^2 + \beta^2}} e^{j\varphi}; \quad \tan \varphi = \frac{\alpha}{\beta}, \quad (8.3.30a - f)$$

$$v_f = \pm c \left[\frac{1}{2} \left(\sqrt{1 + \left(\frac{\kappa}{\omega\epsilon} \right)^2} + 1 \right) \right]^{-1/2},$$

where $c^2 = 1/\mu\epsilon$. All of these characteristics have non-zero values. Using expressions (8.3.30a - d), we can write the phasor equations (8.3.29a,b) for fields of a direct plane wave as:

$$\underline{E}_x^+ = E_0 e^{-\alpha z} e^{-j\beta z}, \quad \underline{H}_y^+ = \frac{E_0}{Z} e^{-\alpha z} e^{-j\beta z} = \frac{E_0}{|Z|} e^{-\alpha z} e^{-j\beta z} e^{-j\varphi}, \quad (8.3.31a,b)$$

while the corresponding real expressions for instantaneous values of field strength are

$$E_{x,Re} = E_0 e^{-\alpha z} \cos(\omega t - \beta z), \quad E_{y,Re} = \frac{E_0}{|Z|} e^{-\alpha z} \cos(\omega t - \beta z - \varphi). \quad (8.3.32a,b)$$

In a conductive material, the attenuation constant is always a value other than zero $\alpha \neq 0$ due to its finite conductivity ($\kappa \neq 0$). Therefore both, electric and magnetic field are attenuated in the direction of wave propagation.

The attenuation of amplitudes of both fields in a plane of constant phase is defined by the attenuation constant α , which in turn depends on the frequency of the field of the wave. The attenuation increases with wave frequency and with increase of conductivity of the material.

In the analysis of electromagnetic waves, we use the notion of *depth of penetration* (or skin depth) for a wave in a conductive material. This is the distance at which the amplitude of the field is reduced to $1/e \approx 0,368$, meaning to approximately 36,8% of its initial amplitude. The depth of penetration of a wave is defined by $z = d$, where $\alpha z = \alpha d = 1$. In applied electromagnetic theory we usually take the distance at which the wave is practically totally attenuated as being $z = 3d$, or $z = 5d$. At this distance the wave is attenuated to approximately 5%, or approximately to 0.7 %, of its initial value.

From expressions (8.3.32a,b) it follows that the *electric and magnetic fields are out of phase*. The magnetic field is retarded in phase by a phase angle φ . The phase angle between the electric and magnetic field increases with increase of conductivity of the material and with decrease of frequency.

It is easy to show that for dielectrics and for materials with negligible conductivity $\varphi \approx 0$, and that for materials with high conductivity, for example metals, $\varphi \approx 45^\circ$. Therefore, for all materials and a whole range of engineering frequencies, the phase angle is within the limits $0 \leq \varphi \leq 45^\circ$.

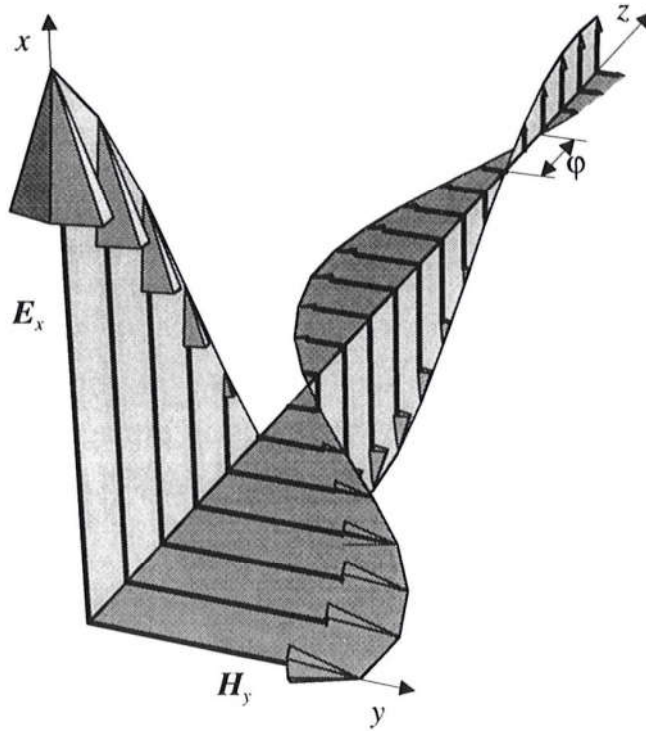


Figure 8.3.4. Vectors of electric and magnetic field of a plane attenuated TEM wave in a conductive lossy material

The finite conductivity of a material $\kappa \neq 0$ results in the attenuation of the travelling TEM wave and it influences the phase velocity of the wave as well. The plane of constant phase propagates with a phase velocity. The phase velocity v_f increases with frequency of the wave, as long as the constitutive parameters ϵ , μ and κ are independent of frequency.

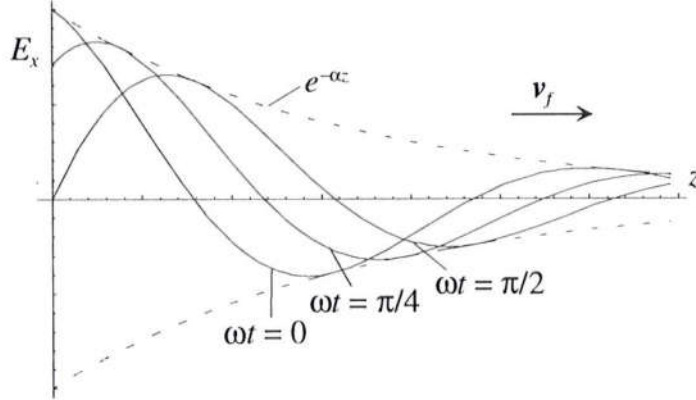


Figure 8.3.5. Propagation of electric field of a linearly polarized TEM plane wave propagating along the z -axis at a few instants of time

The attenuation of a travelling TEM wave propagating in a conductive material is the consequence of losses of energy along the way. Losses in unit volume of the material in the vicinity of a point on the z -axis are in a real domain and are calculated from the divergence of Poynting's vector as:

$$(P_g')_{Re,av} = -\nabla \cdot (N_z)_{Re,av} = \frac{\kappa}{2} E_0^2 e^{-2\alpha z}, \quad (8.3.33)$$

and we measure them in watts per cubic metre. E_0 is the amplitude, i.e. peak value of the electric field at the initial point $z = 0$. The time-averaged power transmitted by a travelling TEM wave through unit surface in the vicinity of a point that is at a distance z from the

origin is determined as being

$$(N_z)_{Re,av} = \frac{1}{2} \frac{E_0^2}{|Z|} e^{-2\alpha z} \cos \varphi . \quad (8.3.34)$$

We shall continue with the analysis of propagation of waves in lossy materials by observing the general case where direct and inverse travelling waves propagate along the z -axis. Suppose that an x -polarized electric field consists of two waves: one that is propagating in the $+z$ direction, and the other propagating in the $-z$ direction. The waves are

$$\begin{aligned} \underline{E}_x &= \underline{E}_x^- + \underline{E}_x^+ = A_1 e^{\alpha z} e^{j\beta z} + A_2 e^{-\alpha z} e^{-j\beta z} , \\ \underline{H}_y &= \underline{H}_y^- + \underline{H}_y^+ = -\frac{A_1}{Z} e^{\alpha z} e^{j\beta z} + \frac{A_2}{Z} e^{-\alpha z} e^{-j\beta z} . \end{aligned} \quad (8.3.35a,b)$$

A superposition of two travelling waves with equal amplitudes ($A_1 = A_2$) propagating in opposite directions results in the occurrence of a standing wave as well. Due to attenuation, it is not possible to have two “equal” waves propagating in opposite directions. They can be equal only at one place. Suppose that this is on the plane $z = 0$. In this case, a *standing wave will occur in the vicinity of the point $z = 0$* , and with increase of distance in both the directions $+z$ and $-z$, the standing wave vanishes, as shown in Figure 8.3.6.

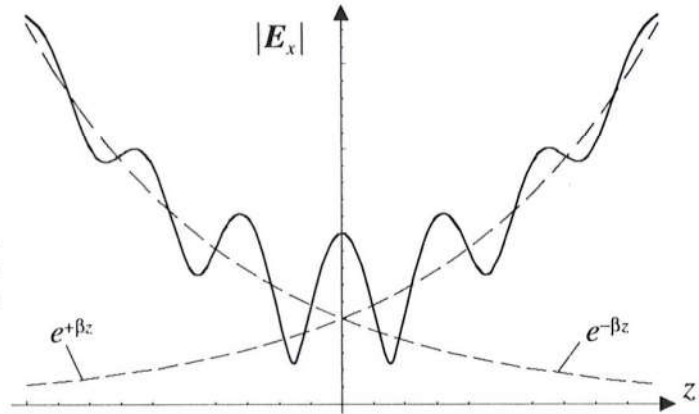


Figure 8.3.6. Standing wave at a point $z = 0$, which has emerged from two “equal” attenuated travelling waves propagating in opposite directions in a lossy material

8.3.3. Classification of materials into insulators and conductors

Examination of expressions determining the fundamental characteristics of an electromagnetic wave in a lossy material shows that the characteristics α , β , γ , Z and v_f are essentially determined by the ratio $\kappa/\omega\epsilon$. The meaning of the ratio $\kappa/\omega\epsilon$ follows from the phasor form of Maxwell’s equation (1.5.15b) $\nabla \times \underline{H} = \underline{J} = \kappa \underline{E} + j\omega\epsilon \underline{E}$. From this equation it follows that $\kappa/\omega\epsilon$ is equal to the ratio of amplitudes of densities of conductive currents and displacement currents.

Two classes of materials are of particular interest in the analysis of the propagation of plane waves. In one class, there are *good dielectrics* in which displacement currents are

large compared to conductive currents, that is $\kappa/\omega\epsilon \ll 1$. The second class are good conductors, for which $\kappa/\omega\epsilon \gg 1$.

It is convenient to introduce normalized values in order to represent the characteristics α , β , $|Z|$ and φ in terms of the ratio between conductive and displacement currents in real materials. Therefore, we express *the normalized attenuation and phase constant* as

$$\frac{\alpha(\omega)}{\kappa Z'} = \frac{1}{\sqrt{2}} x \sqrt{\sqrt{1 + \frac{1}{x^2}} - 1} \quad , \quad \frac{\beta(\omega)}{\kappa Z'} = \frac{1}{\sqrt{2}} x \sqrt{\sqrt{1 + \frac{1}{x^2}} + 1} \quad , \quad (8.3.36a,b)$$

where

$$x = \frac{\omega\epsilon}{\kappa} \quad , \quad Z' = \sqrt{\frac{\mu}{\epsilon}} \quad . \quad (8.3.37a,b)$$

We shall express the normalized value of the wave impedance of a material as

$$\frac{Z}{Z'} = \left(1 + \frac{\kappa}{j\omega\epsilon}\right)^{-1/2} = \left(1 + \frac{1}{x^2}\right)^{-1/4} e^{j\frac{1}{2}\arctan\frac{1}{x}} \quad . \quad (8.3.38)$$

8.3.3.1. Good insulators

Good insulators are materials in which the ratio between conductive and displacement currents is very small at a given frequency ω , i.e. $\kappa/\omega\epsilon \ll 1$. In order to obtain approximate asymptotic expressions for the characteristics γ , α , β , Z , and v_f , we shall rearrange equations (8.3.30a – e) into the more convenient form

$$\gamma = jk' \sqrt{1 + \frac{\kappa}{j\omega\epsilon}} \quad , \quad Z = \frac{j\omega\mu}{\gamma} = Z' \left(1 + \frac{\kappa}{j\omega\epsilon}\right)^{-1/2} \quad , \quad v_f = \frac{\omega}{\beta} \quad . \quad (8.3.39a - c)$$

Here k and k' are separation constants defined by expressions (8.2.18a,b) and Z and Z' are the wave impedances determined by (8.2.39f), and (8.3.37). The phase velocity is given by expression (8.2.28). If we apply a binomial expansion to expressions (8.3.39a,b), we obtain the next series for the propagation constant γ and for the wave impedance Z :

$$\begin{aligned} \gamma &= \alpha + j\beta = jk' \left[1 - j\frac{1}{2} \left(\frac{\kappa}{\omega\epsilon} \right) - \frac{1}{8} \left(\frac{\kappa}{\omega\epsilon} \right)^2 + j\frac{1}{16} \left(\frac{\kappa}{\omega\epsilon} \right)^3 + \dots \right] , \\ Z &= Z' \left[1 + j\frac{1}{2} \left(\frac{\kappa}{\omega\epsilon} \right) - \frac{3}{8} \left(\frac{\kappa}{\omega\epsilon} \right)^2 + j\frac{5}{16} \left(\frac{\kappa}{\omega\epsilon} \right)^3 + \dots \right] . \end{aligned} \quad (8.3.40a,b)$$

Since $\kappa/\omega\epsilon \ll 1$ for good insulators, we can omit terms with indices higher than two and we thus obtain

$$\begin{aligned}
\alpha &\approx \frac{k'}{2} \left(\frac{\kappa}{\omega\epsilon} \right) \left[1 - \frac{1}{8} \left(\frac{\kappa}{\omega\epsilon} \right)^2 + \dots \right] \approx \frac{\kappa}{2} \sqrt{\frac{\mu}{\epsilon}} \left[1 - \frac{1}{8} \left(\frac{\kappa}{\omega\epsilon} \right)^2 + \dots \right], \\
\beta &\approx k' \left[1 - \frac{1}{8} \left(\frac{\kappa}{\omega\epsilon} \right)^2 + \dots \right] \approx \frac{\omega}{c} \left[1 - \frac{1}{8} \left(\frac{\kappa}{\omega\epsilon} \right)^2 + \dots \right], \\
Z &\approx Z' \left\{ \left[1 - \frac{3}{8} \left(\frac{\kappa}{\omega\epsilon} \right)^2 + \dots \right] + j \left[\frac{1}{2} \left(\frac{\kappa}{\omega\epsilon} \right) + \dots \right] \right\}.
\end{aligned} \tag{8.3.41a - c}$$

Analysis of these expressions result in the following conclusions:

1. If we take into consideration only the first term in (8.3.41a), the approximation expression for the attenuation constant in a good insulator is

$$\alpha \approx \frac{\kappa}{2} \sqrt{\frac{\mu}{\epsilon}}. \tag{8.3.42}$$

For materials and frequencies at which is $\kappa/\omega\epsilon \leq 0.2$ the error in the calculation of the attenuation constant using the approximation expression (8.3.42) is only 0.5% or less.

2. The first approximation of the phase constant in good insulators, according to (8.3.41b), is

$$\beta \approx \frac{\omega}{c} = \omega \sqrt{\mu\epsilon}, \tag{8.3.43}$$

and is approximately equal to that of an ideal dielectric. Analysis of accuracy is the same as it was for the attenuation constant.

3. An approximate expression for wave impedance of good insulators is

$$Z \approx Z' \left[1 + j \frac{1}{2} \left(\frac{\kappa}{\omega\epsilon} \right) \right] ; \quad \varphi \approx \arctan \frac{1}{2} \left(\frac{\kappa}{\omega\epsilon} \right), \tag{8.3.44a,b}$$

and we can represent it by a complex number. If ratio $\kappa/\omega\epsilon$ satisfies the condition $\kappa/\omega\epsilon \leq 0.2$, then we can simply write

$$Z \approx Z' = \sqrt{\frac{\mu}{\epsilon}} ; \quad \varphi \approx 0, \tag{8.3.44c,d}$$

and the error is less than or equal to 1.5%. If we neglect the imaginary part, the error is at most 10%. Therefore, if we calculate the impedance according to expression (8.3.44a), and if the ratio $\kappa/\omega\epsilon \leq 0.2$, the error in the real part is at most 1.5%. If we calculate the impedance as a real number, which would be the case in an ideal dielectric, then the additional error in the imaginary part is 10%.

4. We can determine the expression for the phase velocity of a sinusoidal wave in a good insulator using (8.3.39c), (8.3.41b) and (8.3.43), as being

$$v_f \approx \frac{c}{1 - \frac{1}{8} \left(\frac{\kappa}{\omega \epsilon} \right)^2}, \quad (8.3.45a)$$

and if we neglect terms of higher order, we obtain

$$v_f \approx c = \frac{1}{\sqrt{\mu \epsilon}}. \quad (8.3.45b)$$

Obviously, if we calculate the velocity using expression (8.3.45b) for the ratio $\kappa/\omega \epsilon \leq 0.2$, the greatest error is 0.5%.

Finally, we can conclude that, due to small values of conductivity κ , attenuation of the field of a sinusoidal wave is very small, meaning that it is insignificantly attenuated in a good insulator. In all other ways, the wave behaves practically as if it propagates in an ideal dielectric.

8.3.3.2. Good conductors

Good conductors in a sinusoidal field are characterized by the ratio $\kappa/\omega \epsilon \gg 1$. *In a good conductor, displacement currents can be neglected* when compared to conductive currents. We shall derive approximate expressions for the fundamental characteristics γ , α , β , Z and v_f of a material in order to clarify the analysis of the behaviour of the sinusoidal wave in such materials.

Therefore, we first rearrange expressions (8.3.39a,b) and, taking into consideration that $\kappa/\omega \epsilon \gg 1$, or $\omega \epsilon/\kappa \ll 1$, we apply a binomial expansion. After rearrangement we obtain

$$\gamma = \alpha + j\beta = (1+j) \sqrt{\frac{\omega \mu \kappa}{2}} \left[1 + j \frac{1}{2} \left(\frac{\omega \epsilon}{\kappa} \right) + \frac{1}{8} \left(\frac{\omega \epsilon}{\kappa} \right)^2 - j \frac{1}{16} \left(\frac{\omega \epsilon}{\kappa} \right)^3 \dots \right],$$

$$Z = (1+j) \sqrt{\frac{\omega \mu}{2\kappa}} \left[1 - j \frac{1}{2} \left(\frac{\omega \epsilon}{\kappa} \right) + \frac{3}{8} \left(\frac{\omega \epsilon}{\kappa} \right)^2 + j \frac{5}{16} \left(\frac{\omega \epsilon}{\kappa} \right)^3 \dots \right].$$

By omitting terms of second or higher order, we obtain approximate expressions for γ and Z in a good conductor as

$$\gamma \approx (1+j) \sqrt{\frac{\omega \mu \kappa}{2}} \left[1 + j \frac{1}{2} \left(\frac{\omega \epsilon}{\kappa} \right) \right], \quad Z \approx (1+j) \sqrt{\frac{\omega \mu}{2\kappa}} \left[1 - j \frac{1}{2} \left(\frac{\omega \epsilon}{\kappa} \right) \right]. \quad (8.3.46a,b)$$

If displacement currents are only 1% of conductive currents, meaning that the ratio between displacement and conductive currents is $\omega \epsilon/\kappa \leq 0.01$, (*if conductive currents are a hundred times greater than displacement currents*, or $\kappa/\omega \epsilon \geq 100$), then the displacement

currents can be completely neglected in expressions (8.3.46a,b) for good conductors, and we obtain

$$\gamma \approx (1+j)\sqrt{\frac{\omega\mu\kappa}{2}} \quad , \quad Z \approx (1+j)\sqrt{\frac{\omega\mu}{2\kappa}} \quad . \quad (8.3.47a,b)$$

The error in calculation of γ and Z in the case where displacement currents have been neglected, i.e. using expressions (8.3.46a,b) and $\kappa/\omega\epsilon \geq 100$, is less than or equal to 0.5%. On the basis of the afore mentioned facts, we can conclude that:

1. We determine the approximate value of the attenuation constant when neglecting displacement currents in a good conductor using expression (8.3.47a) as

$$\alpha \approx \sqrt{\frac{\omega\mu\kappa}{2}} = \sqrt{\pi f\mu\kappa} \quad . \quad (8.3.48)$$

The attenuation constant is measured in *neper per metre* (1/m). Its value is very high for good conductors. This means that the wave is practically completely attenuated, and that waves do not penetrate deeply into metals. Thus, *metals act as a shield against electromagnetic waves*.

2. Neglecting displacement currents results in a first approximation for the phase constant in a good conductor, according to (8.3.47a), as

$$\beta \approx \alpha \approx \sqrt{\frac{\omega\mu\kappa}{2}} = \sqrt{\pi f\mu\kappa} \quad . \quad (8.3.49)$$

The phase constant is approximately equal to the attenuation constant. The unit of phase constant is 1/m. The values of phase constants for good conductors are high when compared to the values of phase constants of good insulators at the same frequency.

3. Wave impedance in a good conductor is calculated approximately using expression (8.3.47b), and we can write

$$Z \approx (1+j)\sqrt{\frac{\omega\mu}{2\kappa}} = \sqrt{\frac{\omega\mu}{\kappa}} e^{j\pi/4} \quad . \quad (8.3.50)$$

The wave impedance of a good conductor is complex and extremely small at radiofrequencies. The real and imaginary parts are approximately equal, and the *phase shift between the phasors of electric and magnetic field is 45°*.

4. The phase velocity of a sinusoidal wave in a good conductor can be approximately calculated as

$$v_f = \sqrt{\frac{2\omega}{\mu\kappa}} \quad . \quad (8.3.51)$$

The phase velocity in good conductors is much smaller than in free space. The frequency dependence of the phase velocity in good conductors, and in conductive

materials in general, should be emphasised. Therefore, harmonics of higher order of a complex wave are constantly accelerated relative to those of a lower order.

5. The definition of wavelength holds for good conductors as well. Therefore, it is determined as

$$\lambda = \frac{v_f}{f} = \frac{2\pi}{\beta} \approx \frac{2\pi}{\alpha} \approx \frac{2\pi}{\sqrt{\pi f \mu \kappa}} . \quad (8.3.52)$$

The wavelength in a good conductor is much smaller than in a vacuum.

6. Specific losses, or losses in unit time in a good conductor are calculated according to expression (8.3.33), and they are

$$(P_g')_{av,Re} = \frac{\kappa}{2} |E_0|^2 e^{-2\alpha z} = \frac{1}{2} \omega \mu |H_0|^2 e^{-2\alpha z} . \quad (8.3.53)$$

Since κ is very high for good conductors, especially metals, their specific losses are high.

7. The power transmitted by a sinusoidal electromagnetic field per unit surface in a good conductor is

$$N_{av} = \frac{1}{2} \mathbf{a}_z |E_0|^2 \frac{1}{Z} e^{-2\alpha z} = \frac{1}{2} \mathbf{a}_z |H_0|^2 e^{-2\alpha z} (1+j) \sqrt{\frac{\omega \mu}{2\kappa}} . \quad (8.3.54)$$

Poynting's vector is a complex quantity and we often represent it using two real quantities $N_{av} = P_{av} + jQ_{av}$, where the module $|N_{av}|$ is the *apparent power* per unit surface transmitted by a sinusoidal electromagnetic field through space. The quantity P_{sr} is called the *active power*, and the quantity Q_{sr} is called the *reactive power*.

8.3.4. The skin effect

The depth of penetration and wavelength of a sinusoidal wave in a good conductor (metal) are

$$d_k \approx \sqrt{\frac{2}{\omega \mu \kappa}} , \quad \lambda_k \approx \frac{2\pi}{\sqrt{\frac{\omega \mu \kappa}{2}}} . \quad (8.3.55)$$

From this equations follows a characteristic expression for metals $\lambda \approx 2\pi d_k$. In Section 8.3.2, we explained that in practical engineering problems, we may consider that a wave in a metal is attenuated to a negligible value (to approximately 5% of the initial value of the original field at the surface of a metal) at a depth of $3d_k$ under the surface of the metal. Obviously, the electromagnetic field vanishes before (after $3d_k$) the wave has propagated a distance that would be equal to one wavelength ($\lambda_k = 6.28d_k$), and there is no sense in talking about an electromagnetic wave travelling in a metal.

Thus, the sinusoidal electromagnetic field in metals is localised in a thin layer at the surface. Therefore, conductive sinusoidal currents of density $\mathbf{J} = \kappa \mathbf{E}$ flowing in a metal due to a field are concentrated in a thin surface layer of the conductor. This phenomenon is known in electromagnetic theory as the *skin effect*.

All problems in calculating the skin effect in good conductors of various forms carrying any type of time-dependent current can be solved using the equations

$$\Delta \mathbf{E} - \mu \kappa \frac{\partial \mathbf{E}}{\partial t} = 0 \quad , \quad \Delta \mathbf{H} - \mu \kappa \frac{\partial \mathbf{H}}{\partial t} = 0 \quad , \quad \Delta \mathbf{J} - \mu \kappa \frac{\partial \mathbf{J}}{\partial t} = 0 \quad . \quad (8.3.56a - c)$$

These are diffusion equations derived from the attenuated wave equations where the displacement currents have been neglected.

It is convenient to calculate fields in the case of *sinusoidal* currents in a conductor, meaning in the case of a sinusoidal electromagnetic field, using phasors. The system of equations (8.3.56a - c) in this case transforms to $\Delta \underline{\mathbf{E}} = \gamma^2 \underline{\mathbf{E}}$, $\Delta \underline{\mathbf{H}} = \gamma^2 \underline{\mathbf{H}}$ and $\Delta \underline{\mathbf{J}} = \gamma^2 \underline{\mathbf{J}}$. These are the fundamental differential equations used to calculate the distribution of sinusoidal quasi-static electromagnetic fields and currents in good conductors, or in other words, for calculating the skin effect.

The propagation constant γ for metals is determined by the transformation of equations (8.3.56a - c) into a phasor form. We can also determine this using the expression $\gamma^2 = -\omega^2 \mu \epsilon + j\omega \mu \kappa$ and omitting the term $(-\omega^2 \mu \epsilon)$, which is identical to neglecting displacement currents. Thus, we obtain $\gamma^2 \approx j\omega \mu \kappa$, and finally γ for metals is

$$\gamma \approx \sqrt{j\omega \mu \kappa} = (1+j) \sqrt{\frac{\omega \mu \kappa}{2}} = (1+j)\alpha \quad . \quad (8.3.57)$$

We shall now apply this general mathematical model to the solution of the problem of the skin effect in the case of a very simple geometry. The geometry of the problem is shown in Figure 8.3.7.

The half space on the right-hand side ($z \geq 0$) is filled with a metal of given constitutive parameters ϵ_0 , μ_0 and κ , and the half-space on the left-hand side ($z \leq 0$) is filled with a good insulator, for example air with ϵ_0 , μ_0 and $\kappa = 0$. A sinusoidal electromagnetic field at frequency ω is applied, and alternating currents of density $\mathbf{J} = \mathbf{J}(z)$ are flowing in the conductor. We have measured the amplitude of the magnetic field strength at the surface of the conductor and it is H_0 A/m.

The solution of the diffusion differential equations in a phasor domain in a Cartesian coordinate system for the case where phasors of the fields $\underline{\mathbf{E}}$, $\underline{\mathbf{H}}$ depend only on the z coordinate is

$$\underline{\mathbf{E}} = \mathbf{a}_x (E_1 e^{+\gamma z} + E_2 e^{-\gamma z}) \quad , \quad \underline{\mathbf{H}} = \mathbf{a}_y (H_1 e^{+\gamma z} + H_2 e^{-\gamma z}) \quad , \quad (8.3.58a,b)$$

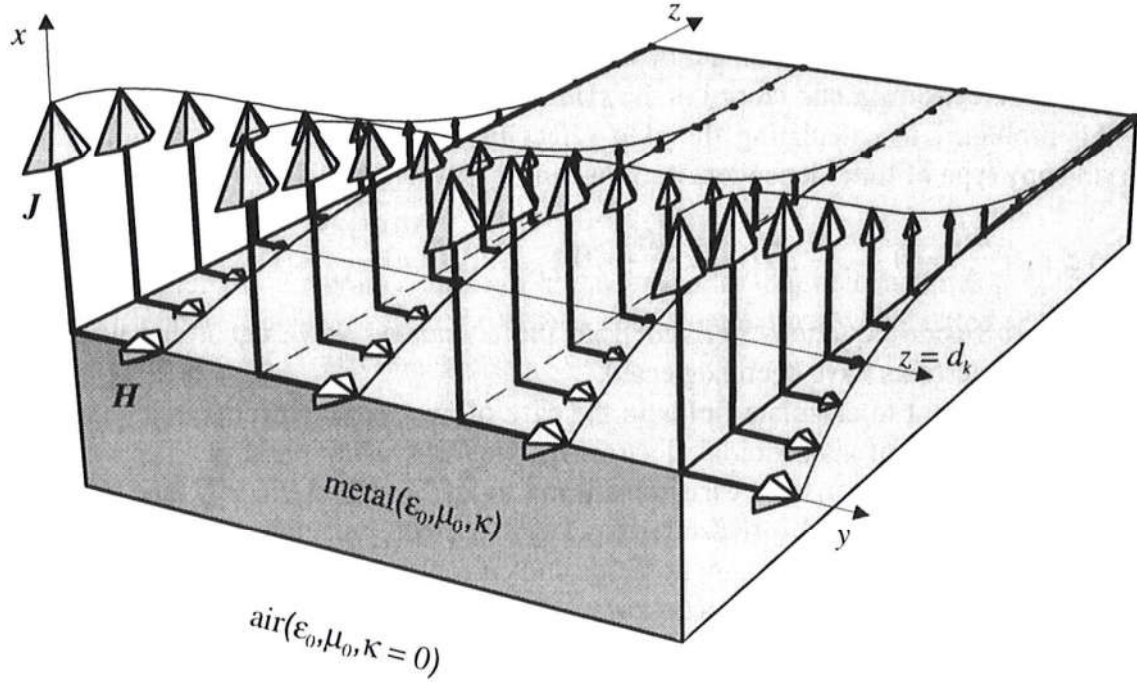


Figure 8.3.7. The pushing of currents and magnetic field strength towards the surface of a metal

The boundary conditions for such a simple geometry are

$$z=0 \Rightarrow \underline{H}(0) = \underline{a}_y H_0 \quad , \quad z \rightarrow \infty \Rightarrow \underline{H}(\infty) = 0 \quad . \quad (8.3.59a,b)$$

We substitute them into equation (8.3.58b) to obtain $H_1 = 0$ and $H_2 = H_0$. The impedance of good conductors are determined by expressions (8.3.50a – c), and, according to expressions (8.2.38a,c) for a direct wave, we obtain

$$\begin{aligned} E_1 = -H_1 Z = 0 \quad , \quad E_2 = H_2 Z = (1+j) \sqrt{\frac{\omega \mu}{2\kappa}} H_0 = E_0 \quad , \\ J_1 = \kappa E_1 = 0 \quad , \quad J_2 = \kappa E_2 = (1+j) \alpha H_0 = J_0 \quad . \end{aligned} \quad (8.3.60e - h)$$

Finally, instantaneous values of the field strengths and of the current density in a real domain are:

$$\begin{aligned} \underline{E}_{Re} &= \underline{a}_x \sqrt{\frac{\omega \mu}{\kappa}} H_0 e^{-z \sqrt{\frac{\omega \mu \kappa}{2}}} \cos \left(\omega t - z \sqrt{\frac{\omega \mu \kappa}{2}} + \frac{\pi}{4} \right) \quad , \\ \underline{H}_{Re} &= \underline{a}_y H_0 e^{-z \sqrt{\frac{\omega \mu \kappa}{2}}} \cos \left(\omega t - z \sqrt{\frac{\omega \mu \kappa}{2}} \right) \quad , \end{aligned} \quad (8.3.61a,b)$$

$$\mathbf{J}_{Re} = \mathbf{a}_x \sqrt{\omega \mu \kappa} H_0 e^{-z \sqrt{\frac{\omega \mu \kappa}{2}}} \cos \left(\omega t - z \sqrt{\frac{\omega \mu \kappa}{2}} + \frac{\pi}{4} \right). \quad (8.3.61c)$$

Since the complete current in the metal is concentrated in the surface layer with thickness $3d_x$, we often talk of a *current sheet* on the surface of a conductor. Using the expression

$$\underline{\mathbf{K}} = \int_0^\infty \mathbf{J} dz = \mathbf{a}_x \int_0^\infty (1+j) \alpha H_0 e^{-(1+j)\alpha z} dz = \mathbf{a}_x H_0 e^{-(1+j)\alpha z} \Big|_0^\infty = \mathbf{a}_x H_0, \quad (8.3.62)$$

we can calculate the phasor of the current sheet, meaning currents in the complete depth of the metal ($z \rightarrow \infty$), per metre of transverse length at the surface of the metal. The *total current per unit transverse length in a metal, meaning the magnitude of the current sheet K , is numerically equal to the value of the magnetic field strength at the surface of the metal H_0 .*

8.4. DISPERSION

So far we have analysed the propagation of a sinusoidal wave in a vacuum, a dielectric and a lossy material. However, difficulties arise in the analysis of propagation of a non-sinusoidal wave or *signal* in real materials and the problem turns into quite a complex one.

In linear materials, such a signal can be constructed by the superimposing of a series of sinusoidal waves of infinite length and duration. Thus, the analysis of propagation of such a signal can be reduced to that of the sinusoidal components (waves). We showed that the phase velocity of these sinusoidal components depends on the constitutive parameters ϵ , μ and κ of a material, which we have so far regarded as constants.

The propagation of a non-sinusoidal signal in a vacuum, or in an ideal dielectric whose constitutive parameters ϵ and μ ($\kappa = 0$) are independent of frequency and whose phase velocities are constant for all sinusoidal components of all frequencies, is without distortion. Therefore, we can conclude that in such *non-dispersive materials*, a non-sinusoidal signal travels undistorted.

All materials in nature show a certain amount of dispersion. The fact is that within a limited range of frequencies most materials show a sufficiently small enough dispersion that the phase velocity can be regarded as constant independent of the frequency of each sinusoidal component of the non-sinusoidal signal.

Even though the constitutive parameters ϵ , μ and κ depend on the frequency, the procedure presented in the previous sections still remains valid for each frequency component separately. We can talk of the *dispersion* of complex non-sinusoidal signals, or pulses, that propagate in materials whose sinusoidal-component of phase velocities are frequency-dependent. In this case, there is a scattering of the individual components, and consequently a *distortion of the signal*.

The dispersion is pronounced in materials that are good conductors, such as metals. The sinusoidal-component phase velocities of a signal in a good conductor are frequency-dependent. All materials in which the phase velocity is frequency-dependent are called *dispersive materials*.

8.4.1. Dispersion in dielectrics; the classical model for $\epsilon(\omega)$

The classical model was founded by Maxwell and Sellmeyer. Lorentz and others improved this model, and it provided very good results in the analysis of dispersion.

All theories of dispersion take the molecular and atomic constitution of matter into account. Particles of matter are regarded as mechanical oscillating systems that possess a natural free resonant frequency, and which are excited by an incident electromagnetic field. At the beginning, the calculation of movements within these systems was based on the obeying of the laws of classical dynamics. However, it is now known that these movements are governed by the more strict principles of quantum mechanics. Since the results obtained by the simple classical model are in good agreement with quantum mechanical formulae for dispersion, we shall briefly describe the classical model.

We shall study the action of a sinusoidal, time-dependent electromagnetic field, expressed in a spectral domain as $\mathbf{E} = \mathbf{E}_0 e^{j\omega t}$ in a dielectric material, such as a gas of permittivity ϵ . We shall restrict our analysis to incident electromagnetic radiation having a wavelength much larger than atomic dimensions. This lessens the influence of the position of an electron relative to the nucleus on the calculation of the forces. This assumption allows us to include ultraviolet radiation.

Each electron in an atom or molecule is displaced by a displacement $\mathbf{r}$ due to the action of a force $q_0 \mathbf{E}_0 e^{j\omega t}$ caused by the incident field. Taking other forces into consideration, such as the inertial force $m(d^2\mathbf{r}/dt^2)$, the restoring force $(m\omega_0^2 \mathbf{r})$, and the damping force $g(d\mathbf{r}/dt)$ that arise because of collisions and radiation of energy, we obtain the equation of motion of an electron of charge q_0 as

$$m \left[\frac{d^2 \mathbf{r}}{dt^2} + g \frac{d\mathbf{r}}{dt} + \omega_0^2 \mathbf{r} \right] = q_0 \mathbf{E}_0 e^{j\omega t} . \quad (8.4.1)$$

The mass of the electron is denoted by m , the natural resonant frequency of the electron is ω_0 , and g is the damping constant that represents the damping frictional forces in the material. The constants ω_0 and g characterise the material and we determine them using experimental results.

The action of the force of the incident electric field on an electron will cause a sinusoidal oscillation of the electron. Atoms and molecules in the material will act as mechanical oscillators. Therefore, we shall suppose the solution for displacement of an electron to be $\mathbf{r} = \mathbf{r}_0 e^{j\omega t}$, which results in the solution of equation (8.4.1) in the form

$$\mathbf{r} = \frac{q_0}{m} [\omega_0^2 - \omega^2 + j\omega g]^{-1} \mathbf{E} . \quad (8.4.2)$$

This electron motion contributes to the dipole moment like so:

$$\mathbf{p} = q_0 \mathbf{r} = \frac{q_0^2}{m} [\omega_0^2 - \omega^2 + j\omega g]^{-1} \mathbf{E} . \quad (8.4.3)$$

Suppose there are Z electrons in a molecule of gas and that there are N molecules per unit volume. If, instead of a single resonant frequency ω_0 and damping constant g , we have n_i electrons with resonant frequencies ω_{0i} and damping constants g_i , the molecular dipole moment can then be written as

$$\mathbf{p}_m = \frac{q_0^2}{m} \sum_i \frac{n_i}{(\omega_{0i}^2 - \omega^2) + j\omega g_i} \mathbf{E} \quad ; \quad Z = \sum_i n_i . \quad (8.4.4)$$

If we introduce the atomic or molecular microscopic *polarizability* α , which is usually defined by the expression $\mathbf{p}_m = \epsilon_0 \alpha \mathbf{E}$, we finally obtain

$$\alpha = \frac{q_0^2}{m\epsilon_0} \sum_i \frac{n_i}{(\omega_{0i}^2 - \omega^2) + j\omega g_i} . \quad (8.4.5)$$

At the same time, the Clausius Mossotti equation relating the macroscopic permittivity ϵ with the microscopic polarizability α of the dielectric molecules is

$$\alpha = \frac{3}{N} \frac{\epsilon_r - 1}{\epsilon_r + 2} . \quad (8.4.6)$$

The polarizability α in this expression relates to *dense materials*. Since neighbouring atoms in such materials are very close, the interaction between them is very strong.

This results in an increase of damping constants g_i and in the widening of the bandwidth of possible resonant frequencies ω_{0i} . Thus, ω_{0i} and g_i for atoms in dense materials will be different from those in free atoms. Even with this reservation, we can still express, at the very least approximately, the polarizability α using expression (8.4.5). Using equations (8.4.5,6), we obtain the result

$$\frac{\epsilon(\omega) - \epsilon_0}{\epsilon(\omega) + 2\epsilon_0} = \frac{N\alpha}{3} = \frac{Nq_0^2}{3m\epsilon_0} \sum_i \frac{n_i}{(\omega_{0i}^2 - \omega^2) + j\omega g_i} , \quad (8.4.7)$$

and we can use it to determine $\epsilon(\omega)$ with respect to the frequency of the incident wave. For a rarefied material, such as gas, in which N is a very small number, we can write equation (8.4.7) as

$$\varepsilon(\omega) \approx \varepsilon_0 \varepsilon_r(\omega) \approx \varepsilon_0 (1 + N\alpha) \quad (8.4.8)$$

After substitution of expression (8.4.8) into this equation we obtain the *relative complex dielectric function* of a gas as

$$\varepsilon_r(\omega) \approx 1 + N\alpha = 1 + \frac{Nq_0^2}{m\varepsilon_0} \sum_i \frac{n_i}{(\omega_{0i}^2 - \omega^2) + j\omega g_i} , \quad (8.4.9)$$

Finally, using expressions (8.4.8,9), we determine

$$\begin{aligned} \operatorname{Re}\{\varepsilon(\omega)\} &= \varepsilon_0 \left[1 + \frac{Nq_0^2}{m\varepsilon_0} \sum_i \frac{n_i(\omega_{0i}^2 - \omega^2)}{(\omega_{0i}^2 - \omega^2)^2 + \omega^2 g_i^2} \right] , \\ \operatorname{Im}\{\varepsilon(\omega)\} &= \frac{Nq_0^2}{m} \sum_i \frac{n_i \omega g_i}{(\omega_{0i}^2 - \omega^2)^2 + \omega^2 g_i^2} . \end{aligned} \quad (8.4.10a,b)$$

and, in fact, these are the *real* and *imaginary* parts of the complex dielectric function.

8.4.2. Dispersion in metals

Metals, as typical examples of good conductors, possess a great number of *free electrons*. If we suppose that in the expression for permittivity of a material (8.4.9) the electron fraction n_0 are free, meaning that all of them possess $\omega_0 = 0$, then the permittivity $\varepsilon(\omega)$ is a singularity at the point $\omega \rightarrow 0$. Therefore, it is convenient to transform expression (8.4.9) to

$$\varepsilon(\omega) = \left[\varepsilon_0 + \frac{Nq_0^2}{m} \sum_i \frac{n_i}{(\omega_i^2 - \omega^2) + j\omega g_i} \right] - j \frac{Nq_0^2}{m\omega} \frac{n_0}{(g_0 + j\omega)} ; \quad (i \neq 0) . \quad (8.4.11)$$

We denote the terms in square brackets as

$$\varepsilon^\circ(\omega) = \varepsilon_0 + \frac{Nq_0^2}{m} \sum_i \frac{n_i}{(\omega_i^2 - \omega^2) + j\omega g_i} ; \quad i \neq 0 . \quad (8.4.12)$$

We can write this in Maxwell's second equation as

$$\nabla \times \mathbf{H} = \kappa \mathbf{E} + \varepsilon^\circ \frac{\partial \mathbf{E}}{\partial t} , \quad (8.4.13)$$

where we introduced the conductivity due to free electrons in a metal κ as $\mathbf{J} = \kappa \mathbf{E}$, analogous to the static case. Taking into account the fact that $\mu = \mu_0$ for all non-ferromagnetic metals, we can write expression (8.4.13) for a sinusoidal field in phasor form as

$$\nabla \times \underline{H} = \kappa \underline{E} + j\omega \epsilon^\circ \underline{E} = j\omega \left(\epsilon^\circ - j \frac{\kappa}{\omega} \right) \underline{E} = j\omega \epsilon \underline{E} . \quad (8.4.14)$$

A comparison of expressions (8.4.11) and (8.4.14) results in an expression for the complex conductivity of a conductor (i.e. metal), which is

$$\kappa(\omega) = \frac{n_0 N \frac{q_0^2}{m}}{g_0 + j\omega} . \quad (8.4.15)$$

This expression is essentially the result obtained from Drude's model for the conduction of electrons in a metal if we consider that $n_0 N$ is the number of *free electrons* per unit volume.

If we analyse the influence of the complex conductivity κ on the propagation of a sinusoidal wave in a metal, we can write the equation for the scalar components of such a field in phasor form as

$$\frac{\partial^2 \underline{w}}{\partial z^2} = \gamma^2 \underline{w} , \quad (8.4.16)$$

where the propagation function γ is determined as being

$$\gamma^2 = -\omega^2 \mu_0 \epsilon_0 + j\omega \mu_0 \kappa = (\alpha + j\beta)^2 . \quad (8.4.17)$$

Using this expression, we can calculate the attenuation $\alpha(\omega)$ and phase $\beta(\omega)$ function in the case of dispersion in a metal for a sinusoidal wave of frequency ω . We have substituted in the constitutive parameters of the metal ϵ_0 , μ_0 and κ . Since it is known that the resonant frequencies of bound electrons ω_{0i} lie far in the violet and ultraviolet region of the frequency spectrum, we can safely replace the permittivity of a metal $\epsilon(\omega)$ with the approximate value ϵ_0 at frequencies of incident wave ω up to visible red, infrared, and certainly radio frequencies, according to expressions (8.4.11) and (8.4.12).

Thus, the *permittivity of a metal is independent of the frequency of incident field up to the infrared frequency range. It is constant and approximately equal to the permittivity of a vacuum $\epsilon = \epsilon_0$.*

At sufficiently low frequencies ω of the incident field, the inertia force $m(d^2\mathbf{r}/dt^2)$ in expression (8.4.15) for complex permittivity is negligible with respect to the viscous force $m g_0(d\mathbf{r}/dt)$. Consequently, *the complex conductivity $\kappa(\omega)$ determined by expression (8.4.15) can be reduced to an approximate constant real value within this range as*

$$\kappa \approx \frac{n_0 N q_0^2}{m g_0} , \quad (8.4.18)$$

which is identical to the *static conductivity* κ . But, if we want to understand the mechanism of electrical conductivity properly, we have to apply a quantum mechanical approach, since the Pauli principle plays an important role here. These studies show that the

distinction between a dielectric and a conductor may be regarded as artificial, but still a useful one. A material that contains a substantial number of free electrons can be regarded as a conductor at low frequencies. But, at ultrahigh frequencies, the material acts as an insulator regardless of whether or not it contains any free electrons.

8.4.3. The group velocity of a signal

We defined the notion of phase velocity as $v_f = \omega/\beta$ and it applies only to a *travelling sinusoidal wave* of infinite duration. In this case v_f represents the velocity with which the plane of constant phase travels in the direction of propagation for a plane wave. The scalar field components of such a wave can be presented by the equation $w(z,t) = w_0 e^{j\omega t - \beta z}$.

A sequence of waves of finite length or of finite duration cannot be simply represented in such a way, and the term phase velocity loses its meaning. Furthermore, there is no sense in using the term phase velocity for a *non-sinusoidal wave*, even of infinite duration, since we cannot speak about the frequency ω . Therefore, we introduce the notion of *group velocity*, which is of special importance for non-sinusoidal waves of both infinite or of finite duration.

Let us consider the simplest case of a wave that emerges from the superposition of two sinusoidal waves whose frequencies and phase constants differ by very small amounts. Let these waves be determined by the equations:

$$w_1(z,t) = W_0 \cos(\omega t - \beta z) \quad , \quad w_2(z,t) = W_0 \cos[(\omega + \delta\omega)t - (\beta + \delta\beta)z] \quad .$$

Their sum is subsequently

$$w = 2W_0 \cos\left(\frac{\delta\omega t - \delta\beta z}{2}\right) \cos\left[\left(\omega + \frac{\delta\omega}{2}\right)t - \left(\beta + \frac{\delta\beta}{2}\right)z\right] \quad . \quad (8.4.19)$$

The field of this wave oscillates at a frequency that differs negligibly from ω , while its amplitude

$$A = 2W_0 \cos\left(\frac{\delta\omega t - \delta\beta z}{2}\right) \quad , \quad (8.4.20)$$

changes slowly between the sum of the two amplitudes of the sinusoidal waves and zero. As a consequence, the field along the z -axis and along the time axis t occurs as a series of periodically repeated groups of waves, as shown in Figure 8.4.1.

The plane of constant amplitude A of the group emerging from the summation of the two plane waves is determined by the equation $\delta\omega t - \delta\beta z = \text{const.}$ and from this equation it follows that the group is moving with velocity

$$v_g = \frac{\delta\omega}{\delta\beta} \quad . \quad (8.4.21)$$

We call this velocity the group velocity of a wave that emerges from the summation of two sinusoidal waves at slightly different frequencies. Here, ω is the carrier frequency and

the amplitude A is the modulating signal. Obviously, the frequency of modulation $\delta\omega$ is much smaller than the carrier frequency ω . The attenuation of the waves has been neglected here ($\alpha \approx 0$).

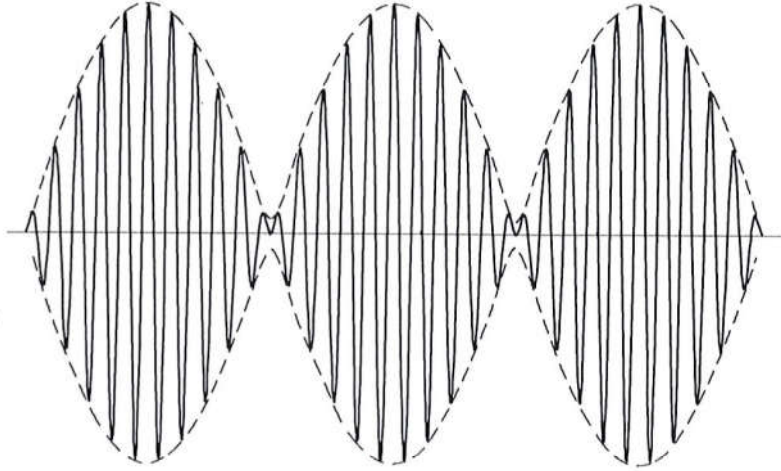


Figure 8.4.1. Group of waves that emerged from the superposition of two sinusoidal waves at slightly different frequencies

In Figure 8.4.1, we may see that the group velocity can be regarded as the phase velocity of the modulation envelope. The group velocity v_g is to the modulation signal what the phase velocity v_f is to the carrier frequency. The notion “group velocity” implies a restriction where *the bandwidth* of a signal is sufficiently narrow that one can make the assumption that all frequency components are travelling with a velocity equal to the group velocity. In non-dispersive materials, $\beta = \omega/v_f$, and $d\beta = d\omega/v_f$, and thus $v_g = d\omega/d\beta = v_f$. In dispersive materials, their values are different.

The situation where a single pulse, or a group of waves of finite duration and of arbitrary shape propagate in a non-dispersive material is more complex. We shall synthesize such a pulse using a Fourier integral, or transform, by selecting the amplitudes of the sinusoidal components in the expression

$$w(z,t) = \int_{-\infty}^{\infty} A(\omega) e^{j(\omega t - \beta z)} d\omega . \quad (8.4.22)$$

If the pulse contains only sinusoidal components within the narrow frequency band $\omega_0 - d\omega \leq \omega \leq \omega_0 + d\omega$, and if outside that band $A(\omega) = 0$, we can write

$$w(z,t) = \int_{\omega_0 - d\omega}^{\omega_0 + d\omega} A(\omega) e^{j(\omega t - \beta z)} d\omega . \quad (8.4.23)$$

Here, we have supposed that $\beta(\omega)$ is a real and known function. In the neighbourhood of the carrier frequency ω_0 , we can expand the function $\beta(\omega)$ using a Taylor's series expansion to

$$\beta(\omega) = \beta(\omega_0) + (\omega - \omega_0) \left(\frac{d\beta}{d\omega} \right)_{\omega = \omega_0} + \dots \quad (8.4.24)$$

If we substitute (8.4.24) into the expression for the wave packet (8.4.23), we obtain

$$w(z, t) = \int_{\omega_0 - d\omega}^{\omega_0 + d\omega} A(\omega) e^{j(\omega_0 t - \beta_0 z) + j(\omega - \omega_0) \left[t - \left(\frac{d\beta}{d\omega} \right)_{\omega_0} z \right]} d\omega ,$$

and if we introduce the mean amplitude w_0 of the packet, defined as

$$w_0 = \int_{\omega_0 - d\omega}^{\omega_0 + d\omega} A(\omega) e^{j(\omega - \omega_0) \left[t - \left(\frac{d\beta}{d\omega} \right)_{\omega_0} z \right]} d\omega ,$$

we finally arrive at

$$w = w_0 e^{j(\omega_0 t - \beta_0 z)} .$$

The mean amplitude w_0 of the wave packet or pulse is constant over the surface determined by

$$\left[t - \left(\frac{d\beta}{d\omega} \right)_{\omega_0} z \right] = \text{const.} ,$$

and differentiation of this expression results in

$$v_g = \frac{dz}{dt} = \left(\frac{d\omega}{d\beta} \right)_{\omega = \omega_0} . \quad (8.4.25)$$

Obviously, one pulse or wave packet moves with a group velocity v_g . If a material is non-dispersive, v_g is equal to the phase velocity v_f (for ω_0), and in the opposite case, the group velocity is a function of the fundamental frequency ω_0 . We should emphasise that the notion of group velocity may only be applied to narrow-band signals and cannot be applied to a broad-band signals (such as a rectangular impulse) in dispersive materials.

8.5. THE PROPAGATION OF A SINUSOIDAL PLANE WAVE IN HALF-BOUNDED REGIONS

In previous units of this chapter, we analysed the propagation of plane waves in an unbounded, source-free space. Unbounded spaces are, of course, an abstraction. The most interesting electromagnetic phenomena in reality are in fact those that happen when a wave is incident to the surface of a discontinuity, from which sudden changes in the physical properties of the materials concerned occur. In such cases, we are talking about the propagation of waves in *bounded, source-free spaces*. Problems of solving fields in such spaces, or at their boundaries, are called *boundary value problems*.

These problems can be classified into two types in accordance with the geometry of the boundaries of the region of interest. The first types are problems involving the propa-

gation of waves in *half-bounded regions*, which result in the reflection and absorption of waves at the boundary. The second types are problems involving the propagation of waves in regions that are *completely bounded*, which means *closed*, as is the case in the transmission of waves in waveguides or the oscillation of waves in resonant cavities.

In this unit, we shall analyse the field and propagation of a plane sinusoidal wave in two half-bounded spaces with a plane boundary. In the process of this analysis, we shall consider the cause-and-effect relationships between fields in both spaces. These relationships are boundary *conditions* that have to be satisfied at the boundary surfaces.

Therefore, we shall consider the classical problem known as reflection of waves. We shall assume that a plane wave from a distant source propagates in a linear material and impinges on a plane of a discontinuity representing the boundary of another linear material. In this case, we have to assume that the linear dimensions of the boundary surface are much larger than the wavelength, which is always the case if the boundary surface is infinite. A part of the *energy of the incident wave is reflected* from the boundary surface, and a part is *transmitted* into the second material. The direction of propagation is in general changed, and this change in direction is called *refraction*. The laws governing reflection and refraction of an electromagnetic wave at a plane boundary are relatively simple and we shall derive them in the next section.

It is worth mentioning that in cases where any, or all the dimensions of the boundary surface are within the order of magnitude of the wavelength of the incident wave, there occurs an abrupt increase in difficulty of mathematical modelling. The disturbance of the field of the incident wave in these cases is called *diffraction*.

8.5.1. Reflection and refraction at a plane boundary

In both this and the following sections, we shall study the electromagnetic approach to a series of problems we have met in early physics courses known as reflection and refraction of waves at boundaries. We shall restrict our analysis to *plane boundaries* and *plane waves in a sinusoidal steady state*. We shall begin with the general case of the incidence of a homogenous plane TEM wave at any angle of incidence on a plane boundary between two lossy materials.

8.5.1.1. The laws of reflection and refraction

We shall analyse the incidence of a sinusoidal homogeneous plane TEM wave to a plane boundary S between two linear materials of constitutive parameters $\epsilon_1, \mu_1, \kappa_1$ and $\epsilon_2, \mu_2, \kappa_2$. In other directions, the materials are unbounded and source-free.

The origin of the Cartesian coordinate system lies in the boundary plane S . The unit vector $\mathbf{a}_z$ directed towards the material (2) represents the normal to the boundary surface S , as shown in Figure 8.5.1.

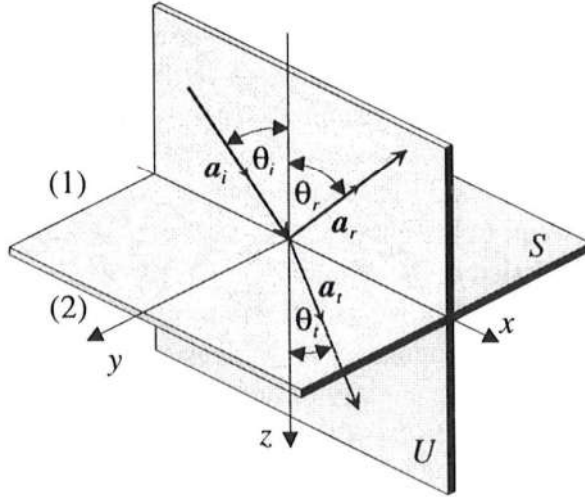


Figure 8.5.1. The incidence of a plane wave to the plane boundary S between materials (1) and (2) with vectors of direction of propagation in the plane of incidence U and with the angles of incidence, reflection and refraction

Since the position vector directed from the origin to any point on the boundary in both regions is given as $\mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$, the boundary surface S is defined by the equation $\mathbf{a}_z \cdot \mathbf{r} = 0$ or $z = 0$.

It should be physically obvious that the primary electromagnetic field of the incident wave causes oscillatory motion of any free and bound charge near the boundary S . These induced currents and charges produce their own secondary field in both materials. As a result, in material (1) a resultant field exists that is the sum of the primary field of the incident wave and the secondary field of the reflected wave radiated by the charge and currents near the boundary. Consequently, due to the radiation of the secondary induced field, we postulate the existence of the field of a refracted wave representing the transmission (and refraction) of the primary field into material (2).

The *plane of incidence* U is defined by the vectors $\mathbf{a}_i$ and $\mathbf{a}_z$ that lie in it. Since the geometry of the problem is symmetrical with respect to the plane of incidence, there is no preferred direction, and there is no reason for any of the waves to turn towards the $+y$ or $-y$ direction. Therefore, we can suppose that the *vectors of direction of propagation* $\mathbf{a}_i$, $\mathbf{a}_r$, and $\mathbf{a}_t$ of all waves lie in the incident plane U , as shown in Figure 8.5.1. and they are independent of y coordinates. If we denote the angles of incidence, reflection and refraction as θ_i , θ_r and θ_t , we can then write the expressions for unit vectors of direction of propagation of the incident, reflected and refracted wave using the geometrical expressions shown in Figure 8.5.1. as

$$\begin{aligned} \mathbf{a}_i &= \mathbf{a}_x \sin \theta_i + \mathbf{a}_z \cos \theta_i, \\ \mathbf{a}_r &= \mathbf{a}_x \sin \theta_r - \mathbf{a}_z \cos \theta_r, \\ \mathbf{a}_t &= \mathbf{a}_x \sin \theta_t + \mathbf{a}_z \cos \theta_t. \end{aligned} \tag{8.5.1a - c}$$

Supposing all three waves are plane homogeneous TEM waves whose complex electric and magnetic field equations are

$$\begin{aligned}
\mathbf{E}_i &= \mathbf{E}_0 e^{j\omega t - \gamma_1 \mathbf{a}_i \cdot \mathbf{r}} \quad ; \quad \mathbf{H}_i = \frac{1}{Z_1} (\mathbf{a}_i \times \mathbf{E}_i) , \\
\mathbf{E}_r &= \mathbf{E}_1 e^{j\omega t - \gamma_1 \mathbf{a}_r \cdot \mathbf{r}} \quad ; \quad \mathbf{H}_r = \frac{1}{Z_1} (\mathbf{a}_r \times \mathbf{E}_r) , \\
\mathbf{E}_t &= \mathbf{E}_2 e^{j\omega t - \gamma_2 \mathbf{a}_t \cdot \mathbf{r}} \quad ; \quad \mathbf{H}_t = \frac{1}{Z_2} (\mathbf{a}_t \times \mathbf{E}_t) ,
\end{aligned} \tag{8.5.2a - f}$$

where $\mathbf{E}_0$, $\mathbf{E}_1$ and $\mathbf{E}_2$ are the complex amplitudes of the incident, reflected and refracted wave. The amplitudes are, by assumption, independent of coordinates x , y , z . The wave propagation constants and wave impedances are

$$\gamma_1 = \alpha_1 + j\beta_1 \quad , \quad \gamma_2 = \alpha_2 + j\beta_2 \quad , \quad Z_1 = \frac{j\omega\mu_1}{\gamma_1} \quad , \quad Z_2 = \frac{j\omega\mu_2}{\gamma_2} .$$

We shall determine boundary relationships for *an oblique wave incidence* to a boundary S by satisfying the boundary conditions for a continuity of tangential components of the vectors of both electric and magnetic field at the boundary. Tangential components of the total fields on both sides of the boundary S can only be equal when the arguments of the exponential terms in expressions (8.5.2a - f) are equal at the boundary surface S . Thus, the equations

$$\gamma_1 \mathbf{a}_i \cdot \mathbf{r} \big|_{z=0} = \gamma_1 \mathbf{a}_r \cdot \mathbf{r} \big|_{z=0} = \gamma_2 \mathbf{a}_t \cdot \mathbf{r} \big|_{z=0} , \tag{8.5.3a,b}$$

have to be satisfied in the boundary plane $z = 0$. These equations, in fact, state that the phase velocities of all three waves on the boundary plane S are equal. By substituting both the equations (8.5.1a - c) and the condition $z = 0$ into the equations (8.5.3a,b), we derive *the law of reflection* as $\theta_r = \theta_i$, and *Snell's law of refraction* as

$$\frac{\sin \theta_t}{\sin \theta_i} = \frac{\gamma_1}{\gamma_2} . \tag{8.5.4}$$

The law of reflection states that the angle of reflection is equal to the angle of incidence, while Snell's law of refraction states that the ratio of sines of the angles of refraction and incidence is equal to the reciprocal ratio of the propagation constants of materials in corresponding half spaces. Both laws are independent of the polarization of the incident wave.

8.5.1.2. Fresnel formulae and the influence of polarization

We shall apply the conditions of equality for tangential components of the total electric and magnetic field strength on both sides of the boundary S in order to determine relationships between the field amplitudes $\mathbf{E}_0$, $\mathbf{E}_1$ and $\mathbf{E}_2$. The amplitudes are in general complex constants since the incident, reflected and refracted TEM waves are presumed to be *homogeneous and linearly polarized*. Therefore, the equations

$$\mathbf{a}_z \times (\mathbf{E}_0 + \mathbf{E}_1) = \mathbf{a}_z \times \mathbf{E}_2 \quad , \quad \mathbf{a}_z \times (\mathbf{H}_0 + \mathbf{H}_1) = \mathbf{a}_z \times \mathbf{H}_2 \quad , \quad (8.5.5a,b)$$

should be satisfied at all points of the plane S . By usage of the identity for a triple cross product $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$ and by a dot product of equation (8.5.5a) with the normal $\mathbf{a}_z$, we obtain

$$[\mathbf{a}_z \cdot (\mathbf{E}_0 + \mathbf{E}_1)]\mathbf{a}_z - (\mathbf{a}_z \cdot \mathbf{a}_z)(\mathbf{E}_0 + \mathbf{E}_1) = (\mathbf{a}_z \cdot \mathbf{E}_2)\mathbf{a}_z - (\mathbf{a}_z \cdot \mathbf{a}_z)\mathbf{E}_2 \quad . \quad (8.5.6)$$

The usage of a triple cross product identity, and substitution of expressions (8.5.2a - f) and (8.5.3a,b) into (8.5.5b) results in

$$\begin{aligned} \frac{1}{Z_1} [(\mathbf{a}_z \cdot \mathbf{E}_0)\mathbf{a}_u - (\mathbf{a}_z \cdot \mathbf{a}_u)\mathbf{E}_0 + (\mathbf{a}_z \cdot \mathbf{E}_1)\mathbf{a}_r - (\mathbf{a}_z \cdot \mathbf{a}_r)\mathbf{E}_1] = \\ = \frac{1}{Z_2} [(\mathbf{a}_z \cdot \mathbf{E}_2)\mathbf{a}_p - (\mathbf{a}_z \cdot \mathbf{a}_p)\mathbf{E}_2] \quad . \end{aligned} \quad (8.5.7)$$

The direction of the vector $\mathbf{E}_0$ is arbitrary, but we can always resolve the vector $\mathbf{E}_0$ into one component orthogonal to the plane of incidence and therefore always tangential to S , and one component that lies in the plane of incidence U . The analysis will be simplified if we analyse these two cases separately.

1. The vector $\mathbf{E}_0$ orthogonal to the plane of incidence U .

In this case, all three vectors of the electric field of the incident, reflected and refracted wave can be written as

$$\mathbf{E}_0 = \mathbf{a}_y E_{0y} \quad ; \quad \mathbf{E}_1 = \mathbf{a}_y E_{1y} \quad ; \quad \mathbf{E}_2 = \mathbf{a}_y E_{2y} \quad . \quad (8.5.8a - c)$$

From here it follows that

$$\mathbf{a}_z \cdot \mathbf{E}_0 = \mathbf{a}_z \cdot \mathbf{a}_y E_{0y} = 0 \quad , \quad \mathbf{a}_z \cdot \mathbf{E}_1 = \mathbf{a}_z \cdot \mathbf{a}_y E_{1y} = 0 \quad , \quad \mathbf{a}_z \cdot \mathbf{E}_2 = \mathbf{a}_z \cdot \mathbf{a}_y E_{2y} = 0 \quad .$$

After substitution of these equations into the expressions for boundary conditions (8.5.6,7), we obtain expressions for the amplitudes of the fields at the boundary S

$$\mathbf{E}_0 + \mathbf{E}_1 = \mathbf{E}_2 \quad , \quad Z_2 \cos \theta_i \mathbf{E}_0 - Z_2 \cos \theta_r \mathbf{E}_1 = Z_1 \cos \theta_t \mathbf{E}_2 \quad . \quad (8.5.9a,b)$$

The directions of the vectors of electric and magnetic fields for all three linearly polarized waves are shown in Figure 8.5.2 a). For a given $\mathbf{E}_0$ and θ_i , we can calculate the amplitudes of reflected and refracted wave $\mathbf{E}_1$ and $\mathbf{E}_2$ from the equations (8.5.9a,b) and using expression (8.5.4) as

$$\begin{aligned} \mathbf{E}_{1\perp} &= \frac{Z_2 \cos \theta_i - Z_1 \cos \theta_t}{Z_2 \cos \theta_r + Z_1 \cos \theta_t} \mathbf{E}_{0\perp} \quad , \\ \mathbf{E}_{2\perp} &= \frac{Z_2 \cos \theta_i + Z_2 \cos \theta_r}{Z_2 \cos \theta_r + Z_1 \cos \theta_t} \mathbf{E}_{0\perp} \quad . \end{aligned} \quad (8.5.10a,b)$$

We obtain equations known as *Fresnel formulae* for the case when $\mathbf{a}_z \cdot \mathbf{E}_0 = 0$. This

means that the electric field of the incident wave is orthogonal to the plane of incidence. If this is the case, we may say that the polarization of the incident wave is orthogonal to the plane of incidence.

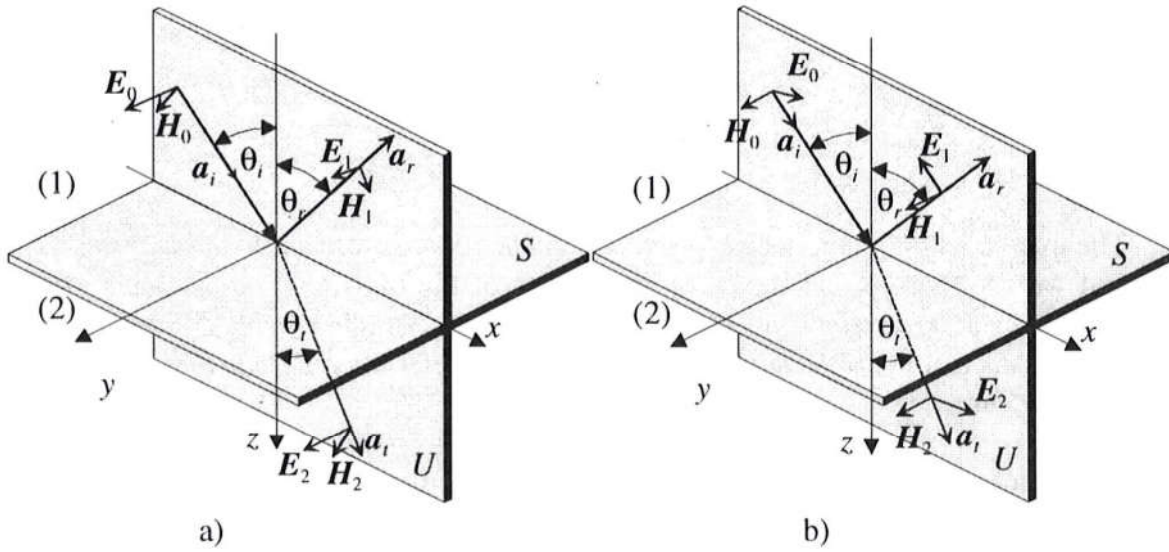


Figure 8.5.2. A plane wave at the boundary between two materials: a) polarization of the waves orthogonal to the plane of incidence, b) polarization parallel to the plane of incidence

The equations are not as simple as they seem at first glance. The angle of refraction θ_r is, according to (8.5.4), *complex* if any of the two materials (1) and (2) is lossy (conductive). The angle θ_r can be complex even in the case where both materials are lossless dielectrics. Using the laws of reflection and refraction we obtain the expressions

$$\cos \theta_r = \cos \theta_i \quad ; \quad \cos \theta_r = \frac{1}{\gamma_2} \sqrt{\gamma_2^2 - \gamma_1^2 \sin^2 \theta_i} \quad , \quad (8.5.11a,b)$$

and we can thus eliminate the angles of reflection and refraction from Fresnel formulae (8.5.10a,b) and obtain an alternative form of these expressions

$$E_{1\perp} = \frac{Z_2 \cos \theta_i - \frac{Z_1}{\gamma_2} \sqrt{\gamma_2^2 - \gamma_1^2 \sin^2 \theta_i}}{Z_2 \cos \theta_i + \frac{Z_1}{\gamma_2} \sqrt{\gamma_2^2 - \gamma_1^2 \sin^2 \theta_i}} E_{0\perp} \quad , \quad (8.5.12a,b)$$

$$E_{2\perp} = \frac{2Z_2 \cos \theta_i}{Z_2 \cos \theta_i + \frac{Z_1}{\gamma_2} \sqrt{\gamma_2^2 - \gamma_1^2 \sin^2 \theta_i}} E_{0\perp} \quad .$$

Equations (8.5.2a – c) and (8.5.8a – c) show that when the vectors E_i , E_r and E_t are orthogonal to the plane of incidence U , only y -components exist. We have denoted their amplitudes as E_{0y} , E_{1y} , E_{2y} . These are the tangential components with respect to

the boundary plane S . It is often convenient to express Fresnel formulae as the *ratios between the amplitudes of the tangential components of the electric fields of the reflected or refracted wave and incident wave*, i.e. as

$$\rho_{\perp} = \frac{E_{1y}}{E_{0y}} = \frac{\frac{Z_2}{\cos \theta_t} - \frac{Z_1}{\cos \theta_i}}{\frac{Z_2}{\cos \theta_t} + \frac{Z_1}{\cos \theta_i}} \quad , \quad \tau_{\perp} = \frac{E_{2y}}{E_{0y}} = \frac{2 \frac{Z_2}{\cos \theta_t}}{\frac{Z_2}{\cos \theta_t} + \frac{Z_1}{\cos \theta_i}} \quad (8.5.13a,b)$$

The quantities $\rho_{\perp}$ and $\tau_{\perp}$ are known as *the coefficients of reflection and transmission* and we use them to calculate the amplitudes of the tangential components of the reflected and refracted wave. These coefficients are generally complex quantities, and this means that the *reflected and refracted waves differ in amplitude and phase from the incident wave*.

2. The vector $\mathbf{E}_0$ parallel to the plane of incidence U .

In the previous case, the vectors of electric field were tangential to the boundary S . In this case, the vectors of magnetic field are all orthogonal to the plane of incidence U and are tangential to the boundary S . Therefore,

$$\mathbf{H}_0 = \mathbf{a}_y H_{0y} \quad ; \quad \mathbf{H}_1 = \mathbf{a}_y H_{1y} \quad ; \quad \mathbf{H}_2 = \mathbf{a}_y H_{2y} \quad , \quad (8.5.14a - c)$$

and

$$\mathbf{a}_z \cdot \mathbf{H}_0 = \mathbf{a}_z \cdot \mathbf{a}_y H_{0y} = 0 \quad , \quad \mathbf{a}_z \cdot \mathbf{H}_1 = \mathbf{a}_z \cdot \mathbf{a}_y H_{1y} = 0 \quad , \quad \mathbf{a}_z \cdot \mathbf{H}_2 = \mathbf{a}_z \cdot \mathbf{a}_y H_{2y} = 0 \quad .$$

Using expressions (8.5.2a - f) we obtain

$$\mathbf{E}_0 = -Z_1(\mathbf{a}_i \times \mathbf{H}_0) \quad , \quad \mathbf{E}_1 = -Z_1(\mathbf{a}_r \times \mathbf{H}_1) \quad , \quad \mathbf{E}_2 = -Z_2(\mathbf{a}_t \times \mathbf{H}_2) \quad . \quad (8.5.15a - c)$$

Substitution of these expressions into the equations for the boundary conditions (8.5.5a,b), and using (8.5.2a - f) results in

$$\mathbf{H}_0 + \mathbf{H}_1 = \mathbf{H}_2 \quad , \quad Z_1 \cos \theta_i \mathbf{H}_0 - Z_1 \cos \theta_r \mathbf{H}_1 = Z_2 \cos \theta_t \mathbf{H}_2 \quad , \quad (8.5.16a,b)$$

where the vectors $\mathbf{H}_0$, $\mathbf{H}_1$ and $\mathbf{H}_2$ are the amplitudes of the magnetic fields of the incident, reflected and refracted wave ($\mathbf{H}_i$, $\mathbf{H}_r$ and $\mathbf{H}_t$). The directions of the vectors of electric and magnetic fields for all three waves are shown in Figure 8.5.2. b). Solutions of the system (8.5.16a,b) are:

$$\left. \begin{aligned} \mathbf{H}_{1\perp} &= \frac{Z_1 \cos \theta_i - Z_2 \cos \theta_t}{Z_1 \cos \theta_r + Z_2 \cos \theta_t} \mathbf{H}_{0\perp} \quad , \\ \mathbf{H}_{2\perp} &= \frac{Z_1 \cos \theta_i + Z_1 \cos \theta_r}{Z_1 \cos \theta_r + Z_2 \cos \theta_t} \mathbf{H}_{0\perp} \quad . \end{aligned} \right\} \quad \text{for } \mathbf{a}_z \cdot \mathbf{H}_0 = 0 \quad . \quad (8.5.17a,b)$$

We can calculate the amplitudes of the magnetic fields of the reflected $\mathbf{H}_1$ and re-

fracted $\mathbf{H}_2$ waves using equations (8.5.17a,b) if the amplitude of the magnetic field vector $\mathbf{H}_0$ of the incident wave propagating in material (1) is given. Generally, those are complex quantities and are shifted in phase with respect to $\mathbf{H}_0$.

In the year 1823, Fresnel derived expressions (8.5.10a,b) and (8.5.17a,b) in a slightly less general form using the dynamic properties of a hypothetical aether. If we eliminate the angles θ_r and θ_t from (8.5.17a,b) using equations (8.5.4), we obtain

$$\mathbf{H}_{1\perp} = \frac{Z_1 \cos \theta_i - \frac{Z_2}{\gamma_2} \sqrt{\gamma_2^2 - \gamma_1^2 \sin^2 \theta_i}}{Z_1 \cos \theta_i + \frac{Z_2}{\gamma_2} \sqrt{\gamma_2^2 - \gamma_1^2 \sin^2 \theta_i}} \mathbf{H}_{0\perp}, \quad (8.5.18a,b)$$

$$\mathbf{H}_{2\perp} = \frac{2Z_1 \cos \theta_i}{Z_1 \cos \theta_i + \frac{Z_2}{\gamma_2} \sqrt{\gamma_2^2 - \gamma_1^2 \sin^2 \theta_i}} \mathbf{H}_{0\perp}.$$

In the previous case 1, the total electric field was tangential to the boundary plane S . If the linearly polarized incident field $\mathbf{E}_0$ is *in the plane of incidence*, as it is in this case 2, only the component $\mathbf{E}_{0x}$ is tangential to the boundary S . Therefore, the wave is partially reflected and partially refracted without any change of polarization, and the coefficients of reflection and transmission that relate the amplitudes of the *tangential* components of the reflected, refracted and incident electric fields together are

$$\rho_{||} = \frac{E_{1x}}{E_{0x}} = \frac{\frac{Z_2}{\cos \theta_i} - \frac{Z_1}{\cos \theta_r}}{\frac{Z_2}{\cos \theta_i} + \frac{Z_1}{\cos \theta_r}}, \quad \tau_{||} = \frac{E_{2x}}{E_{0x}} = \frac{2 \frac{Z_2}{\cos \theta_i}}{\frac{Z_2}{\cos \theta_i} + \frac{Z_1}{\cos \theta_r}}. \quad (8.5.19a,b)$$

Fresnel formulae show that the part of the incident wave that is reflected and the part that is refracted depend (in amplitude and in phase) not only on the materials at the border and the angle of incidence, but also on the nature of the polarization of the incident wave as well.

3. *The orthogonal incidence of a wave on the boundary S .*

If the incident wave is orthogonal to the boundary surface S , then $\theta_i = 0$. In this case there is no difference between the cases 1. and 2. The angle of reflection is $\theta_r = 0$, and the angle of refraction is $\theta_t = 0$. The amplitudes of the reflected and refracted waves are

$$\left. \begin{aligned} E_1 &= \frac{Z_2 - Z_1}{Z_2 + Z_1} E_0, \\ E_2 &= \frac{2Z_2}{Z_2 + Z_1} E_0, \end{aligned} \right\} \quad \text{for } \mathbf{a}_i = \mathbf{a}_z. \quad (8.5.20a,b)$$

Since the electric and magnetic fields are tangential to the boundary S , the same result would be obtained using both expressions (8.5.13a,b), and (8.5.19a,b). The coefficients of reflection and transmission for orthogonal incidence of a wave to the boundary surface S are

$$\rho = \frac{E_1}{E_0} = \frac{Z_2 - Z_1}{Z_2 + Z_1}, \quad \tau = \frac{E_2}{E_0} = \frac{2Z_2}{Z_2 + Z_1}. \quad (8.5.21a,b)$$

Thus, in the case of orthogonal incidence, the nature of the polarization of the wave is not important. There is no reflection ($\rho = 0$) if $Z_2 = Z_1$. Total reflection $|\rho| = 1$ occurs when Z_2 is equal to zero, or infinite, or purely imaginary.

8.5.2. The boundary between two dielectrics

We shall analyse the incidence of a plane homogeneous TEM wave on a plane boundary S between two dielectrics with constitutive parameters $\epsilon_1, \mu_1 = \mu_0$ and $\epsilon_2, \mu_2 = \mu_0$. The dielectrics are ideal lossless materials and their conductivities are thus $\kappa_1 = \kappa_2 = 0$. Although this is an idealised case that cannot really exist, the analysis is relatively simple and the results are very perceptible. They can be used as an approximation for many good insulators in reality. In this case, the law of reflection remains unchanged $\theta_r = \theta_i$, while Snell's law of refraction can be written as

$$\frac{\sin \theta_r}{\sin \theta_i} = \frac{\gamma_1}{\gamma_2} = \frac{c_2}{c_1} = \frac{v_{f2}}{v_{f1}} = \sqrt{\frac{\epsilon_{r1}}{\epsilon_{r2}}} = \frac{n_1}{n_2} = n_{12}. \quad (8.5.22)$$

Here, v_{f1} and v_{f2} are the phase velocities, and n_1 and n_2 are the indices of refraction of dielectrics (1) and (2), while n_{12} is the relative index of refraction between these two dielectrics.

If the relative permittivities of the materials satisfy $\epsilon_{r2} > \epsilon_{r1}$, then the relative index of refraction is $n_{12} < 1$. This means that there will be a real angle of refraction θ_r that necessarily corresponds to every angle of incidence θ_i , since $(n_{12} \sin \theta_i) < 1$.

If $\epsilon_{r2} < \epsilon_{r1}$, which is the case for any example where a wave propagates from a liquid or solid dielectric to air, the angle of refraction θ_r is real only for values of angle θ_i for which $(n_{12} \sin \theta_i) \leq 1$. When $(n_{12} \sin \theta_i) > 1$, the phenomenon of *total reflection* occurs and the incident wave is totally reflected.

8.5.3. Transmission of a wave into a lossy material

The laws of reflection and refraction, as well as the coefficients of reflection and transmission derived in Section 8.5.1. are generally valid and they are valid in the case when a homogeneous plane TEM wave impinges from an ideal dielectric (1) of constitutive parameters $\epsilon_1, \mu_1 = \mu_0, \kappa_1 = 0$ and transmits into a linear lossy material (2) of constitutive parameters $\epsilon_2, \mu_2, \kappa_2 \neq 0$ as well.

We shall suppose that material (1) is lossless in order to simplify the problem. Most frequently, this material is either a vacuum or air. The phenomena of reflection and refraction are significantly modified by the presence of a lossy (conductive) material at the boundary. All laws derived in Section 8.5.1. are valid in a purely formal way but the complex values of propagation constants γ and of the wave impedances Z result in complex values of the angle of refraction θ_r , as well as of the coefficients of reflection ρ and of transmission τ .

Therefore, the physical interpretation is completely different when compared to that when the materials at the boundary are dielectrics. The field in dielectric (1) is, in fact, of the same form as it was in the case when the material (2) was lossless (dielectric). Losses in material (2) introduce a major change to the field of the transmitted wave, which subsequently becomes a *non-homogeneous plane wave*. In continuation, we shall derive this transmitted field and its wave.

The attenuation and phase constants in material (1) are

$$\alpha_1 = 0 \quad , \quad \beta_1 = \frac{\omega}{c_1} = \omega \sqrt{\mu_1 \epsilon_1} \quad , \quad \gamma_1 = j\beta_1 \quad . \quad (8.5.23a - c)$$

The propagation constant for material (2) is

$$\gamma_2 = \alpha_2 + j\beta_2 \quad . \quad (8.5.24)$$

Let a homogeneous plane TEM wave ($\mathbf{E}_i, \mathbf{H}_i$) at a frequency of ω with a given angle of incidence θ_i be incident on the boundary S as shown in Figure 8.5.3. From the continuity of phase velocities at the boundary S it follows that

$$\gamma_1 \mathbf{a}_i \cdot \mathbf{r} \big|_{z=0} = \gamma_2 \mathbf{a}_t \cdot \mathbf{r} \big|_{z=0} \quad . \quad (8.5.25)$$

After substitution we obtain two real equations

$$\alpha_2 \sin \theta_t = 0 \quad , \quad \beta_1 \sin \theta_i = \beta_2 \sin \theta_t \quad . \quad (8.5.26a,b)$$

If we introduce the phase constants β_{1x}, β_{2x} and β_{2z} in the direction of the x and z axes we obtain:

$$\beta_{1x} = \beta_1 \sin \theta_i \quad , \quad \beta_{2x} = \beta_2 \sin \theta_t \quad , \quad \beta_{2z} = \beta_2 \cos \theta_t \quad . \quad (8.5.27a - c)$$

We then introduce the attenuation constants α_{2x} and α_{2z} along the x and z axes in a similar way as

$$\alpha_{2x} = \alpha_2 \sin \theta_t, \quad \alpha_{2z} = \alpha_2 \cos \theta_t. \quad (8.5.28a,b)$$

Now we can write expressions (8.5.26a,b) in a more convenient form as

$$\alpha_{2x} = 0, \quad \beta_{1x} = \beta_{2x} = \frac{\omega}{c_1} \sin \theta_i. \quad (8.5.29a,b)$$

Therefore, we have determined the attenuation and phase constants of the transmitted wave in the direction of the x -axis from the condition of field-continuity at the boundary S ($z = 0$). According to (8.5.2e), the electric field of the transmitted wave is

$$\mathbf{E}_t = \mathbf{E}_2 e^{j\omega t - \gamma_2 \mathbf{a}_t \cdot \mathbf{r}}, \quad (8.5.30)$$

and the indices of the propagation factor in material (2) is

$$\gamma_2 \mathbf{a}_t \cdot \mathbf{r} = (\alpha_2 + j\beta_2)(\mathbf{a}_x \sin \theta_t + \mathbf{a}_z \cos \theta_t) \cdot (\mathbf{a}_x x + \mathbf{a}_y y + \mathbf{a}_z z),$$

which, after rearrangement, results in

$$\gamma_2 \mathbf{a}_t \cdot \mathbf{r} = \alpha_2 \sin \theta_t x + \alpha_2 \cos \theta_t z + j(\beta_2 \sin \theta_t x + \beta_2 \cos \theta_t z).$$

Finally, taking into consideration (8.5.27 – 29) we obtain

$$\gamma_2 \mathbf{a}_t \cdot \mathbf{r} = \alpha_{2z} z + j(\beta_{2x} x + \beta_{2z} z). \quad (8.5.31)$$

Let us introduce the vector of transmitted-wave propagation, which is generally defined by

$$\gamma_t = \mathbf{a}_x (\alpha_{2x} + j\beta_{2x}) + \mathbf{a}_y (\alpha_{2y} + j\beta_{2y}) + \mathbf{a}_z (\alpha_{2z} + j\beta_{2z}), \quad (8.5.32)$$

and, according to (8.5.31), in this case we get

$$\gamma_t = \gamma_2 \mathbf{a}_t = \mathbf{a}_z \alpha_{2z} + j(\mathbf{a}_x \beta_{2x} + \mathbf{a}_z \beta_{2z}). \quad (8.5.33)$$

Finally, the electric field of the transmitted wave in the lossy material (2), according to (8.5.30) and (8.5.33), can be written as

$$\mathbf{E}_t = \mathbf{E}_2 e^{j\omega t - \gamma_t \cdot \mathbf{r}} = \mathbf{E}_2 e^{-\alpha_{2z} z + j(\omega t - \beta_{2x} x - \beta_{2z} z)}. \quad (8.5.34)$$

The magnetic field of the transmitted wave is determined by expression (8.5.2f). On the basis of equation (8.5.34), we can conclude that:

1. The transmitted wave is attenuated in the direction of the z -axis, which is normal to the boundary surface S . The amplitude of the wave depends only on the depth passed by the transmitted wave.
2. The plane of constant amplitude in the lossy material (2) is defined by the equation $z = \text{const.}$ and is normal to the z -axis.
3. The phase of the transmitted wave propagates in the direction $\mathbf{a}_t$, which is in the direction of the x and z axes.
4. The plane of constant phase is defined by the equation $\beta_{2x} x + \beta_{2z} z = \text{const.}$, and is normal to the vector $\mathbf{a}_t$. The angle ψ between the normal vector to the boundary plane

$S \mathbf{a}_z$ and the unit direction vector for the transmitted wave into the material (2) $\mathbf{a}_t$ is the true angle of refraction and is defined by

$$\sin \psi = \frac{\beta_{2x}}{\sqrt{\beta_{2x}^2 + \beta_{2z}^2}} . \quad (8.5.35)$$

The relationship between the planes of constant amplitude and planes of constant phase for the wave transmitted into material (2) is shown in Figure 8.5.3.

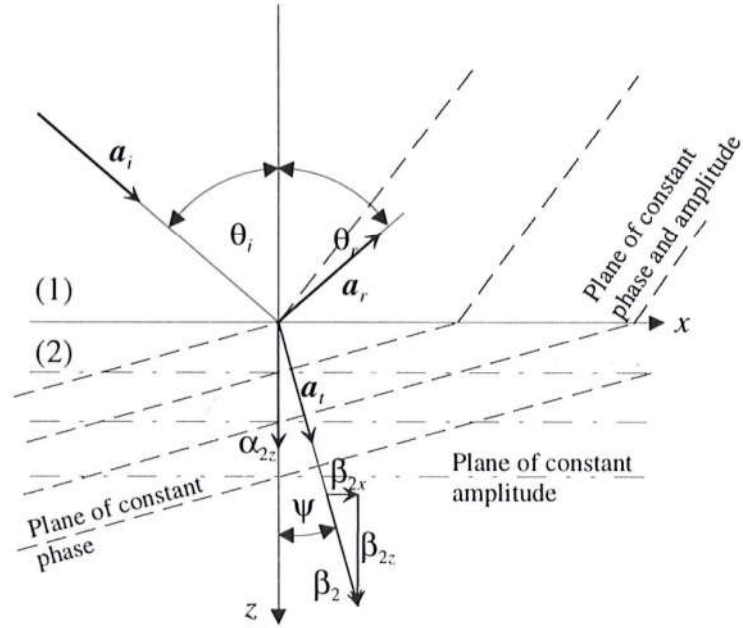


Figure 8.5.3. The transmission of a wave into a lossy material (2)

5. The transmitted wave is a plane *non-homogeneous* wave.

Using the equation $\gamma^2 = -\omega^2\mu\epsilon + j\omega\mu\kappa$ and squaring expression (8.5.93) results in

$$-\omega^2\mu_2\epsilon_2 + j\omega\mu_2\kappa_2 = \alpha_{2z}^2 + j2\alpha_{2z}\beta_{2z} - \beta_{2x}^2 - \beta_{2z}^2 ,$$

and from here two real equations follow

$$\alpha_{2z}^2 - \beta_{2z}^2 = \beta_{1x}^2 - \frac{\omega^2}{c_2^2} ,$$

$$2\alpha_{2z}\beta_{2z} = \omega\mu_2\kappa_2 .$$

Solving these equations results in expressions for the unknown constants in the direction of the z -axis

$$\alpha_{2z} = \frac{\omega}{\sqrt{2}} \left\{ \left(\frac{1}{c_2^2} - \frac{\sin^2 \theta_i}{c_1^2} \right) \left[\left(1 + \left(\frac{\kappa_2}{\omega \epsilon_2} \right)^2 \left(\frac{c_1^2}{c_1^2 - c_2^2 \sin^2 \theta_i} \right)^2 \right)^{1/2} - 1 \right] \right\}^{1/2},$$

$$\beta_{2z} = \frac{\omega}{\sqrt{2}} \left\{ \left(\frac{1}{c_2^2} - \frac{\sin^2 \theta_i}{c_1^2} \right) \left[\left(1 + \left(\frac{\kappa_2}{\omega \epsilon_2} \right)^2 \left(\frac{c_1^2}{c_1^2 - c_2^2 \sin^2 \theta_i} \right)^2 \right)^{1/2} + 1 \right] \right\}^{1/2}. \quad (8.5.36a,b)$$

The propagation factor for the transmitted wave in expression (8.5.34) is completely determined because the attenuation constant α_{2z} is calculated in (8.5.36a), and the phase constants β_{2x} and β_{2z} are calculated by expressions (8.5.29b) and (8.5.36b). In the case of a normally incident wave ($\theta_i = 0$) to the boundary S, equations (8.5.36a,b) are reduced to those we calculated from equations (8.3.30a,b).

8.6. PROPAGATION OF GUIDED PLANE WAVES

In the preceding units, we studied the propagation of plane electromagnetic waves in unbounded or half-bounded spaces. In this unit, we shall analyse the propagation of electromagnetic waves in the vicinity of conductive and dielectric boundaries of complex forms in the case where the form of these boundaries causes the effect of guidance of the waves.

In guidance of waves, we suppose that the direction of flow of energy is primarily along the guidance system. In all of these systems, this wave guidance can be attributed to a direct connection between waves, currents and charge at the conductive surfaces, or to special conditions of reflection at the dielectric boundaries.

Systems for guidance of electromagnetic waves serve in transmission of “electric” energy and information. The simplest, and most commonly used guiding system is “a pair of wires”, or “a two-wire line”, or a *transmission line*. The next most common system is “a coaxial cable” in which one wire is replaced by a metal shield around another wire. In such a way, the field is confined to a certain region, and radiation losses are significantly decreased in comparison with an open two-wire line.

Another possibility is to allow the flow of electromagnetic energy into a hollow metallic pipe called a *waveguide*. This is used because, in modern *microwave* devices, electromagnetic energy is produced at frequencies of $3 \cdot 10^9$ Hz or higher, which corresponds to wavelengths of 10 cm and shorter. At these frequencies, transmission by means of wires would incur intolerable radiation losses. For millimetre waves, dielectric waveguides are thus used. All these systems can be referred to as transmission lines, but we shall refer only to systems with one or more conductors as transmission lines.

To start with, it is suitable to classify wave-guiding systems on the basis of the form of fields or *modes* in which they can be transmitted. Thus, all systems can be classified into two groups:

1. Systems that transmit *TEM modes*. The vectors $\mathbf{E}$ and $\mathbf{H}$ in such modes are transverse and possess no components in the direction of propagation. All lines with two or more conductors, such as a two-wire transmission line or coaxial line, are in this group.
2. Systems that can transmit only *higher-order modes*. These modes always possess at least one component in the direction of propagation. Hollow metallic waveguides and dielectric rods are in this group.

In our mathematical analysis, we wish to find solutions to the wave equations (field vectors, potentials or super-potentials) that will satisfy the boundary conditions at conductive and dielectric boundaries of the wave guiding systems. We shall focus on solutions that represent the transmission of energy along the transmission system. These solutions will be closely tied to the system by the currents flowing in it and through induced charge. We shall describe and analyse some essential characteristics of fields inside electromagnetic resonant cavities.

8.6.1. Transmission lines

We shall derive a theory for transmission lines from electromagnetic field theory using guided waves that propagate along the lines, and will subsequently connect this theory to circuit theory.

An *open two-wire* or Lecher transmission line consisting of two conductors of diameter d at a separation distance D is shown in Figure 8.6.1. The conductors are perfectly conductive ($\kappa \rightarrow \infty$) and are lying parallel to the z -axis. The transmission line is ideal (lossless), uniform and infinitely long.

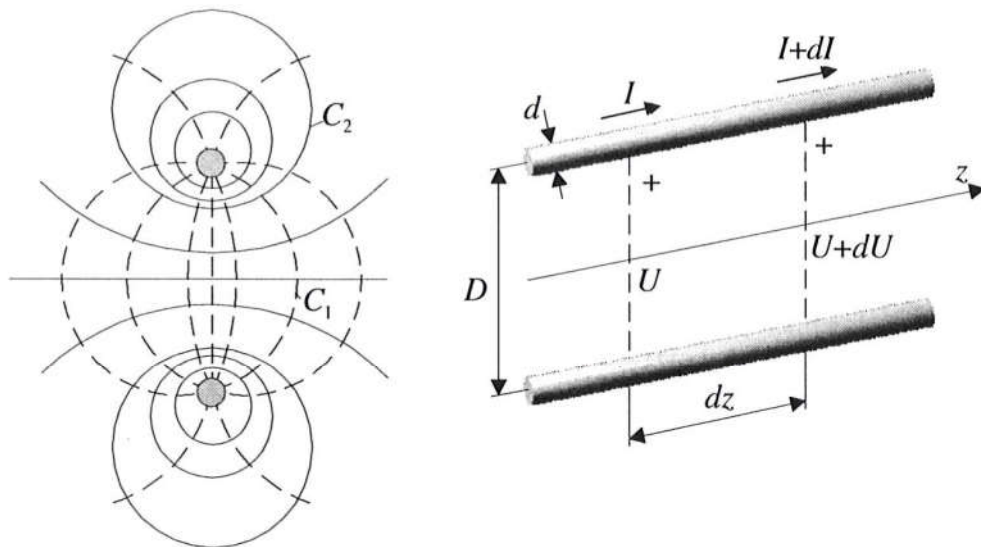


Figure 8.6.1. The electric and magnetic field between conductors of a two-wire transmission line

The electromagnetic field between the conductors must satisfy the boundary conditions at the surface of perfect conductors. Only surface currents and charge can exist at the boundary surface between an (ideal) dielectric and a perfect conductor in an ideal transmission line.

These conditions are approximately valid in real transmission lines. Thus, longitudinal components of field vectors $\mathbf{E}$ and $\mathbf{H}$ do not exist in an ideal line ($E_z = H_z = 0$). These represent the TEM fields of a direct wave, which is assumed to propagate along the line in the $+z$ -direction.

This is not the only wave that can propagate along the transmission line. From Maxwell's equations it follows that an infinite number of different types of wave can exist within the transmission line. Each possible wave is called a *mode*, and the TEM wave is particularly called a *transmission-line mode*. All other waves that necessarily possess E_z or H_z or both are called *higher-order modes*. These modes are usually important only in the vicinity of discontinuities on the line. We shall restrict our study here to the analysis of TEM modes.

Since total similarity exists between a sinusoidal plane TEM wave in an unbounded space and a guided wave in an ideal transmission line, we shall assume that along the line, according to expressions (8.3.6c,d), a direct travelling plane TEM wave propagates

$$\underline{\mathbf{E}}_T = \mathbf{E}_0 e^{-\gamma z} \quad , \quad \underline{\mathbf{H}}_T = \mathbf{H}_0 e^{-\gamma z} \quad , \quad (8.6.1a,b)$$

where $\mathbf{E}_0$ and $\mathbf{H}_0$ are complex vectors, $\underline{\mathbf{E}}_T$ and $\underline{\mathbf{H}}_T$ are complex phasors of transverse fields, and γ is the complex constant of propagation. This assumption will be valid under the condition that the fields (8.6.1a,b) are solutions of wave equations (8.1.3a,b), which were derived from the system of Maxwell's equations.

For *sinusoidal fields in a steady state*, equations (8.1.3a,b) transform into the phasor Helmholtz equations $\Delta \underline{\mathbf{E}}_T = \gamma^2 \underline{\mathbf{E}}_T$ and $\Delta \underline{\mathbf{H}}_T = \gamma^2 \underline{\mathbf{H}}_T$, which must be satisfied together with the corresponding boundary conditions to make the solutions (8.6.1a,b) valid. If we substitute the solutions (8.6.1a,b) into the right-hand side of the phasor Helmholtz equations it follows that both are equal to zero, and we obtain the equations

$$\frac{\partial^2 \underline{\mathbf{E}}_T}{\partial z^2} - \gamma^2 \underline{\mathbf{E}}_T = 0 \quad , \quad \frac{\partial^2 \underline{\mathbf{H}}_T}{\partial z^2} - \gamma^2 \underline{\mathbf{H}}_T = 0 \quad , \quad (8.6.2,3)$$

and these are one-dimensional Helmholtz phasor equations whose solutions are expressions (8.6.1a,b). The conditions which the transverse fields $\underline{\mathbf{E}}_T$ and $\underline{\mathbf{H}}_T$ have to satisfy in order to enable the existence of a plane TEM wave on a transmission line are

$$\frac{\partial^2 \underline{\mathbf{E}}_T}{\partial x^2} + \frac{\partial^2 \underline{\mathbf{E}}_T}{\partial y^2} = 0 \quad , \quad \frac{\partial^2 \underline{\mathbf{H}}_T}{\partial x^2} + \frac{\partial^2 \underline{\mathbf{H}}_T}{\partial y^2} = 0 \quad . \quad (8.6.4a,b)$$

Therefore, the transverse electromagnetic fields $\underline{\mathbf{E}}_T$ and $\underline{\mathbf{H}}_T$ on a transmission line have to satisfy a "pure" static Laplace's equation and the corresponding boundary conditions. The boundary conditions for this problem are $(\mathbf{E}_T)_{\text{tangential}} = 0$ and $(\mathbf{H}_T)_{\text{normal}} = 0$ on the surface of a perfect conductor. This is a very important solution which can be generalized as follows: *we can add the propagation factor $e^{(j\omega t - \gamma z)}$ to every two-dimensional electrostatic field and obtain the electric field of a plane TEM wave. The same is valid for a static magnetic field.* The form of the electromagnetic transverse field depends on the shape of the cross-section of the conductors. This field is non-homogeneous, and the plane TEM wave is non-homogeneous too.

These solutions give exact results under the condition that the distance between the conductors D of a transmission line is much smaller than the wavelength λ of any wave that propagates along the line ($D \ll \lambda$). Obviously, at very high frequencies, the wavelength becomes small enough to enable a strong radiating of electromagnetic energy in a direction transverse to the direction of the line, which is not taken into consideration here. The idea of a quasi-static transverse electromagnetic field in this case does not match reality either.

We shall determine the relationships between the components of the transverse field by substitution of the solutions (8.6.1a,b) into the main Maxwell's equations $\nabla \times \underline{E}_T = -j\omega\mu \underline{H}_T$ and $\nabla \times \underline{H}_T = j\omega\varepsilon \underline{E}_T$. Before this, we shall resolve the transverse fields (8.6.1a,b) into the components

$$\underline{E}_T = \underline{a}_x E_{x0} e^{-\gamma z} + \underline{a}_y E_{y0} e^{-\gamma z}, \quad \underline{H}_T = \underline{a}_x H_{x0} e^{-\gamma z} + \underline{a}_y H_{y0} e^{-\gamma z}, \quad (8.6.5a,b)$$

and substitute them into the main Maxwell's equations obtaining

$$\nabla \times (\underline{a}_x E_{x0} e^{-\gamma z} + \underline{a}_y E_{y0} e^{-\gamma z}) = -j\omega\mu (\underline{a}_x H_{x0} e^{-\gamma z} + \underline{a}_y H_{y0} e^{-\gamma z}),$$

$$\nabla \times (\underline{a}_x H_{x0} e^{-\gamma z} + \underline{a}_y H_{y0} e^{-\gamma z}) = j\omega\varepsilon (\underline{a}_x E_{x0} e^{-\gamma z} + \underline{a}_y E_{y0} e^{-\gamma z}).$$

Solutions of these equations can be written in short form as

$$\frac{E_{x0}}{H_{y0}} = -\frac{E_{y0}}{H_{x0}} = \frac{j\omega\mu}{\gamma} = Z. \quad (8.6.6)$$

The propagation constant and wave impedance in a linear lossless dielectric are

$$\gamma = j\beta = j\omega\sqrt{\mu\varepsilon}, \quad Z = \frac{j\omega\mu}{\gamma} = \sqrt{\frac{\mu}{\varepsilon}} = Z'. \quad (8.6.7)$$

The density of complex electromagnetic power transmitted along the line is equal to the time-averaged Poynting's vector

$$\begin{aligned} N_{av} &= \frac{\underline{E}_T \times \underline{H}_T^*}{2} = \frac{E_{x0} H_{y0} - E_{y0} H_{x0}}{2} \underline{a}_z = \frac{1}{2} \underline{a}_z \left(\frac{E_{x0}^2}{Z'} + \frac{E_{y0}^2}{Z'} \right) = \\ &= \underline{a}_z \frac{|E_0(x,y)|^2}{2Z'}. \end{aligned} \quad (8.6.8)$$

The total power transmitted by the line can be determined by integration of the power density over the surface of the unbounded transverse plane.

Since the transverse electromagnetic field is obtained as a solution of the static Laplace equations (8.6.4a,b) and corresponding boundary conditions we can use "static" *capacitances and inductances* for transmission lines, despite the field being sinusoidal in time. Therefore, we shall directly link the line voltage $\underline{U}$ and line current $\underline{I}$ to the quasi-static fields $\underline{E}_T$ and $\underline{H}_T$ by the integrals

$$\underline{U} = \int_{C_1} \underline{E}_T \cdot d\mathbf{l} \quad , \quad I = \oint_{C_2} \underline{H}_T \cdot d\mathbf{l} \quad , \quad (8.6.9a,b)$$

where C_1 and C_2 are shown in Figure 8.6.1. It is easy to show, using equations (8.6.5,6), that a direct wave in a transmission line satisfies the expressions

$$\underline{E}_T = Z'(\underline{H}_T \times \mathbf{a}_z) \quad ; \quad \underline{H}_T = \frac{1}{Z'}(\mathbf{a}_z \times \underline{E}_T) \quad . \quad (8.6.10a,b)$$

By usage of (8.6.9b) and (8.6.10b), we obtain

$$I = \frac{1}{Z'} \int_{C_2} (\mathbf{a}_z \times \underline{E}_T) \cdot d\mathbf{l} = \frac{1}{Z'} \oint_{C_2} (E_0)_n e^{-\gamma z} dl \quad . \quad (8.6.11)$$

Capacitance in a static field is defined by (4.4.4), and the capacitance per unit length of transmission line is

$$C' = \frac{Q}{U} = \frac{\epsilon}{U} \int \underline{E}_T \cdot \mathbf{n} dS = \frac{\epsilon}{U} \oint_{C_2} (E_0)_n e^{-\gamma z} dl \quad , \quad (8.6.12)$$

measured in farads per metre.

We shall introduce the new notion of *characteristic impedance* of a transmission line and define it as the ratio of line voltage $\underline{U}$ and line current I . Using (8.6.11,12), we obtain the expression for characteristic impedance

$$Z_0 = \frac{U}{I} = Z' \frac{\epsilon}{C'} \quad . \quad (8.6.13)$$

Similarly, using expressions (8.6.9a) and (8.6.10a), we can link the characteristic impedance of a line to the inductance of the line per unit length

$$\underline{U} = Z' \int_{C_1} (\underline{H}_T \times \mathbf{a}_z) \cdot d\mathbf{l} = Z' \int_{C_1} (H_0)_n e^{-\gamma z} dl \quad . \quad (8.7.14)$$

Inductance of the line per unit length in a static magnetic field is

$$L' = \frac{\Psi}{I} = \frac{\mu}{I} \int \underline{H}_T \cdot \mathbf{n} dS = \frac{\mu}{I} \int_{C_1} (H_0)_n e^{-\gamma z} dl \quad , \quad (8.6.15)$$

and we can calculate the characteristic impedance of a line using (8.6.14,15) as being

$$Z_0 = \frac{U}{I} = Z' \frac{L'}{\mu} \quad . \quad (8.6.16)$$

From (8.6.13) and (8.6.16) it follows that the parameters of an ideal transmission line L' and C' are mutually linked by the expression

$$L'C' = \mu\epsilon = \frac{1}{c^2} . \quad (8.7.17)$$

After determination of the electrostatic capacitance C' , or the static magnetic inductance L' , we can determine the characteristic impedance Z_0 of an ideal transmission line using (8.6.13) or (8.6.16).

The static electric and magnetic fields here have vectors $\mathbf{E}_T$ and $\mathbf{H}_T$ that are everywhere mutually orthogonal and we call such fields *conjugate fields*.

We shall now apply circuit theory to a two-wire line shown in Figure 8.6.1. Let us consider an infinitesimal part of the line of length dz as shown in Figure 8.6.2. in order to derive equations of distribution of voltage and current along the line.

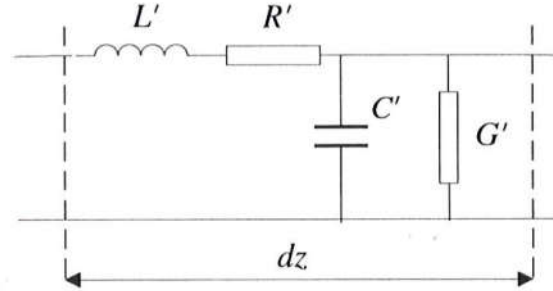


Figure 8.6.2. An equivalent circuit of a part of the transmission line

If we denote the series impedance of a line per unit length as Z_v' , and the shunt admittance of a line per unit length as Y_v' , from circuit theory it follows that

$$d\underline{U} = -\underline{I}Z_v' dz , \quad d\underline{I} = -\underline{U}Y_v' dz . \quad (8.6.18a,b)$$

These expressions enable us to write equations for a transmission line as

$$\frac{d\underline{U}}{dz} = -\underline{I}Z_v' , \quad \frac{d\underline{I}}{dz} = -\underline{U}Y_v' . \quad (8.6.19a,b)$$

These equations imply the assumption that there are no mutual inductive connections between the elementary parts of a line, and that the shunt currents dI are flowing in planes transverse to the z -axis. In field theory, this is equivalent to the assumption that the E_z and H_z components of the field do not exist. The transmission line is homogeneous (uniform) if Z_v' and Y_v' are independent of z coordinates. After differentiation and rearrangement of equations (8.6.19a,b), we obtain

$$\frac{d^2\underline{U}}{dz^2} - Z_v'Y_v'\underline{U} = 0 , \quad \frac{d^2\underline{I}}{dz^2} - Z_v'Y_v'\underline{I} = 0 , \quad (8.6.20a,b)$$

which are two *one-dimensional Helmholtz equations* for phasors of the line voltage and line current. These equations are of the same form as field equations. General solutions of these equations (8.6.20a,b) represent the sum of the direct and inverse travelling waves, and if we introduce the propagation constant

$$\gamma = \sqrt{Z_V' Y_V'} , \quad (8.6.21)$$

we can write them in *phasor form* as

$$\underline{U} = U_1 e^{\gamma z} + U_2 e^{-\gamma z} , \quad \underline{I} = I_1 e^{\gamma z} + I_2 e^{-\gamma z} . \quad (8.6.22a,b)$$

We choose a direct wave that propagates in the + z direction, and we obtain the solution

$$\underline{U}^+ = U_0 e^{-\gamma z} , \quad \underline{I}^+ = I_0 e^{-\gamma z} , \quad (8.6.23a,b)$$

where U_0 and I_0 are general complex quantities. In an ideal line, U_0 and I_0 are real quantities. If we substitute the solutions (8.6.23a,b) into (8.6.19a,b) we obtain

$$\frac{\underline{U}^+}{\underline{I}^+} = \frac{Z_V'}{\gamma} = \frac{\gamma}{Y_V'} = \sqrt{\frac{Z_V'}{Y_V'}} . \quad (8.6.24)$$

Now we can introduce *the characteristic impedance of a line* as being

$$Z_0 = \frac{\underline{U}^+}{\underline{I}^+} = \sqrt{\frac{Z_V'}{Y_V'}} . \quad (8.6.25)$$

According to Figure 8.6.2, it is suitable to write elements of the equivalent circuit of a transmission line as being

$$Z_V' = R' + j\omega L' , \quad Y_V' = G' + j\omega C' , \quad (8.6.26a,b)$$

expressed in ohms and siemens per metre of line length. Obviously, a complete similarity and analogy exists between a linearly polarized TEM plane wave and a transmission line.

If we introduce a lossy insulator instead of the dielectric, with the conductors still assumed to be perfect, all our equations are still valid. The characteristics Z_0 and γ become *complex* numbers.

The most important effect of losses in the insulator is the attenuation of the wave in the direction of propagation. The attenuation and phase constants of a good insulator can be calculated as

$$\alpha \approx \frac{\kappa}{2} \sqrt{\frac{\mu}{\epsilon}} , \quad \beta \approx \omega \sqrt{\mu \epsilon} = \frac{\omega}{c} \quad \Rightarrow \quad \gamma = \alpha + j\beta , \quad Z \approx Z' = \sqrt{\frac{\mu}{\epsilon}} . \quad (8.6.27)$$

When the conductors are real (imperfect), *the field is no longer exactly TEM*, and exact solutions are impractical. Electromagnetic waves will still be characterized by the propagation constant $\gamma = \alpha + j\beta$, but the attenuation constant α will be greater.

Voltage and current in a lossy transmission line can be written in a *complex domain* according to (8.6.22a,b) as

$$U = U_1 e^{az + j(\omega t + \beta z)} + U_2 e^{-az + j(\omega t - \beta z)} , \quad I = -\frac{U_1}{Z_0} e^{az + j(\omega t + \beta z)} + \frac{U_2}{Z_0} e^{-az + j(\omega t - \beta z)} ,$$

where U_1 , U_2 and Z_0 are complex quantities. The flow of the complex time-averaged power is given by

$$P_N = UI^* = \frac{1}{2} \left[-\frac{|U_1|^2}{Z_0^*} e^{2az} + \frac{|U_2|^2}{Z_0^*} e^{-2az} \right] , \quad (8.6.28)$$

while the flow of time-averaged active power along the line is $P_{N,Re} = \text{Re}\{P_N\}$, and if on an infinite line only a direct wave exists, we get

$$P_{N,Re} = \text{Re} \left\{ \frac{|U_0|^2}{2Z_0^*} e^{-2az} \right\} = \text{Re} \left\{ \frac{|U_0|^2}{2|Z_0|} e^{-2az + j\varphi_0} \right\} = \frac{|U_0|^2}{2|Z_0|} e^{-2az} \cos \varphi_0 . \quad (8.6.29)$$

Power losses per unit length in the line are

$$P_g' = -\nabla \cdot \mathbf{P}_{N,Re} = +2\alpha P_{N,Re} = 2\alpha P_0 e^{-2az} \cos \varphi_0 , \quad (8.6.30)$$

where we have denoted

$$P_0 = \frac{|U_0|^2}{2|Z_0|} ; \quad Z_0^* = |Z_0| e^{-j\varphi_0} . \quad (8.6.31a,b)$$

From expression (8.6.30), we can calculate the attenuation constant of a real transmission line as being

$$\alpha = \frac{P_g'}{2P_{N,Re}} . \quad (8.6.32)$$

This is an exact expression for determining the attenuation constant. Losses and active transmitted power can be measured on a real line.

Another way of obtaining the wave propagation constant and characteristic impedance of a real (lossy) line is to use an equivalent circuit. So, using (8.6.2b) and (8.6.26a,b), we obtain

$$\gamma = \sqrt{(R' + j\omega L')(G' + j\omega C')} \Rightarrow Z_0 = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}} . \quad (8.6.33)$$

Here R' is the series (longitudinal) resistance of a line per unit length, and G' is the shunt (transverse) conductance per unit length. The series inductance per unit length is L' , and the shunt capacitance per unit length is C' . We calculate the quantities L' and C' using field theory from Chapters 4 and 5. If R' and G' can be neglected, the line is ideal (lossless). If R' and G' are small, but not negligible, we can approximately write

$$Z_0 \approx \sqrt{\frac{L'}{C'}} \left[1 + j \left(\frac{G'}{2\omega C'} - \frac{R'}{2\omega L'} \right) \right]. \quad (8.6.34)$$

If Heaviside's condition

$$\frac{G'}{C'} = \frac{R'}{L'}, \quad (8.6.35)$$

is satisfied, then Z_0 is real, and such a transmission line is called a *distortionless line*.

8.6.1.1. A terminated transmission line

So far we have analysed lines of infinite length. Now we shall analyse the situation where a line of characteristic impedance Z_0 is terminated at a load impedance Z_l , as shown in Figure 8.6.3.

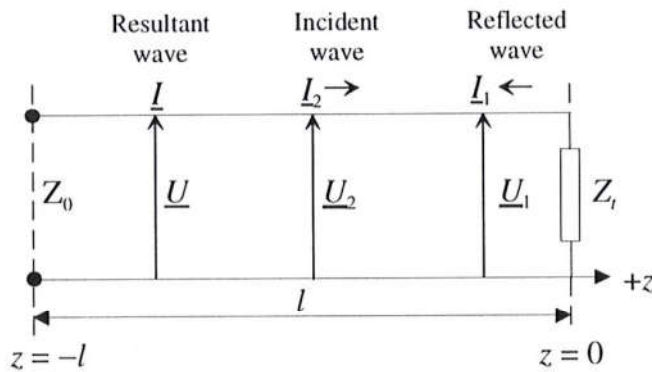


Figure 8.6.3. Terminated transmission line

According to the diagram, the load is at $z = 0$, and positive distance is measured to the left along the line. The total voltage and total current are the sum of two travelling waves moving in opposite directions, as on an infinite transmission line. The wave that propagates to the right is taken as the incident (direct) wave, whereas the wave that propagates to the left is the reflected (inverse) wave.

The load impedance determines the relationship between the reflected and incident waves. At a distance d from the load Z_l (the end of the line), the total voltage and current on the line are $\underline{U}(-d)$ and $\underline{I}(-d)$. $\underline{U}_2$ and $\underline{I}_2$ are the voltage and current at the same place due to the incident travelling wave, while $\underline{U}_1$ and $\underline{I}_1$ are the voltage and current due to the reflected wave. The sum of $\underline{U}_2$ and $\underline{U}_1$ must be equal to the voltage $\underline{U}_l$ that occurs across the load Z_l at $z = 0$, i.e., $\underline{U}_1 + \underline{U}_2 = \underline{U}_l$. Similarly, the sum of currents from the direct and reflected wave along the line at the point of discontinuity must be equal to the current that flows through the load Z_l , i.e., $\underline{I}_1 + \underline{I}_2 = \underline{I}_l$. We can write this equation as

$$\frac{\underline{U}_2}{Z_0} - \frac{\underline{U}_1}{Z_0} = \frac{\underline{U}_l}{Z_l}. \quad (8.6.36)$$

We introduce the *coefficient of reflection* as the ratio of the voltage of the reflected wave and the voltage of the incident wave as being

$$\rho = \frac{U_1}{U_2} = \frac{Z_l - Z_0}{Z_l + Z_0}, \quad (8.6.37)$$

and the *coefficient of transmission* as the ratio of the voltage across the load and voltage of the incident wave as being

$$\tau = \frac{U_l}{U_2} = \frac{2Z_l}{Z_l + Z_0}. \quad (8.6.38)$$

The most interesting case occurs when $Z_l = Z_0$. In that case, $\rho = 0$, meaning that *there is no reflected wave*. The total energy of the incident wave is, in this case delivered to the load. Thus, the coefficient of transmission is $\tau = 1$, which means the line acts as if it is infinite.

In order to obtain the ratio of the total voltage and current U/I at a distance d from the end expressed in terms of the coefficient of reflection $\rho = U_1/U_2$ at $z = 0$, we divide (8.6.22a) by (8.6.22b). This ratio is called the line impedance at any point d and it is

$$Z_d = \frac{U(-d)}{I(-d)} = Z_0 \frac{1 + \rho e^{-2\gamma d}}{1 - \rho e^{-2\gamma d}} = Z_0 \frac{Z_l \cosh \gamma d + Z_0 \sinh \gamma d}{Z_0 \cosh \gamma d + Z_l \sinh \gamma d}. \quad (8.6.39)$$

We may calculate the impedance at the beginning of the line if we substitute $d = l$. We call this the *input impedance of the line* expressed by

$$Z_u = Z_0 \frac{Z_l \cosh \gamma l + Z_0 \sinh \gamma l}{Z_0 \cosh \gamma l + Z_l \sinh \gamma l}.$$

It is suitable to introduce the *voltage standing wave ratio* (VSWR) on the transmission line. For a lossless line, this is given by

$$\text{VSWR} = \frac{U_{\max}}{U_{\min}} = \frac{U_2 + U_1}{U_2 - U_1} = \frac{1 + |\rho|}{1 - |\rho|}.$$

8.6.2. Waveguides

The systems that guide electromagnetic waves are called waveguides. Waveguides are hollow pipes of highly-conductive metal and are most frequently of rectangular or circular cross-section. The rectangular infinitely long ideal waveguide shown in Figure 8.6.4. is the most commonly used among hollow-pipe waveguides.

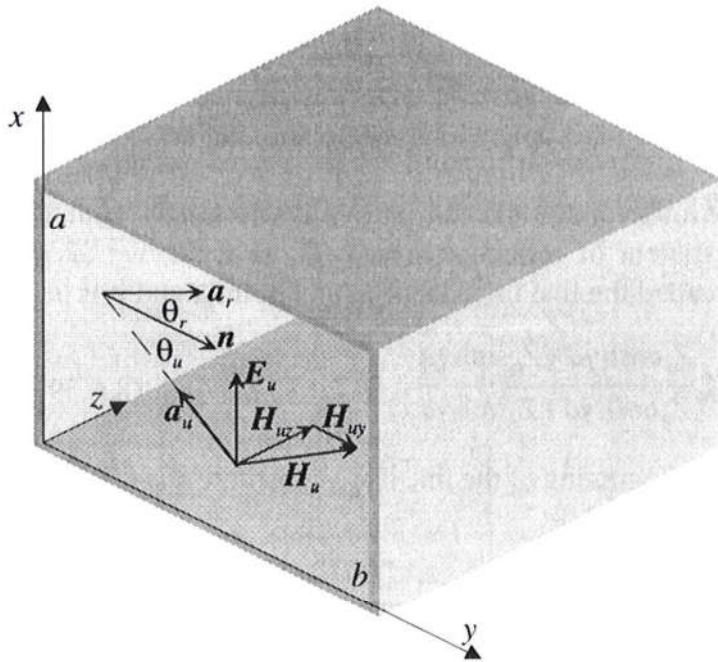


Figure 8.6.4. A rectangular waveguide with a TEM incident wave

The waveguide is a rectangular hollow pipe made of a perfect conductor ($\kappa_k \rightarrow \infty$) and filled with an ideal dielectric ($\epsilon, \mu, \kappa = 0$). The wave characteristics of the dielectric are $\alpha = 0, \beta = \omega(\mu\epsilon)^{1/2}, \gamma = j\beta, Z = j\omega\mu/\gamma = \omega\mu/\beta = (\mu/\epsilon)^{1/2}$, (the most common dielectric used in waveguides is air). Suppose that a wave incident at an angle of incidence θ_i with respect to the normal $\mathbf{n} = \mathbf{a}_y$ impinges on the $x - z$ plane. The wave will be reflected at the perfectly conductive boundary in the $x - z$ plane at an angle of reflection equal to the angle of incidence ($\theta_r = \theta_i$).

The total electric field after reflection at the wall of the waveguide is

$$E_{Tx} = E_{0x} \sin(\beta y \cos \theta_i) e^{-j\beta z \sin \theta_i} \quad (8.6.40)$$

In this coordinate system, we then introduce a new term into this equation, that being $\underline{E}_{0x} = -j2E_0$, assuming that the field E_{0x} is normal to the plane of incidence U . After that we introduce the following:

$$\beta_y = \beta \cos \theta_i = \omega \sqrt{\mu\epsilon} \cos \theta_i = \beta_c \quad , \quad j\beta_z = j\beta \sin \theta_i = j\omega \sqrt{\mu\epsilon} \sin \theta_i = \gamma_v \quad .$$

If we square these two equations and subtract one equation from the other, we obtain $\gamma_v^2 = \beta_c^2 - \beta^2$, and the total electric field (8.6.40) in the waveguide can thus be written as

$$\underline{E}_{Tx} = \underline{E}_{0x} \sin(\beta_c y) e^{-\gamma_v z} . \quad (8.6.41)$$

Let us now analyse the possible existence of such a field in an ideal rectangular waveguide. Firstly, only the x -component of electric field E_{Tx} exists at the surfaces $x = 0$ and $x = a$, meaning that there is no tangential component of the field. This is in accordance with the boundary conditions at the surface of a perfect conductor. Secondly, the tangential component of electric field should be equal to zero at the surfaces $y = 0$ and $y = b$. Therefore, the condition

$$\beta_c b = n\pi \quad ; \quad n = 1, 2, 3, \dots \quad (8.6.42)$$

has to be satisfied as well. From this condition it follows that the allowed values of β_c

$$\beta_c = \frac{n\pi}{b} , \quad (8.6.43)$$

are those that satisfy the boundary conditions in the waveguide.

Here we may observe the possible fields in the waveguide, because every choice from (8.6.43) determines a possible field or mode. The modes in the waveguide are usually classified with respect to the existence of a longitudinal z -component of the field. A mode without an E_z component is called a *TE mode*, and one without a H_z component is called a *TM mode*. All modes in the rectangular waveguide belong to either of these two types. The modes represented by (8.6.41) and (8.6.43) do not possess an E_z component and they are thus TE modes. For $n = 1, 2, 3, \dots$ we obtain TE_{0n} modes. The subscript 0 denotes that there is no dependence on x -coordinates, and subscript n denotes the choice from (8.6.43).

If the dielectric is lossless, $\beta = \omega/c$ is a real quantity, and we can generally introduce the propagation constant $\gamma_v = \alpha_v + j\beta_v$ for the waveguide as being

$$\begin{aligned} \gamma_v = \alpha_v &= \sqrt{\left(\frac{n\pi}{b}\right)^2 - \beta^2} \quad ; \quad \beta < \frac{n\pi}{b} , \\ \gamma_v = j\beta_v &= j\sqrt{\beta^2 - \left(\frac{n\pi}{b}\right)^2} \quad ; \quad \beta > \frac{n\pi}{b} . \end{aligned} \quad (8.6.44a,b)$$

Obviously, the propagation of a wave in the z -direction along the waveguide will happen if $\beta > \beta_c$, or $\gamma_v = j\beta_v$, and such modes are called *propagation modes*. The field exponentially decreases along the z -axis and there is no propagation in the waveguide if $\beta < \beta_c$, or $\gamma_v = \alpha_v$. We can say that under this condition no modes are propagated. The transition from one way of behaviour to the other happens suddenly when $\beta = \beta_c = n\pi/b$ ($\alpha_v = \beta_v = 0$). If we substitute $\beta = \omega/c$ into this condition, we obtain the *cutoff (critical) frequency* and the corresponding *cutoff (critical) wavelength* of the TE_{0n} mode as being

$$f_c = \frac{n}{2b\sqrt{\mu\epsilon}} , \quad \lambda_c = \frac{2b}{n} . \quad (8.6.45)$$

The mode propagates at wave-frequencies in the propagation range $f > f_c$ ($\lambda < \lambda_c$), and the mode does not propagate at frequencies within the attenuation range $f < f_c$ ($\lambda > \lambda_c$).

Knowledge of f_c or λ_c leads us to knowing β_c in the waveguide, because

$$\beta_c = \frac{2\pi}{\lambda_c} = 2\pi f_c \sqrt{\mu\epsilon} . \quad (8.6.46)$$

The quantities f_c and λ_c are determined by the boundary value problem. The propagation constant of the waveguide (8.6.44a,b) can be now expressed as

$$\begin{aligned} \gamma_v = \alpha_v = \beta_c \sqrt{1 - \left(\frac{f}{f_c}\right)^2} & \quad ; \quad f < f_c , \\ \gamma_v = j\beta_v = j\beta_c \sqrt{1 - \left(\frac{f_c}{f}\right)^2} & \quad ; \quad f > f_c . \end{aligned} \quad (8.6.47a,b)$$

Obviously, the phase constant β_v of a particular propagating mode in the waveguide is always smaller than the phase constant β of the dielectric. The constant β_v tends to β as $f \rightarrow \infty$. The attenuation constant α_v of a mode that does not propagate is always smaller than β_c . The constant α_v tends to β_c as $f \rightarrow 0$.

It is possible to determine the wavelength λ_v and the phase velocity v_v along the waveguide for a mode that is propagating in the waveguide. Thus, *the wavelength of propagation* is determined as the length at which the phase of the electric field increases to 2π , or $\beta_v \lambda_v = 2\pi$. If we substitute for β_v from (8.6.47b), we obtain

$$\lambda_p = \frac{\lambda}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}} . \quad (8.6.48)$$

This expression shows that the wavelength of propagation is always smaller than the wavelength $\lambda = 2\pi/\beta$ in the dielectric. The phase velocity of propagation v_v along the waveguide is determined as

$$v_p = \frac{\omega}{\beta_v} = \frac{v_f}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}} , \quad (8.6.49)$$

where $v_f = \omega/\beta$ is the phase velocity of the mode in the dielectric. The phase velocity of propagation along the waveguide is always greater than the phase velocity in the dielectric.

The existence of a *characteristic wave impedance* is especially important for modes in the waveguide. Therefore, let us write all the components of the field of the propagating wave in the waveguide

$$\underline{H}_T = \underline{a}_z \frac{E_{0x}}{j} \frac{\cos \theta_i}{Z} \cos(\beta y \cos \theta_i) e^{-j\beta z \sin \theta_i} + \underline{a}_y \frac{E_{0x}}{Z} \frac{\sin \theta_i}{\sin(\beta y \cos \theta_i)} e^{-j\beta z \sin \theta_i}.$$

Finally we obtain expressions for all field components as being

$$\begin{aligned} E_{Tx} &= E_{0x} \sin(\beta_c y) e^{-\gamma_v z}, \\ H_{Ty} &= \frac{\gamma_v}{j\omega\mu} E_{0x} \sin(\beta_c y) e^{-\gamma_v z}, \quad H_{Tz} = \frac{\beta_c}{j\omega\mu} E_{0x} \cos(\beta_c y) e^{-\gamma_v z}. \end{aligned} \quad (8.6.50a - c)$$

The wave impedance of the waveguide in the direction of the z -axis is

$$Z_z = \frac{E_{Tx}}{H_{Ty}} = \frac{j\omega\mu}{\gamma_v} = Z_0, \quad (8.6.51)$$

and this is called the *characteristic impedance of the mode*. It is denoted as Z_0 . If we substitute (8.6.47a,b) into expression (8.6.51), we obtain the characteristic impedance for TE modes as

$$Z_0 = \frac{jZ}{\sqrt{\left(\frac{f_c}{f}\right)^2 - 1}} \quad ; \quad f < f_c \quad , \quad Z_0 = \frac{Z}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}} \quad ; \quad f > f_c. \quad (8.6.52a,b)$$

The characteristic impedance of a TE mode that does not propagate is *imaginary* and tends to zero as $f \rightarrow 0$. The characteristic impedance of a TE_{0m} propagating mode is always greater than the wave impedance of the dielectric Z and it tends to Z as $f \rightarrow \infty$.

So far we have used only direct waves propagating in the direction of the $+z$ axis in our analysis. Waves that propagate in the $-z$ direction, which are obtained if we substitute $+\gamma$ instead of $-\gamma$ into equations (8.6.50a - c), are possible as well. The simultaneous existence of both waves of the same mode results in *standing waves along the waveguide*.

The mode of lowest cutoff frequency is called the *dominant mode*. In a waveguide where is $b > a$, the dominant mode is the TE_{01} mode. The cutoff wavelength of the TE_{01} mode is $\lambda_c = 2b$. Thus, propagation of a wave in a rectangular waveguide can only occur if the width of the waveguide is greater than $\lambda/2$. A plot of the electromagnetic field in such a waveguide at one moment in time is shown in Figure 8.6.5. The field plot travels in the direction of the $+z$ -axis as time passes. The rectangular waveguide usually works in such a way that only the TE_{01} mode propagates. This is the only wave of significant amplitude along the waveguide. We shall study it in more detail due to its importance.

The complex power that is transmitted along the waveguide is obtained by the integration of the longitudinal component of Poynting's vector over the cross-section of the waveguide as

$$P_N = \frac{1}{2} \int_0^a \int_0^b E_{Tx} H_{Ty}^* dx dy = \frac{1}{4} |E_{0x}|^2 \frac{ab}{Z_0^*} \quad (8.6.53)$$

This is the time-averaged complex power that is transmitted by the waveguide. It is real at frequencies higher than the cutoff frequency. At frequencies lower than the cutoff frequency, this power is imaginary, and consequently, there is no transfer of active power by the waveguide.

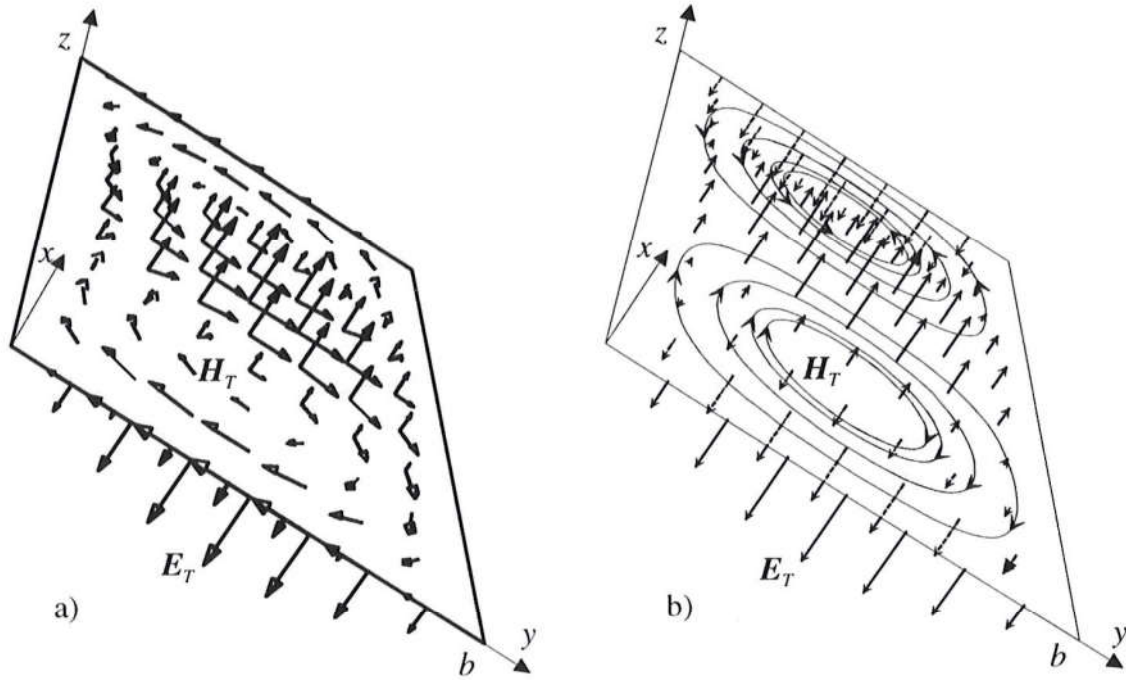


Figure 8.6.5. Field plot of the TE_{01} mode in a rectangular waveguide in a time domain, a) presentation of field vectors, b) presentation of field lines

So far, we have been analysing an ideal lossless waveguide. If we want to consider only losses in an insulator, all expressions remain valid, but most of the parameters will be complex. If this is the case, a real cutoff frequency will not exist, and γ_v is never equal to zero. The characteristic impedance is complex at all frequencies. The dependence of the propagation constant $\gamma_v = \alpha_v + j\beta_v$ on frequency in the case of small losses is shown in Figure 8.6.6.

The most important effect of losses in the insulator is the existence of an attenuation constant at all frequencies. In the case of small losses, this amounts to:

$$\gamma_v = \alpha_v + j\beta_v \approx j\beta \sqrt{1 - \left(\frac{f_c}{f}\right)^2}, \quad (8.6.54)$$

if we ensure that f is not too close to f_c . The attenuation constant of the waveguide due to losses in a good insulator can be calculated as

$$\alpha_i \approx \frac{\kappa}{2} \sqrt{\frac{\mu}{\epsilon}} \sqrt{1 - \left(\frac{f_c}{f}\right)^2}, \quad (8.6.55)$$

where κ is the conductivity of the insulator.

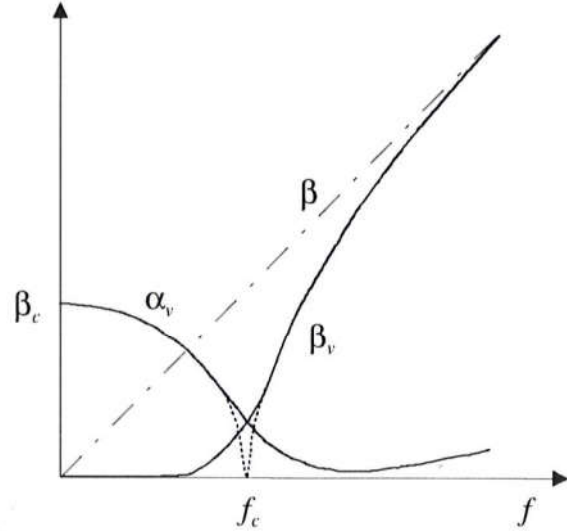


Figure 8.6.6. The propagation constant of a lossy waveguide

Much more significant attenuation of a wave in the waveguide arises due to the finite conductivity of the walls. The previously obtained solutions are no longer exact since the boundary conditions have changed. The tangential component of $\mathbf{E}_T$ at the surface of a good conductor is not equal to zero, but is of very small magnitude. Thus, the field varies only slightly from a lossless one. Consequently, the lossless solution is used to approximate $\mathbf{H}_T$ at the conductor. Now, we can approximately calculate the power loss in the conductor. The power loss per unit length in the wall $y = 0$ of the waveguide obtained in such a way is

$$(P_{l,av})_{y=0} = \frac{1}{2} \sqrt{\frac{\omega\mu}{2\kappa}} \int_0^a |\mathbf{H}_{Tz}|^2 dx = \frac{1}{2} \sqrt{\frac{\omega\mu}{2\kappa}} |\mathbf{E}_{0x}|^2 \left(\frac{f_c}{Zf}\right)^2 a. \quad (8.6.57a)$$

The same amount of power is lost in the wall $y = b$. The power that is lost per unit length in the wall $x = 0$ is

$$\begin{aligned} (P_{l,av})_{x=0} &= \frac{1}{2} \sqrt{\frac{\omega\mu}{2\kappa}} \int_0^b (|\mathbf{H}_{Ty}|^2 + |\mathbf{H}_{Tz}|^2) dy = \\ &= \frac{1}{2} \sqrt{\frac{\omega\mu}{2\kappa}} |\mathbf{E}_{0x}|^2 \left[\frac{b}{2Z_0^2} + \left(\frac{f_c}{Zf}\right)^2 \frac{b}{2} \right], \end{aligned} \quad (8.6.57b)$$

and the same amount is lost in the wall $x = a$. The total power lost in all walls of the waveguide is the sum of all these losses

$$P_{l,av} = \frac{1}{2} \sqrt{\frac{\omega\mu}{2\kappa}} |E_{0x}|^2 \left[\frac{b}{Z_0^2} + \left(\frac{f_c}{Zf} \right)^2 (2a+b) \right]. \quad (8.6.58)$$

Equation (8.6.32) is valid for any travelling wave, and using (8.6.53) and (8.6.58), we obtain the attenuation constant of the waveguide caused by losses in its walls as being

$$\alpha_s = \frac{Z_0}{ab} \sqrt{\frac{\omega\mu}{2\kappa}} \left[\frac{b}{Z_0^2} + \left(\frac{f_c}{Zf} \right)^2 (2a+b) \right]. \quad (8.6.59)$$

The total attenuation constant in the waveguide in the case where there are losses in the walls and losses in the insulator is $\alpha_v = \alpha_i + \alpha_s$. Similar analysis can be carried out for TM modes as well. Here, the field in the waveguide possesses a longitudinal E_z component, and the procedure is practically the same.

8.6.2.1. Modes in a rectangular waveguide

We shall solve the problem of determining modes in a rectangular waveguide by application of vector potentials and antipotentials. In the previous section, we analysed the dominant mode, and now we shall analyse the complete spectrum of modes. TM modes ($H_z = 0$) and TE modes ($E_z = 0$) may exist in an ideal rectangular waveguide. We shall begin with an analysis of TM modes in the waveguide. It is suitable to introduce the magnetic vector potential $\mathbf{A}$ when presenting fields with TM modes. The magnetic vector potential possesses only a longitudinal component ($A_z = A$) since the magnetic field is transverse and possesses only H_x and H_y components. The magnetic vector potential in a dielectric satisfies the wave equation

$$\Delta \mathbf{A} - \mu\epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = 0, \quad (8.6.60)$$

which transforms into Helmholtz equation (8.2.1) by a separation of variables. For sinusoidal fields, this becomes a scalar phasor Helmholtz equation $\Delta \underline{A} + k^2 \underline{A} = 0$, where $k^2 = k_x^2 + k_y^2 + k_z^2$, which is a separation equation in a Cartesian coordinate system. Solutions of the scalar phasor Helmholtz equation can be written as

$$\underline{A} = \underline{A}_0 \sin k_x x \sin k_y y e^{-jk_z z}. \quad (8.6.61)$$

The magnetic field in the waveguide may be calculated by $\underline{\mathbf{B}} = \nabla \times \underline{\mathbf{A}}$, which results in

$$\underline{\mathbf{B}} = \mathbf{a}_x \frac{\partial \underline{A}}{\partial y} - \mathbf{a}_y \frac{\partial \underline{A}}{\partial x} = \mu (\mathbf{a}_x \underline{H}_x + \mathbf{a}_y \underline{H}_y + \mathbf{a}_z \underline{H}_z), \quad (8.6.62)$$

and components of the phasor of the magnetic field for TM modes are

$$\begin{aligned}
H_x &= \frac{A_0}{\mu} k_y \sin k_x x \cos k_y y e^{-jk_z z}, \\
H_y &= -\frac{A_0}{\mu} k_x \cos k_x x \sin k_y y e^{-jk_z z}, \\
H_z &= 0.
\end{aligned} \tag{8.6.63a - c}$$

We shall determine the components of electric field for TM modes using the first main Maxwell's equation in a dielectric written in phasor form as $\nabla \times \underline{H} = j\omega\epsilon \underline{E}$. In such a way, we obtain the components

$$\underline{E}_x = -\frac{1}{j\omega\epsilon} \frac{\partial H_y}{\partial z}, \quad \underline{E}_y = \frac{1}{j\omega\epsilon} \frac{\partial H_x}{\partial z}, \quad \underline{E}_z = \frac{1}{j\omega\epsilon} \left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right]. \tag{8.6.64a - c}$$

After substitution of (8.6.63a - c) into (8.6.64a - c), and after introduction of the notation

$$\underline{E}_{0z} = (k_x^2 + k_y^2) \frac{A_0}{j\omega\mu\epsilon}, \tag{8.6.65}$$

we obtain the components of electric field for TM modes as

$$\begin{aligned}
\underline{E}_x &= -j \frac{k_x k_z}{k_x^2 + k_y^2} \underline{E}_{0z} \cos k_x x \sin k_y y e^{-jk_z z}, \\
\underline{E}_y &= j \frac{k_y k_z}{k_x^2 + k_y^2} \underline{E}_{0z} \sin k_x x \cos k_y y e^{-jk_z z}, \\
\underline{E}_z &= \underline{E}_{0z} \sin k_x x \sin k_y y e^{-jk_z z}.
\end{aligned} \tag{8.6.66a - c}$$

The tangential components of electric field must be equal to zero ($E_z = 0$) at the perfectly conductive walls of the waveguide ($x = 0$, $x = a$, $y = 0$, and $y = b$), and from these boundary conditions it follows that

$$k_x = \frac{m\pi}{a}; \quad m = 1, 2, 3, \dots \quad \text{and} \quad k_y = \frac{n\pi}{b}; \quad n = 1, 2, 3, \dots \tag{8.6.67a,b}$$

The explanation for every mode is similar to that for the dominant TE_{01} mode in Section 8.6.2. From the equation $k^2 = k_x^2 + k_y^2 + k_z^2$ and (8.6.65a,b), we obtain

$$\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 + k_z^2 = k^2. \tag{8.6.68}$$

We introduce the constant of propagation along the waveguide as

$$(\gamma_v)_{mn}^2 = (k_c)_{mn}^2 - k^2, \quad (8.6.69)$$

and from field equations (8.6.63a – c) and (8.6.66a – c) it is obvious that $\gamma_v = jk_z$. The constant of propagation γ_v becomes equal to zero in the case where k in equation (8.6.68) is equal to

$$(k_c)_{mn} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}. \quad (8.6.70)$$

The quantity $(k_c)_{mn}$ is called *the cutoff (critical) wave number of the mn-mode*. Thus, for all values of k we get

$$\begin{aligned} (\gamma_v)_{mn} = jk_z = (\alpha_v)_{mn} &= \sqrt{(k_c)_{mn}^2 - k^2} \quad ; \quad k < k_c, \\ (\gamma_v)_{mn} = jk_z = j(\beta_v)_{mn} &= j\sqrt{k^2 - (k_c)_{mn}^2} \quad ; \quad k > k_c. \end{aligned} \quad (8.6.71a,b)$$

For $k > k_c$ the mode propagates, and for $k < k_c$ the mode is non-propagated along the waveguide. The cutoff frequencies of the mn -modes follow from (8.6.70) as

$$(f_c)_{mn} = \frac{k_c}{2\pi\sqrt{\mu\epsilon}} = \frac{1}{2\sqrt{\mu\epsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}, \quad (8.6.72)$$

and the cutoff wavelengths are

$$(\lambda_c)_{mn} = \frac{2\pi}{k_c} = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}. \quad (8.6.73)$$

We can express the constants of propagation for the mn -modes using the cutoff frequency as

$$\begin{aligned} (\gamma_v)_{mn} = (\alpha_v)_{mn} &= (k_c)_{mn} \sqrt{1 - \left(\frac{f}{f_c}\right)^2} \quad ; \quad f < f_c, \\ (\gamma_v)_{mn} = j(\beta_v)_{mn} &= jk \sqrt{1 - \left(\frac{f_c}{f}\right)^2} \quad ; \quad f > f_c. \end{aligned} \quad (8.6.74a,b)$$

The wave number for a dielectric is $k^2 = \omega^2\mu\epsilon$. Finally, we can write the phasor components of the field of the TM_{mn} modes as

$$\begin{aligned}
 (E_x)_{mn} &= -jE_{0z} \frac{k_x k_z}{k_c^2} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} e^{-\gamma_v z}, \\
 (E_y)_{mn} &= -jE_{0z} \frac{k_y k_z}{k_c^2} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} e^{-\gamma_v z}, \quad (E_z)_{mn} = E_{0z} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} e^{-\gamma_v z}, \\
 (H_x)_{mn} &= j\omega\epsilon E_{0z} \frac{k_y}{k_c^2} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} e^{-\gamma_v z}, \\
 (H_y)_{mn} &= -j\omega\epsilon E_{0z} \frac{k_x}{k_c^2} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} e^{-\gamma_v z}, \quad (H_z)_{mn} = 0,
 \end{aligned}$$

for $f > f_c$. The characteristic impedance of the TM modes is

$$Z_0 = \frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{k_z}{\omega\epsilon} = Z \sqrt{1 - \left(\frac{f_c}{f}\right)^2} \quad ; \quad f < f_c, \quad (8.6.75)$$

where $Z = (\mu/\epsilon)^{1/2}$. The physical size of the waveguide relative to the wavelength determines which modes propagate along the waveguide. Waveguides are usually constructed so that only one mode propagates. Therefore, usually $b/a > 1$. The most common ratio is $b/a = 2$. Greater ratios are impractical for transmitting power.

In a similar way, by the introduction of the electric vector potential A^* , we can calculate higher-order TE modes in a waveguide. The TM_{11} mode in a rectangular waveguide is shown in Figure 8.6.7.

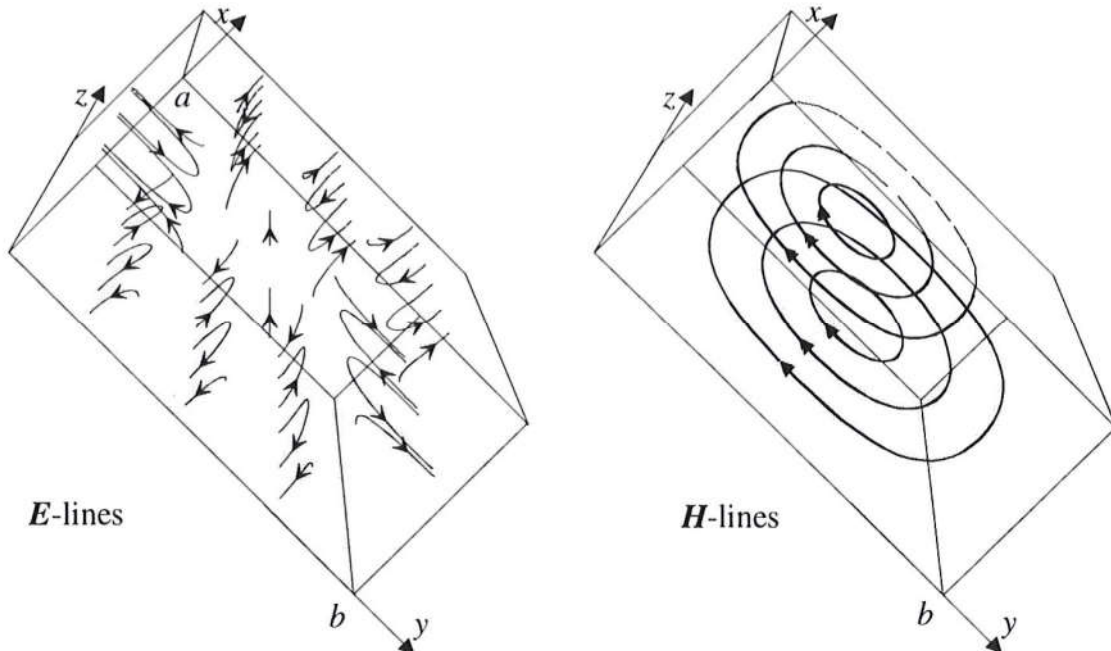


Figure 8.6.7. Electric and magnetic field of the TM_{11} mode in a rectangular waveguide

8.6.3. Resonant cavities

We shall next study the effects of an electromagnetic wave in a dielectric space totally enclosed by conductors. Under ideal (lossless) conditions, an electromagnetic field can indeed exist inside source-free regions enclosed by perfect conductors. However, the field can only exist at certain frequencies known as *resonant frequencies*. This means that certain conditions must be satisfied.

The most important one is that the relationship between the dimensions of the cavity and the wavelength enable the existence of a *standing wave* of the electromagnetic field inside the cavity..

Cavity resonators are used only at extremely high frequencies (the microwave region). Resonators for optical frequencies exist as well. Only they differ from the microwave resonators in their dimensions, which are much greater than the wavelength of the light used, but we shall not consider such resonators here. A *rectangular cavity resonator* is shown in Figure 8.6.8. In this example, we shall assume the conductors and dielectric to be perfect.

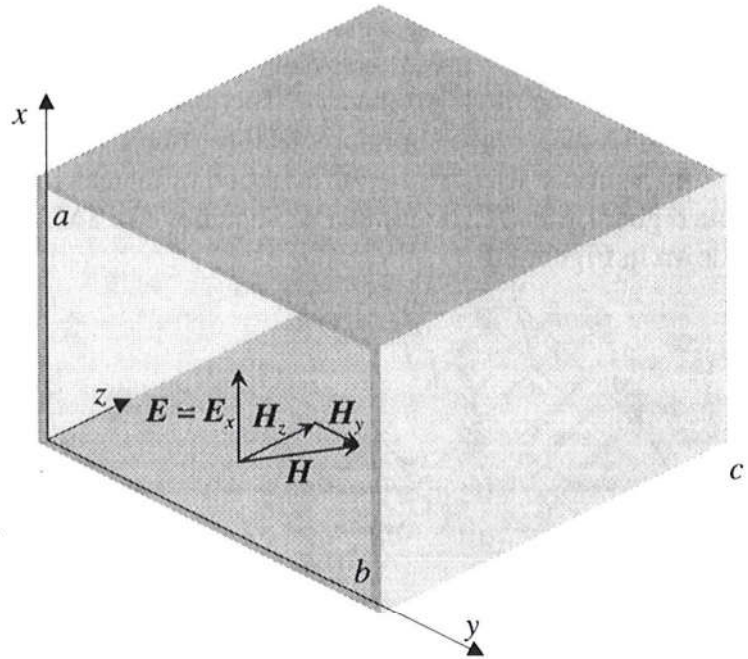


Figure 8.6.8. Rectangular cavity resonator with the TE_{011} mode

We are seeking a solution for the electric field in which the tangential components at all six walls of the resonator are equal to zero. In Section 8.6.2. we saw that the TE_{01} mode satisfies this condition at all four side walls of the waveguide. Thus, we shall assume that the field in the resonator is a standing wave along the z -axis of the form

$$\underline{E}_x = A \sin \frac{\pi y}{b} (e^{-j\beta_v z} + e^{+j\beta_v z}) \Rightarrow \underline{E}_x = \underline{E}_0 \sin \frac{\pi y}{b} \sin \beta_v z . \quad (8.6.76)$$

The boundary conditions along the z -axis are that at $z = 0$ and $z = c$ the field is $E_x = 0$, which results in $\beta_v c = \pi$, or $\beta_v = \pi/c$. The electric field of the TE_{011} mode in the resonator is

$$\underline{E}_x = \underline{E}_0 \sin \frac{\pi y}{b} \sin \frac{\pi z}{c} . \quad (8.6.77)$$

The propagation constant of an ideal waveguide is given by (8.6.44b), and we obtain

$$\beta_v = \frac{\pi}{c} = \omega \sqrt{\mu \epsilon} \sqrt{1 - \left(\frac{f_c}{f} \right)^2} ; \quad f > f_c , \quad (8.6.78)$$

where the cutoff frequency for the TE_{01} mode is, according to (8.6.45),

$$f_c = \frac{1}{2b \sqrt{\mu \epsilon}} . \quad (8.6.79)$$

If we solve equation (8.6.78) for the resonant frequency $f = f_r$, we obtain

$$f_r = \frac{1}{2bc} \sqrt{\frac{b^2 + c^2}{\mu \epsilon}} , \quad (8.6.80)$$

or expressed in terms of the wavelength of the resonant standing wave as

$$\lambda = \frac{c}{f_r} = \frac{2bc}{\sqrt{b^2 + c^2}} . \quad (8.6.81)$$

When a is the smallest dimension of the cavity resonator, this equation expresses the resonant frequency and wavelength of the dominant mode, called the TE_{011} mode. The additional subscript 1 indicates that we have chosen the first null-point of $\sin \beta_v z$. The choice of subsequent null-points result in *higher-order modes*, i.e. modes with higher resonant frequencies. By usage of the second main Maxwell's equation in phasor form $\nabla \times \underline{E} = -j\omega \mu \underline{H}$ written for a dielectric in which we have substituted (8.6.77), we obtain

$$\underline{H}_x = 0 , \quad \underline{H}_y = -\frac{1}{j\omega \mu} \frac{\partial \underline{E}}{\partial z} , \quad \underline{H}_z = \frac{1}{j\omega \mu} \frac{\partial \underline{E}}{\partial y} . \quad (8.6.82a - c)$$

After differentiation of expression (8.6.77) and substitution of (8.6.82) into it we obtain components of the field of the standing TE_{011} wave in the resonator as

$$\begin{aligned} \underline{H}_y &= j \frac{b}{\sqrt{b^2 + c^2}} \frac{\underline{E}_0}{Z} \sin \frac{\pi y}{b} \cos \frac{\pi z}{c} , \\ \underline{H}_z &= -j \frac{c}{\sqrt{b^2 + c^2}} \frac{\underline{E}_0}{Z} \cos \frac{\pi y}{b} \sin \frac{\pi z}{c} , \\ \underline{E}_x &= \underline{E}_0 \sin \frac{\pi y}{b} \sin \frac{\pi z}{c} . \end{aligned} \quad (8.6.83a - c)$$

It should be emphasised that the electric field E_x and the components of magnetic fields H_y and H_z are out of a phase by $\pm \pi/2$, which proves that there is no flow of active power in

an ideal resonator. The distribution of instantaneous field lines at a moment in time when both electric and magnetic fields are non-zero values is shown in Figure 8.6.9.

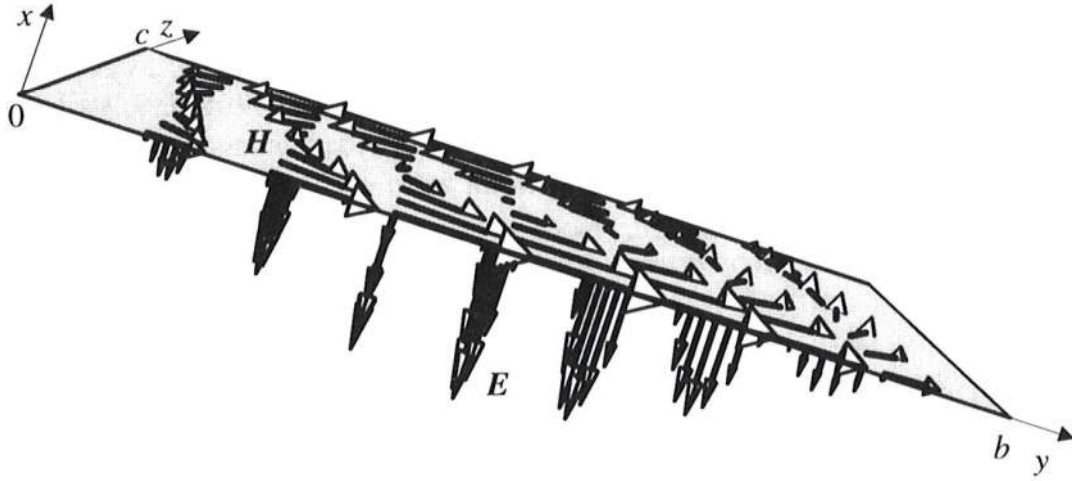


Figure 8.6.9. Electric and magnetic field of the TE_{011} mode in a rectangular cavity resonator

From the law of energy conservation in sinusoidal fields it follows that $W_{e,av} = W_{m,av}$. The time-averaged stored electric and magnetic energy are

$$W_{e,av} = W_{m,av} = \frac{\epsilon}{2} \int_V |\underline{E}|^2 dV = \frac{\epsilon}{8} |\underline{E}_0|^2 abc . \quad (8.6.84)$$

From the same law it is obvious that the total energy within the cavity resonator is independent of time, i.e. it is constant

$$W = W_{e,av} + W_{m,av} = \frac{\epsilon}{4} |\underline{E}_0|^2 abc . \quad (8.6.85)$$

For a lossy resonator, as the ones we meet are in reality, we introduce the *quality factor of a resonator* as an amount governed by the angular frequency multiplied by the stored energy and divided by the time-averaged dissipated power

$$Q = \frac{\omega W}{P_{l,av}} . \quad (8.6.86)$$

If the insulator material is lossy, then we can calculate the quality factor as being

$$Q_i = \frac{\omega \epsilon \int_V |\underline{E}|^2 dV}{\kappa \int_V |\underline{E}|^2 dV} = \frac{\omega \epsilon}{\kappa} .$$

This relation is valid for any mode in the resonator. If the walls are of a good lossy conductor, then we calculate the losses in the walls as being

$$P_{l,av} = \frac{1}{2} \sqrt{\frac{\omega\mu}{2\kappa}} \oint_{S_{cavity}} |H|^2 dS = \frac{\sqrt{\frac{\omega\mu}{2\kappa}} |E_0|^2}{2Z^2(b^2 + c^2)} [bc(b^2 + c^2) + 2a(b^3 + c^3)] ,$$

and finally we obtain the quality factor due to losses in the walls of the resonator as

$$Q_s = \frac{\pi Z}{2 \sqrt{\frac{\omega\mu}{2\kappa}}} \frac{a(b^2 + c^2)^{3/2}}{bc(b^2 + c^2) + 2a(b^3 + c^3)} . \quad (8.6.90)$$

From the symmetry of Q_s in b and c , it is evident that when $b = c$ a maximum in Q_s occurs. For a square-base cavity ($b = c$) we obtain

$$f_r = \frac{1}{b\sqrt{2\mu\epsilon}} \quad ; \quad Q_s = \frac{1.11Z}{\sqrt{\frac{\omega\mu}{2\kappa}} \left(1 + \frac{b}{2a}\right)} . \quad (8.6.91a,b)$$

The quality factor increases as a increases, but if $a > b$ we no longer have a dominant mode in the resonator. For a cubic cavity constructed of copper we get

$$Q_s = \frac{1.07 \cdot 10^9}{\sqrt{f}} ,$$

which, at microwave frequencies, amounts to several thousand.

8.7. ELECTROMAGNETIC WAVE RADIATION

The analysis in the preceding units has been concerned only with the propagation of electromagnetic waves, and not with the generation of such waves. In this unit, we shall study elementary sources of electromagnetic waves. Our fundamental problem will be the determination of the magnitude and distribution of the fields of electromagnetic waves produced by sources in unbounded spaces. We shall use the retarded integral solutions of Hertz vectors and vector potentials that were derived in Chapter 2. We shall also analyse electric dipole radiation and radiation from linear thin-wire antennae.

8.7.1. Electric dipole radiation

We usually calculate field quantities around thin linear conductors carrying sinusoidal currents at high frequencies on the basis of an *assumed* distribution of current along the conductors. Such conductors serve as sources of electromagnetic waves and we call them antennae.

The simplest case for calculating radiated power and the electromagnetic field distribution around a conductor is that of a linear conductor that is so short that the current can be considered uniform over its length. More complex shapes of antennae can be considered to be made up of a large number of such short elements.

We shall study only cases where the elements are placed in an unbounded space and carry a sinusoidal current. This current may be expressed in a complex domain as $I = I_0 e^{j\omega t}$. Therefore, we shall suppose that the current sinusoidally oscillates within a line element of the wire that has been placed at the origin of a spherical coordinate system, as shown in Figure 8.7.1. The space is unbounded and free with respective constitutive parameters ϵ_0 , μ_0 and $\kappa = 0$.

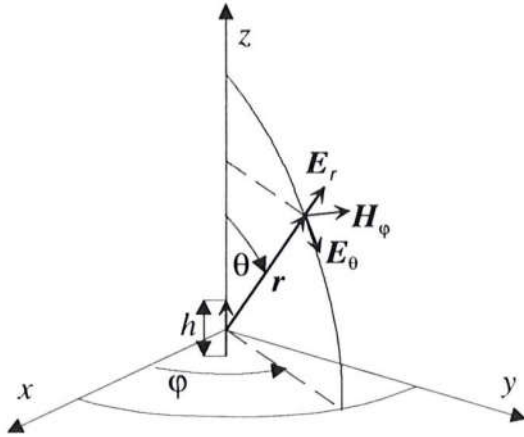


Figure 8.7.1. Components of the field around an electric dipole

We assume that the length of the wire element carrying the sinusoidal current is very small when compared with the wavelength ($h \ll \lambda$). Here h is the length of the element, and the wavelength is $\lambda = c_0/f$. We shall assume that the current I suddenly vanishes at the ends of the wire, causing a store of charge at points $z' = \pm h/2$. The stored charge at these points is

$$q = \pm \int I dt = \pm \frac{I}{j\omega} = \pm \frac{I_0}{j\omega} e^{j\omega t} . \quad (8.7.1)$$

We can see that the charge linked to the current element (Ih) is that associated with an *oscillating* electric dipole. So the element is frequently called a *radiation electric dipole*, or simply *Hertzian dipole*. Its electric dipole moment is

$$p = qh = \frac{Ih}{j\omega} = \frac{I_0 h}{j\omega} e^{j\omega t} . \quad (8.7.2)$$

The field of a Hertzian dipole can be calculated in various ways. One is by calculation of the general integral of the Hertz vector $\mathbf{Z}$ in an unbounded space from the inhomogeneous wave equation (2.5.1c). We shall adjust the integral of the Hertz vector (2.5.4c) to suit our purpose by writing the dipole moment as

$$\mathbf{p} = \mathbf{a}_z \frac{I_0 h}{j\omega} e^{j\omega t}, \quad (8.7.3)$$

and its *retarded value* observed from a point N as

$$[\mathbf{p}] = \mathbf{p} \left(0, t - \frac{r}{c_0} \right), \quad (8.7.4)$$

which in a complex domain is expressed as

$$[\mathbf{p}] = \mathbf{a}_z \frac{I_0 h}{j\omega} e^{j\left(\omega t - \frac{\omega r}{c_0}\right)} = \mathbf{a}_z \frac{I_0 h}{j\omega} e^{j(\omega t - kr)}, \quad (8.7.5)$$

and instead of the integration in (2.5.4c), we obtain the solution for the Hertz vector of the electric oscillating dipole in the form

$$\mathbf{Z} = \mathbf{a}_z \frac{1}{4\pi\epsilon_0 r} \frac{I_0 h}{j\omega} e^{j(\omega t - kr)}, \quad (8.7.6)$$

where $k = \omega/c_0$. Obviously, the Hertz vector is parallel to the electric dipole, i.e. to the z -axis. It decreases by $1/r$ and possesses a wave character. According to (2.3.3a), the magnetic vector potential of an electric oscillating dipole can be calculated in a sinusoidal field as the complex quantity $\mathbf{A} = j\omega\mu_0\epsilon_0\mathbf{Z}$, which results in

$$\mathbf{A} = \mathbf{a}_z \frac{\mu_0}{4\pi r} I_0 h e^{j(\omega t - kr)}, \quad (8.7.7)$$

a quantity independent of azimuthal angle ϕ . This is a *spherical outward-travelling wave* since the surfaces of constant phase are spheres. We shall determine the nature of the electromagnetic wave around the oscillating electric dipole from the magnetic vector potential (8.7.7). We shall begin by resolving expression (8.7.7) into its components in a spherical system (r, θ, ϕ) by the introduction of expression

$$\mathbf{a}_z = \mathbf{a}_r \cos\theta - \mathbf{a}_\theta \sin\theta. \quad (8.7.8)$$

Thus, the magnetic vector potential resolved in spherical coordinates is

$$\mathbf{A} = (\mathbf{a}_r \cos\theta - \mathbf{a}_\theta \sin\theta) \frac{\mu_0}{4\pi r} I_0 h e^{j(\omega t - kr)}. \quad (8.7.9)$$

According to (2.1.2), the magnetic field of the dipole is

$$\mathbf{H} = \frac{1}{\mu_0} \nabla \times \mathbf{A} = \mathbf{a}_\phi \frac{I_0 h}{4\pi} e^{j(\omega t - kr)} \left(\frac{jk}{r} + \frac{1}{r^2} \right) \sin\theta. \quad (8.7.10)$$

We can determine the electric field of an oscillating electric dipole using a complex form

of Maxwell's first equation for a field in a dielectric $\nabla \times \underline{H} = j\omega \epsilon \underline{E}$. From this equation, we obtain the electric field resolved in spherical coordinates as

$$\underline{E} = \frac{1}{j\omega \epsilon_0} \nabla \times \underline{H} = \frac{I_0 h}{4\pi} Z_0 e^{j(\omega t - kr)} \left[\underline{a}_r \left(\frac{1}{r^2} + \frac{1}{jkr^3} \right) 2 \cos \theta + \underline{a}_\theta \left(\frac{jk}{r} + \frac{1}{r^2} + \frac{1}{jkr^3} \right) \sin \theta \right]. \quad (8.7.11)$$

The wave impedance of free space Z_0 and the real separation constant k are

$$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 376.6 \Omega \quad ; \quad k = \omega \sqrt{\mu_0 \epsilon_0} = \frac{\omega}{c_0}.$$

The general structure of the oscillating electric dipole field can easily be obtained from expressions (8.7.10) and (8.7.11). We note that the vector of electric field lies in a meridian plane through the axis of the dipole, and the vector of the magnetic field lies in a plane perpendicular to the meridian plane, i.e. it possesses only an azimuthal component. The corresponding magnetic field lines are circles coaxial to the dipole.

Note that this is the simplest case of calculation of a radiating electromagnetic field from a very short line element carrying an alternating current independent of z coordinates. The results obtained become more and more accurate the shorter the length of the element h becomes relative to the wavelength λ . The introduced assumptions are unsustainable, i.e. the results are not valid, if $h/\lambda > 0.1$.

1. Near induction field

The electric and magnetic field in equations (8.7.10,11) include terms that vary by $1/r^3$, $1/r^2$ and $1/r$. For a region near the current element ($r \ll \lambda$), the predominant term in $\underline{H}$ is one that varies by $1/r^2$, and the predominating terms in $\underline{E}$ are those varying by $1/r^3$. According to (8.7.3), we can write the magnitude of the dipole moment as being

$$p = \frac{I_0 h}{j\omega} e^{j\omega t}, \quad (8.7.12)$$

and since $r \ll \lambda$, we obtain approximate expressions for fields in the *near zone* as being

$$\left. \begin{aligned} \underline{E} &\approx \frac{p}{4\pi \epsilon_0 r^3} (\underline{a}_r 2 \cos \theta + \underline{a}_\theta \sin \theta), \\ \underline{H} &\approx \underline{a}_\phi \frac{I_0 h e^{j\omega t}}{4\pi} \frac{\sin \theta}{r^2}. \end{aligned} \right\} r \ll \lambda \quad (8.7.13a,b)$$

If we compare the reduced expression (8.7.13a) for the electric field in the near zone of a dipole with expression (4.7.21) for the field of a static electric dipole, we notice that they are identical. Similarly, we notice the identicalness of the reduced expression (8.7.13b)

for the magnetic field of the oscillating dipole and expression (6.8.7) for the field of a direct-current element calculated from the Biot-Savart law.

The field in the near zone is quasi-static and we call it an *induction field* or *near field*. Note that the electric and magnetic fields in this zone are 90° out of a phase, so there is no flow of time-averaged active electromagnetic energy. Only the flow of time-averaged reactive energy from the density given by Poynting's vector exists, that being

$$N_{av} \approx j \frac{1}{2} \left(\frac{I_0 h}{4\pi} \right)^2 \frac{1}{\omega \epsilon_0 r^5} [a_\theta \sin 2\theta - a_r \sin^2 \theta] . \quad (8.7.14)$$

2. Far radiation field

The field equations (8.7.10,11) in a region far away ($r \gg \lambda$) from the current element (radiating source) are reduced to the approximate expressions

$$\left. \begin{aligned} E_\theta &\approx \frac{jkI_0 h}{4\pi r} Z_0 e^{j(\omega t - kr)} \sin \theta , \\ H_\phi &\approx \frac{jkI_0 h}{4\pi r} e^{j(\omega t - kr)} \sin \theta , \end{aligned} \right\} r \gg \lambda \quad (8.7.15a,b)$$

where $E_\theta/H_\phi = Z_0$. This field is that of a spherical outward-travelling wave and we call it a *radiation field*. At large distances from the source every part of the surface of the spherical wave is a plane wave and so we can expect it to be a typical homogeneous plane wave. The fields E_θ and H_ϕ are in phase and linked by Z_0 . Spatially, they are perpendicular to each other and to the direction of propagation. Poynting's vector is completely radial. The time-averaged flow of active energy is interesting. Its density is

$$N_{r,av} = \frac{1}{2} \left(\frac{I_0 h}{4\pi} \right)^2 \left(\frac{k}{r} \right)^2 Z_0 \sin^2 \theta . \quad (8.7.16)$$

We obtain the *total flow of radiated active energy* from the source if we integrate Poynting's vector (8.7.16) over a sphere of radius r , for $r \gg \lambda$. Thus, according to Figure 8.7.1. we obtain

$$P_{N,av} = \oint_S N_{r,av} dS = \int_0^\pi N_{r,av} 2\pi r \sin \theta r d\theta = 40\pi^2 I_0^2 \left(\frac{h}{\lambda} \right)^2 , \quad (8.7.17)$$

where $\lambda = 2\pi/k$, and $Z_0 \approx 120\pi$. The radiated active power is independent of radius of integration sphere.

It is suitable to introduce the *radiation resistance* of a source. We define it as the active resistance in which the same amount of total power is dissipated radiated by the source if the same current flows through it. Therefore, the equivalent radiation resistance of a Hertzian dipole is

$$R = \frac{2P_{N,av}}{I_0^2} = 80\pi^2 \left(\frac{h}{\lambda} \right)^2, \quad (8.7.18)$$

expressed in ohms. This equation shows that a short linear current element is a very inefficient source of radiation, since the radiation resistance is proportional to the square of the ratio of the element length and the wavelength. Also, for a short element, $h/\lambda \ll 1$, e.g. a thin wire one-hundredth of a wavelength long radiates 0.08 watts per ampere squared. At a frequency of 100 kHz, one-hundredth of the wavelength is 30 m.

The field plot of a current element is interesting as well. The magnetic field is represented by coaxial circles. The electric field is more complex and we have presented it in Figure 8.7.2.

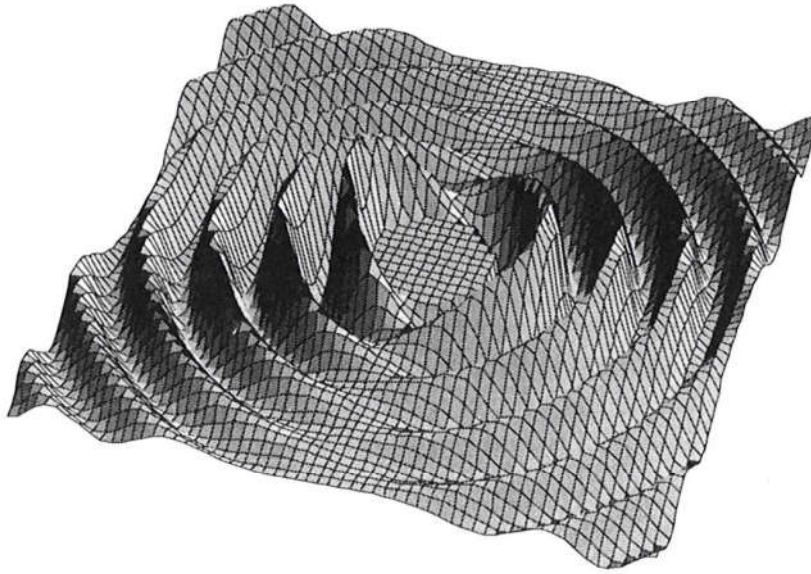


Figure 8.7.2. The distribution of a far radiation electric field E_0 in the plane $\varphi = \text{const.}$ for a Hertzian dipole in a time domain at $t = 0$

All sources of radiation are more or less directed. That is, they radiate more power in one direction than another. We represent this by a *radiation pattern* of the source. The directivity of a source (antenna) can be represented in two ways: by a field pattern or by a power radiation pattern. The field pattern gives the amplitude of the far electric or far magnetic field as a function of angles θ and φ at a fixed radius r . It is usually normalized with respect to its maximum value. Thus, for an electric dipole, we can write

$$E_\theta = Z_0 H_\varphi = jZ_0 \frac{khI_0 e^{j(\omega t - kr)}}{4\pi r} F(\theta, \varphi), \quad (8.7.19)$$

where the space factor or directivity factor of the source is

$$F(\theta, \varphi) = \sin \theta = \frac{E_\theta(\theta, \varphi)}{E_{\theta,m}} = \frac{H_\varphi(\theta, \varphi)}{H_{\varphi,m}}. \quad (8.7.20)$$

For a dipole antenna, $F(\theta, \varphi)$ is independent of angle φ . The field pattern is presented by a polar plot as shown in Figure 8.7.3.a).

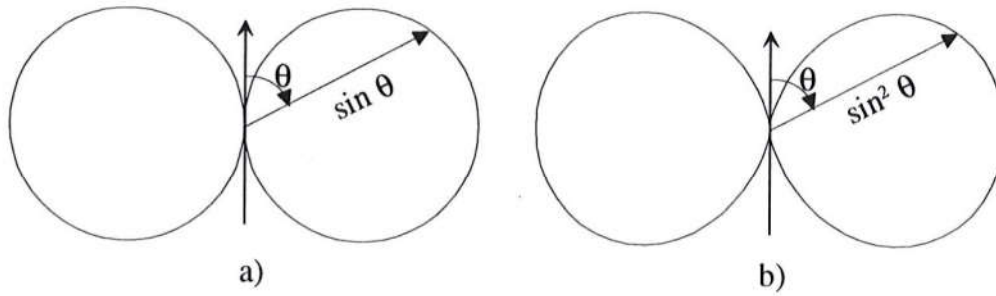


Figure 8.7.3. a) The radiation field pattern and b) the radiated power pattern of a Hertzian dipole

The power radiation pattern is usually a normalized plot of the function $N_{r,sp}(\theta, \phi)$ and it possesses the shape of the field pattern squared. For an elementary dipole, it is shown in Figure 8.7.3.b).

8.7.2. The radiation field of a linear conductor

The electromagnetic field of a solitary current-carrying linear conductor of finite length is analytically solvable only in the case where the distribution of the current in the conductor is given. Most frequently the current distribution is approximated by a sinusoidal distribution along the length of the conductor. If the assumed sinusoidal distribution results in unsatisfactorily inaccurate results, then we can approximate the distribution by a Fourier series for the sinusoidal terms. These terms can be determined, for example, by analysis of an experimentally determined distribution of current.

The strict determination of the current distribution along the conductor is a boundary value problem. In the year 1898, Abraham proved by analysis of the solutions of a conducting rotational ellipsoid that the distribution of current along an insulated wire is essentially sinusoidal and that the ratio of wire length to the wavelength is half an integer as long as the cross-sectional dimensions are small when compared with the length. A much greater cause of error than the finite diameter of the conductor is the attenuation due to radiation and dissipation of energy as heat in the conductor. The current distribution is influenced by the vicinity of the ground as well as other objects.

Devices whose primary function is connected with radiation or receiving of electromagnetic energy are usually called *antennae* in radiocommunication engineering. A thin-wire current-carrying conductor is an example of a linear antenna. If it is excited at its midpoint, we call it a dipole antenna. Analysis and calculations that will be presented here are the basis for determining fields and radiation patterns of antennae.

In this unit we shall study the radiation from a completely isolated linear antenna in free space. We shall accept the assumption that there is a sinusoidal distribution of current along a thin-wire line conductor and thus shall calculate the electromagnetic field caused by this assumed distribution of current.

A conductor of length l is placed at the origin of a coordinate system shown in Figure 8.7.4. The current at any point at a distance z' from the centre of the wire is sinusoidal in time and we represent it by the complex quantity $I = I(z')e^{j\omega t}$. The retarded value at a point

at a distance R is

$$[I] = I(z') e^{j\omega\left(t - \frac{R}{c_0}\right)} = I(z') e^{j(\omega t - kR)}, \quad (8.7.21)$$

where the separation constant is $k = \omega/c_0$. The distribution of current along the line conductor is assumed to be *sinusoidal*, and we thus write

$$I(z') = I_0 \sin k(h - |z'|). \quad (8.7.22)$$

Continuity of current flow requires that at every moment in time, the current that leaves one half of the conductor must be exactly that entering the other half of the conductor. Therefore, we write $|z'|$ instead of z' in expression (8.7.22).

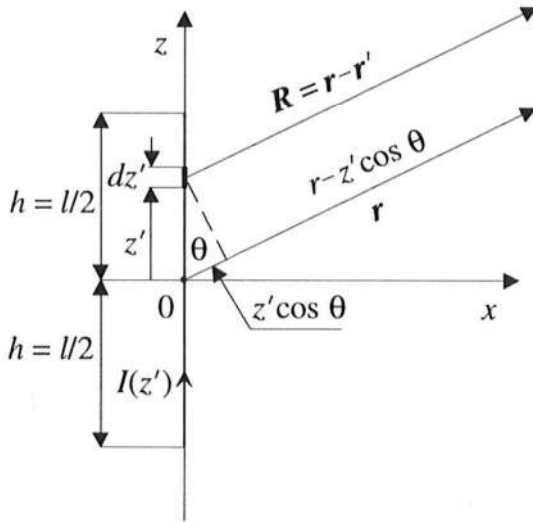


Figure 8.7.4. Phase shift in the radiation field of a line conductor

We shall determine the magnetic vector potential of a line conductor of length $l = 2h$ that carries a current $I(z')$ at a distant point from the conductor, $r \gg l$. We shall use the general integral of the wave equation, thus

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int_{V'} \frac{[\mathbf{J}] dV'}{R}.$$

For a *thin-wire conductor*, we may introduce $\mathbf{J}(\mathbf{r}') dV' = \mathbf{J}(\mathbf{r}') dS' dz' = I(z') dz' \mathbf{a}_z$ and transform this integral to

$$\mathbf{A}(\mathbf{r}) = \mathbf{a}_z \frac{\mu_0}{4\pi} \int_{-h}^{+h} \frac{I(z') e^{j(\omega t - kR)}}{R} dz'.$$

This is the expression for calculating the magnetic vector potential at a point a distance $\mathbf{r}$ from the origin O . A thin-wire line conductor of length $l = 2h$ oriented along the z -axis is placed as shown in Figure 8.7.4. According to Figure 8.7.4, we take that $\mathbf{r}' = z' \mathbf{a}_z$ and $R = |\mathbf{r}' - \mathbf{r}|$. After that, we integrate the expression

$$\mathbf{A}(\mathbf{r}) = \mathbf{a}_z \frac{\mu_0 I_0}{4\pi} \int_{-h}^{+h} \frac{\sin k(h - |z'|)}{R} e^{j(\omega t - kR)} dz' , \quad (8.7.23)$$

which we shall simplify by the introduction of certain approximations. Firstly, we shall write the ratio for the far radiation field ($r \gg h$) as

$$\frac{1}{R} = \frac{1}{|\mathbf{r} - \mathbf{r}'|} \approx \frac{1}{r} , \quad (8.7.24)$$

and exclude it from the integration with respect to z' . The second factor that influences the magnetic vector potential is the difference in phase of each contribution to that of the wave at a point $\mathbf{r}$ due to elemental sources at $\mathbf{r}'$, which is expressed by the factor $e^{jk(\mathbf{r} - \mathbf{r}')}.$ If, in accordance with Figure 8.7.4, we write the approximation into the indices

$$R = |\mathbf{r} - \mathbf{r}'| = |\mathbf{r} - \mathbf{a}_z z'| \approx r - z' \cos \theta , \quad (8.7.25)$$

for $r \gg h$, we finally obtain the integral for the magnetic vector potential as

$$\mathbf{A}(\mathbf{r}) = \mathbf{a}_z \frac{\mu_0 I_0}{4\pi} \frac{e^{j(\omega t - kr)}}{r} \int_{-h}^{+h} \sin k(h - |z'|) e^{jkz' \cos \theta} dz' . \quad (8.7.26)$$

The magnetic vector potential for the far radiation field of a thin-wire line conductor of length $l = 2h$ is obtained by integration as

$$\mathbf{A}(\mathbf{r}) = \mathbf{a}_z \frac{\mu_0 I_0}{2\pi k r} \frac{\cos(kh \cos \theta) - \cos(kh)}{\sin \theta} e^{j(\omega t - kr)} . \quad (8.7.27)$$

The far field vectors are calculated from the magnetic vector potential using Maxwell's first main equation in a complex domain as

$$\mathbf{E}(\mathbf{r}) = \frac{1}{j\omega\mu_0\epsilon_0} \nabla \times \mathbf{B}(\mathbf{r}) = \frac{1}{j\omega\mu_0\epsilon_0} \nabla \times \nabla \times \mathbf{A}(\mathbf{r}) . \quad (8.7.28a,b)$$

If we solve these equations in a spherical coordinate system and neglect all terms with $1/r^2$ and $1/r^3$, then the radiation field of a line conductor is

$$E_\theta = Z_0 H_\phi = Z_0 \frac{jI_0}{2\pi r} \frac{\cos(kh \cos \theta) - \cos(kh)}{\sin \theta} e^{j(\omega t - kr)} . \quad (8.7.29)$$

For antennae at high frequencies, we use lengths of conductors that are multiples of half-wavelengths. So, we get

$$l = 2h = m \frac{\lambda}{2} \quad ; \quad kh = m \frac{\pi}{2} \quad ; \quad k = \frac{m\pi}{l} . \quad (8.7.30a - c)$$

If we substitute expression (8.7.30b) into (8.7.29), we obtain an expression for the fields of a spherical radiation wave as

$$E_{\theta} = Z_0 H_{\phi} = Z_0 \frac{jI_0}{2\pi r} \cdot e^{j(\omega t - kr)} \left[\frac{\cos\left(\frac{m\pi}{2} \cos \theta\right) - \cos \frac{m\pi}{2}}{\sin \theta} \right]. \quad (8.7.31)$$

The expression in the square bracket is *the space factor* $F(\theta)$, and we can thus write the field equations as

$$E_{\theta} = Z_0 H_{\phi} = Z_0 \frac{jI_0}{2\pi r} F(\theta) e^{j(\omega t - kr)}, \quad (8.7.32)$$

where

$$F(\theta) = \frac{\cos\left(\frac{m\pi}{2} \cos \theta\right) - \cos \frac{m\pi}{2}}{\sin \theta}. \quad (8.7.33)$$

The value of the space factor differs for various conductors according to the length, which is always an integer multiple of either half-wavelength or wavelength. Since $m = l/(\lambda/2)$, we obtain the next expressions for the space factor as being

$$\begin{aligned} F(\theta) &= \frac{\cos\left[\frac{(2n-1)\pi}{2} \cos \theta\right]}{\sin \theta} ; \quad \text{for } m = (2n-1), n = 1, 2, 3, \dots \\ F(\theta) &= \frac{2 \cos^2\left[\frac{(2n-1)\pi}{2} \cos \theta\right]}{\sin \theta} ; \quad \text{for } m = 2(2n-1), n = 1, 2, 3, \dots \\ F(\theta) &= \frac{2 \sin^2(n\pi \cos \theta)}{\sin \theta} ; \quad \text{for } m = 4n, n = 1, 2, 3, \dots \end{aligned} \quad (8.7.34a - c)$$

The obtained expressions for field strength (8.7.32) and for the space factor (8.7.34a - c), show that the strength of the radiation field does not depend on the length of the conductor, but only upon the mode of distribution of current along the conductor. In such a way, we have represented the field of a conductor as that of a dipole located at the origin of a particular coordinate system and whose amplitude is modified by a space factor $F(\theta)$.

The intensity of radiation, i.e. the density of flow of electromagnetic power in the line-conductor radiation field is determined from the time-averaged value of Poynting's vector as being

$$N_{av} = \frac{1}{2} \mathbf{E} \times \mathbf{H}^* = \mathbf{a}_r \frac{Z_0 I_0^2}{8\pi^2 r^2} F^2(\theta), \quad (8.7.35)$$

measured in watts per square metre. We then obtain the total power radiated from the conductor by integration of expression (8.7.35) over the surface of a sphere of radius r as

$$P_{N,av} = \oint_S \mathbf{N}_{av} \cdot \mathbf{n} dS = \int_0^\pi N_{av} 2r^2 \pi \sin \theta d\theta ,$$

which results in

$$P_{N,av} = 30 I_0^2 \int_0^\pi F^2(\theta) \sin \theta d\theta , \quad (8.7.36)$$

where we have substituted that $Z_0 = 120\pi$ and $kh = m\pi/2$. Integration of expression (8.7.36) after certain substitutions and transformations results in the expressions

$$\begin{aligned} P_{N,av} &= 15 I_0^2 \text{Cin}(2m\pi) \quad ; \quad \text{when } m \text{ is odd} , \\ P_{N,av} &= I_0^2 [60 \text{Cin}(m\pi) - 15 \text{Cin}(2m\pi)] \quad ; \quad \text{when } m \text{ is even} , \end{aligned} \quad (8.7.37a,b)$$

where

$$\begin{aligned} \text{Cin } x &= \int_0^x \frac{1 - \cos x}{x} dx = C + \ln x + \int_x^\infty \frac{\cos x}{x} dx , \\ \text{Ci } x &= - \int_x^\infty \frac{\cos x}{x} dx , \end{aligned} \quad (8.7.38a,b)$$

are cosine integrals, and $C = 0.577$ is Euler's constant. The radiation resistance can be calculated from (8.7.37a,b) and from the expression

$$R_0 = \frac{2P_{N,av}}{I_0^2} , \quad (8.7.39)$$

and for characteristic cases of half-wave and full-wave dipole antennae we obtain

$$\begin{aligned} R_0 &\approx 73 \Omega \quad ; \quad m = 1 , \\ R_0 &\approx 199 \Omega \quad ; \quad m = 2 . \end{aligned} \quad (8.7.40a,b)$$

PROBLEMS

- 8.1. Determine that the time-averaged power transmitted by a wave through unit surface in a dielectric is real and constant!
- 8.2. Analyse a plane standing wave with a circularly polarized field!
Hint: suppose that the constants in the equations (8.3.13a – d) are given as

$$A_1 = \frac{E_0}{2} ; \alpha_1 = -\frac{\pi}{2} , \quad A_2 = \frac{E_0}{2} ; \alpha_2 = +\frac{\pi}{2} ,$$

$$B_1 = \frac{E_0}{2} ; \beta_1 = 0 , \quad B_2 = \frac{E_0}{2} ; \beta_2 = \pi .$$

- 8.3. Determine the frequency up to which sea water ($\kappa = 5 \text{ S/m}$, $\epsilon = 81\epsilon_0$ and $\mu = \mu_0$) is a good conductor!
- 8.4. Consider a situation where the vector E_0 of an electromagnetic wave is normal to the plane of incidence U and impinges on the boundary S between two ideal dielectrics and then determine the total electromagnetic field in dielectrics (1) and (2)! Prove that the field in dielectric (1) has the properties of a travelling wave in the direction of the x -axis and of a combination of a travelling and standing wave in the direction of the z -axis!
- 8.6. Determine the condition under which *there is no reflection* at the boundary between two ideal dielectrics! (The angle of incidence that satisfies this condition is called *Brewster's angle*.)
- 8.7. Determine the conditions under which a plane TEM wave obliquely incident to a boundary S between two (ideal) dielectrics is *totally reflected*!
- 8.8. Determine the transmitted wave propagating into a lossy material (2) when the vector E_0 is parallel to the plane of incidence U and impinges from an ideal dielectric (1) and transmits into the lossy material!
- 8.9. Determine the input impedance of an open-circuited and short-circuited transmission line!
- 8.10. Prove expression (8.6.40) for the total electric field after reflection at the wall of an ideal rectangular waveguide!

9. THE FINITE ELEMENT METHOD

The finite element method (from now on, referred to as FEM) is nowadays a widely accepted tool for solving field problems. A number of general purpose FEM computer programmes, as well as FEM programmes specially adopted for solving electromagnetic field problems can be purchased on the market. It is our intention to make the reader familiar with the fundamentals of FEM and its application in electromagnetism within this chapter.

Difficulties in describing and mastering FEM originate from its multi-disciplinary character. One must understand the physical principles involved in a problem, approximation theory, mathematical modelling with variational formulations, matrix algebra, numerical methods, computer programming etc. Thus, we have chosen a step-by-step approach based on the describing of solving actual electromagnetic problems.

We shall begin with a detailed description of all the essential steps in the solution of simple, two-dimensional static problems. In continuation, we shall outline non-linear problems and finally we shall describe the solving of dynamic quasi-static fields.

We hope that, having read this chapter, the reader will be able to understand and use most existing FEM programmes. We shall present the fundamental algorithms in the form of easily understandable procedural programming routines in order to enable the producing of one's own computer codes.

9.1. A DEFINITION OF FEM

In previous chapters, we showed that problems in electromagnetism, as well as in other branches of engineering sciences, can be described by partial differential equations. In general, one has to determine the unknown function u that satisfies the set of partial differential equations:

$$\mathbf{D}(u) = 0, \quad (9.1.1)$$

within an Ω domain, together with given boundary conditions at a boundary in the domain Γ

$$\mathbf{R}(u) = 0. \quad (9.1.2)$$

FEM is based on two fundamental steps used in *obtaining an approximate solution* of a problem defined by (9.1.1,2), which are:

1. Subdivision of the Ω domain into sub-domains Ω^e called finite elements.
2. An approximation of the unknown function u for each element in the form

$$u \approx \sum_i N_i u_i . \quad (9.1.3)$$

Here, N_i are the interpolation functions that we have chosen, and u_i are values of the function u at chosen points of an element called interpolation nodes.

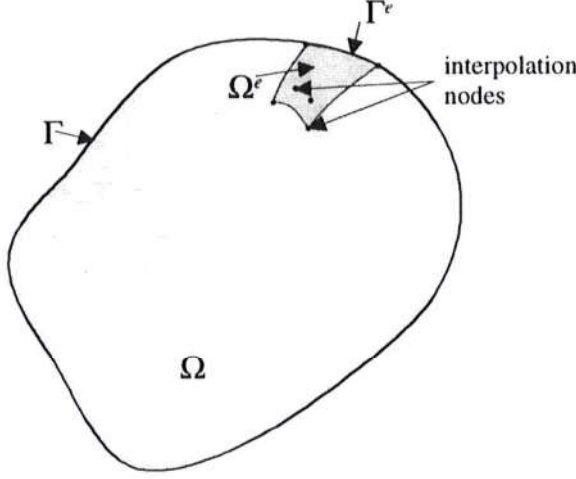


Figure 9.1.1. the Ω domain and its boundary Γ

In order to employ a finite element approximation (9.1.3), we have to find an *integral form* that is equivalent to our system of partial differential equations (9.1.1,2):

$$\int_{\Omega} \mathbf{O}(u) d\Omega + \int_{\Gamma} \mathbf{G}(u) d\Gamma = 0 . \quad (9.1.4)$$

Thus, if the functions $\mathbf{O}$ and $\mathbf{G}$ can be integrated over Ω and Γ , we can write

$$\int_{\Omega} \mathbf{O}(u) d\Omega + \int_{\Gamma} \mathbf{G}(u) d\Gamma = \sum_{e=1}^N \left\{ \int_{\Omega^e} \mathbf{O}(u) d\Omega + \int_{\Gamma^e} \mathbf{G}(u) d\Gamma \right\} = \sum_{e=1}^N W^e . \quad (9.1.5)$$

Here, N is the total number of finite elements, Ω^e is the domain of the e -th element, Γ^e is the part of the boundary belonging to the e -th element, and W^e is the element integral form of the e -th element.

The finding of an equivalent integral form is most frequently based on the application of an energetic functional, or on the usage of a weighted residual method. If the partial differential equations are linear, we can write the system (9.1.1,2) as

$$\begin{aligned} \mathbf{D}(\mathbf{u}) &= [\mathbf{D}]\mathbf{u} + \mathbf{d} = 0 , \\ \mathbf{R}(\mathbf{u}) &= [\mathbf{R}]\mathbf{u} + \mathbf{r} = 0 . \end{aligned} \quad (9.1.6)$$

By subdivision of the domain into finite elements, and by substitution of the approximation (9.1.3) into the corresponding integral form, we obtain a system of linear algebraic equations for unknown coefficients u_i

$$[\mathbf{K}]\mathbf{u} = \mathbf{F} . \quad (9.1.7)$$

Terms of the matrix $[\mathbf{K}]$ and of the vector $\mathbf{F}$ on the right-hand side are calculated as being

$$K_{ij} = \sum_{e=1}^N K_{ij}^e \quad ; \quad F_i = \sum_{e=1}^N F_i^e . \quad (9.1.8)$$

After solution of the system (9.1.7), an approximation of the unknown function u over the whole domain Ω is defined by the chosen geometry of each finite element and by the chosen form of the interpolation functions.

9.1.1. The integral form and its approximation

9.1.1.1. A functional as an integral form

In this section, we wish to introduce the reader to the fundamentals of the variation calculus necessary for understanding FEM, while at the same time avoiding any detailed mathematical studies. A functional Π is a scalar function of a set of functions and their derivatives

$$\Pi = \Pi(u, \frac{\partial u}{\partial x}, \dots) . \quad (9.1.9)$$

We define the first variation of Π as

$$\delta \Pi = \frac{\partial \Pi}{\partial u} \delta u + \frac{\partial \Pi}{\partial \left(\frac{\partial u}{\partial x} \right)} \delta \left(\frac{\partial u}{\partial x} \right) + \dots \quad (9.1.10)$$

where δu and $\delta(\partial u / \partial x)$ are arbitrary variations of functions u and $\partial u / \partial x$. The fundamental properties of the variation operator are that:

$$\begin{aligned} \delta \left(\frac{\partial u}{\partial x} \right) &= \frac{\partial}{\partial x} \delta u \quad , \quad \delta \left(\int_V u dV \right) = \int_V \delta u dV , \\ \delta(u + v) &= \delta u + \delta v \quad , \quad \delta(uv) = u \delta v + v \delta u . \end{aligned} \quad (9.1.11a - d)$$

For certain problems of the form (9.1.1,2), we may establish an energetic functional

$$\Pi = \int_{\Omega} \mathbf{F}(\mathbf{u}, \frac{\partial \mathbf{u}}{\partial x}, \dots) d\Omega + \int_{\Gamma} \mathbf{E}(\mathbf{u}, \frac{\partial \mathbf{u}}{\partial x}, \dots) d\Gamma , \quad (9.1.12)$$

for which the function u satisfying the condition that the functional Π is stationary $\delta \Pi = 0$ is the solution of the system (9.1.1,2) as well. If we apply this condition to (9.1.12) we obtain

$$\delta \Pi = \int_{\Omega} \delta u \mathbf{A}(u) d\Omega + \int_{\Gamma} \delta u \mathbf{B}(u) d\Gamma = 0 . \quad (9.1.13)$$

Since this equation must be valid for any variation δu , it follows that

$$\mathbf{A}(\mathbf{u}) = 0 \quad \text{in } \Omega \quad , \quad \mathbf{B}(\mathbf{u}) = 0 \quad \text{on } \Gamma \quad . \quad (9.1.14a,b)$$

If this system is identical to system (9.1.1,2), then the solution of equation $\delta\Pi = 0$ is the solution of the initial system as well. Equations (9.1.15a,b) are called *Euler's differential equations*. They correspond to the condition where the functional (9.1.12) is stationary. *We can find Euler's differential equations for any functional, but only some systems of partial differential equations are actually Euler's equations of a functional.* After substitution of the approximation (9.1.3) into the condition $\delta\Pi = 0$, we obtain

$$\delta\Pi = \sum_i \frac{\partial\Pi}{\partial u_i} \delta u_i = 0 \quad . \quad (9.1.15)$$

Since this equation must hold for any variation δu_i , unknown coefficients u_i must satisfy the system of equations

$$\frac{\partial\Pi}{\partial u_i} = 0 \quad ; \quad i = 1, \dots, N \quad . \quad (9.1.16)$$

We shall present an application of this approach in the unit on solving two-dimensional static fields.

9.1.1.2. Weighted residual method

The system of partial differential equations (9.1.1) must be equal to zero at every point in the Ω domain and therefore we get that

$$\int_{\Omega} \mathbf{v}^T \mathbf{D}(\mathbf{u}) d\Omega = \int_{\Omega} (v_1 D_1(\mathbf{u}) + v_2 D_2(\mathbf{u}) + \dots) d\Omega = 0 \quad . \quad (9.1.17)$$

In this expression $\mathbf{v}$ is a set of arbitrary functions v_i . The number of these functions is equal to the number of equations, or the number of components of $\mathbf{u}$ in the problem.

We can formulate this statement in another way: if equation (9.1.17) is satisfied *for any set of functions $\mathbf{v}$* , then differential equations (9.1.1) *must be valid at any point* in the Ω domain. The proof of this statement is obvious if we assume the possibility that $\mathbf{D}(\mathbf{u}) \neq 0$ at any point in the domain. In this case, we may immediately find a set of functions $\mathbf{v}$ for which the integral in (9.1.17) is not equal to zero, and thus prove the statement.

If we must satisfy the boundary conditions (9.1.2) at every point of the boundary of the domain Γ , we analogously get

$$\int_{\Omega} \mathbf{v}^T \mathbf{R}(\mathbf{u}) d\Gamma = \int_{\Omega} (v_1 R_1(\mathbf{u}) + v_2 R_2(\mathbf{u}) + \dots) d\Gamma = 0 \quad , \quad (9.1.18)$$

for any set of functions $\mathbf{v}$. Accordingly, the integral form

$$\int_{\Omega} \mathbf{v}_1^T \mathbf{D}(\mathbf{u}) d\Omega + \int_{\Gamma} \mathbf{v}_2^T \mathbf{R}(\mathbf{u}) d\Gamma = 0 \quad , \quad (9.1.19)$$

which is satisfied for any set of functions $\mathbf{v}_1$ and $\mathbf{v}_2$, is equivalent to the differential form (9.1.1,2). If we approximate the unknown function using the form (9.1.3) as

$$u \approx \sum_i N_i u_i , \quad (9.1.20)$$

the integral formulation (9.1.19) enables the substitution of previously prescribed *weighting functions*

$$\left. \begin{array}{l} v_1 = w_i \\ v_2 = \bar{w}_i \end{array} \right\} \quad i = 1, \dots, n , \quad (9.1.21)$$

instead of arbitrary functions v_1 and v_2 . Here, n is the number of unknown coefficients u_i . The integral form (9.1.19) in this case takes the form

$$\int_{\Omega} w_i^T D \left(\sum_i N_i u_i \right) d\Omega + \int_{\Gamma} \bar{w}_i^T R \left(\sum_i N_i u_i \right) d\Gamma = 0 \quad ; \quad i = 1, \dots, n . \quad (9.1.22)$$

If we note that $D(\sum N_i u_i)$ represents a residue, or an error, due to the substitution of the approximation into the differential equations, and that $R(\sum N_i u_i)$ represents a residue in the boundary conditions, then equation (9.1.19) is a weighted integral of such a residue. Therefore, we call such an approximation the *weighted residual method*. We most frequently use Galerkin's method, for which $w_i = N_i$.

Example 9.1.1.1. Application of the weighted residual method

Let a partial differential equation be given in the form

$$\nabla \cdot (\alpha \nabla u(\mathbf{r})) = q(\mathbf{r}) \quad ; \quad \mathbf{r} \in \Omega , \quad (9.1.23)$$

which is satisfied by the scalar function u , together with Neumann boundary condition

$$\alpha \frac{\partial u(\mathbf{r})}{\partial n} - p(\mathbf{r}) = 0 \quad ; \quad \mathbf{r} \in \Gamma_1 , \quad (9.1.24)$$

and Dirichlet boundary condition

$$u(\mathbf{r}) - u_0 = 0 \quad ; \quad \mathbf{r} \in \Gamma_2 . \quad (9.1.25)$$

Γ_1 and Γ_2 constitute the complete boundary of the Ω domain. The weighted integral of the residue of this equation, according to equation (9.1.22), is

$$\int_{\Omega} w_i [\nabla \cdot (\alpha \nabla u - q)] d\Omega + \int_{\Gamma_1} \bar{w}_i \left(\alpha \frac{\partial u}{\partial n} - p \right) d\Gamma + \int_{\Gamma_2} \bar{\bar{w}}_i (u - u_0) d\Gamma = 0 . \quad (9.1.26)$$

Application of Green's theorem to the first integral in this expression results in

$$-\int_{\Omega} (\nabla w_i) \cdot (\alpha \nabla u) d\Omega + \int_{\Gamma_1 + \Gamma_2} w_i \alpha \frac{\partial u}{\partial n} d\Gamma - \int_{\Omega} w_i q d\Omega . \quad (9.1.27)$$

Thus, the order of highest derivative of the approximation on an element is reduced and the approximation should be differentiable only once. If we choose the weighting func-

tions in such a way that

$$\bar{w}_i = -w_i, \quad (9.1.28)$$

is valid, then substitution of (9.1.27) into (9.1.28) results in

$$\begin{aligned} & - \int_{\Omega} (\nabla w_i) \cdot (\alpha \nabla u) d\Omega + \int_{\Gamma_2} w_i \alpha \frac{\partial u}{\partial n} d\Gamma - \int_{\Omega} w_i q d\Omega + \\ & + \int_{\Gamma_1} w_i p d\Gamma + \int_{\Gamma_2} \bar{w}_i (u - u_0) d\Gamma = 0. \end{aligned} \quad (9.1.29)$$

The Dirichlet boundary condition on the boundary Γ_2 is usually given, and we remove the residues on this boundary from this form. The weighting functions w_i are identical to the interpolation functions N_i in Galerkin's method. Since the interpolation functions are defined locally on the element containing the unknown value of function u_i at the i -th node, expression (9.1.29) defines a system of algebraic equations of the form $[A]U = C$. Terms of the matrix $[A]$ are calculated using

$$A_{ij} = \int_{\Omega^e} (\nabla N_i) \cdot (\alpha \nabla N_j) d\Omega. \quad (9.1.30)$$

The vector U contains the unknown values of the function u at the interpolation nodes, and using the expression

$$C_i = \int_{\Omega^e} N_i q d\Omega, \quad (9.1.31)$$

we calculate the terms of the vector C .

9.2. TWO-DIMENSIONAL STATIC FIELDS

We shall explain the application of FEM in solving linear two-dimensional static electric and magnetic fields within this unit. The fundamental procedures that will be described here are equivalent to those that are used in the application of FEM in more complex three-dimensional and dynamic problems, but the simplicity of the graphical presentation and expressions in two-dimensional problems enables its easier understanding.

9.2.1 An integral form equivalent to Poisson's equation

In Chapters 4. and 6. we showed that the potentials in two-dimensional static electric and magnetic non-linear fields satisfy Poisson's differential equation:

$$\frac{\partial}{\partial x} \left(v \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(v \frac{\partial u}{\partial y} \right) + f_v = 0, \quad (9.2.1)$$

together with given boundary conditions at the boundary Γ of the domain of the problem

$$u - u_0 = 0, \quad v \frac{\partial u}{\partial n} - q = 0. \quad (9.2.2a,b)$$

Here:

x, y – are coordinates of a two-dimensional Cartesian system,

u – is an unknown potential function (φ or A_z),

v – is the constitutive parameter of the material (ε or $1/\mu$),

f_v – are given field sources (ρ or J_z).

The Dirichlet boundary conditions are given at the part of the boundary Γ_0 by equation (9.2.2a), and the Neumann conditions are given at the part of the boundary Γ_n by equation (9.2.2b). The functional whose stationary condition gives us the solution of Poisson's equation is

$$\Pi = \int_{\Omega} \left[\frac{v}{2} \left(\frac{\partial u}{\partial x} \right)^2 + \frac{v}{2} \left(\frac{\partial u}{\partial y} \right)^2 - f_v u \right] d\Omega - \int_{\Gamma_n} q u d\Gamma. \quad (9.2.3)$$

The variation of this functional is

$$\delta \Pi = \int_{\Omega} \left[v \frac{\partial u}{\partial x} \delta \left(\frac{\partial u}{\partial x} \right) + v \frac{\partial u}{\partial y} \delta \left(\frac{\partial u}{\partial y} \right) - f_v \delta u \right] d\Omega - \int_{\Gamma_n} q \delta u d\Gamma. \quad (9.2.4)$$

Application of expression (9.1.11a) and of Green's theorem result in

$$\begin{aligned} \int_{\Omega} \psi \left[\frac{\partial}{\partial x} \left(v \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(v \frac{\partial u}{\partial y} \right) \right] dx dy = \\ = - \int_{\Omega} \left[\frac{\partial \psi}{\partial x} \left(v \frac{\partial u}{\partial x} \right) + \frac{\partial \psi}{\partial y} \left(v \frac{\partial u}{\partial y} \right) \right] dx dy + \oint_{\Gamma} \psi \left(v \frac{\partial u}{\partial n} \right) d\Gamma, \end{aligned} \quad (9.2.5)$$

and with the assumption that $\psi = \delta u$, equation (9.2.4) transforms to

$$\delta \Pi = \int_{\Omega} \delta u \left[\frac{\partial}{\partial x} \left(v \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(v \frac{\partial u}{\partial y} \right) + f_v \right] dx dy + \oint_{\Gamma_n} \delta u \left(v \frac{\partial u}{\partial n} - q \right) d\Gamma = 0, \quad (9.2.6)$$

if $\delta u = 0$ at the part of the boundary Γ_0 . Equation (9.2.6) must be valid for any variation δu . If the unknown function u is given in such a way that $u = u_0$ and $\delta u = 0$ at the part of the boundary Γ_0 , then expression (9.2.3) is the integral form whose stationary condition is the solution of Poisson's equation as well.

9.2.2. Application of FEM to the integral form equivalent to Poisson's equation

The application is performed in two steps:

1. Subdivision of the domain Ω into finite elements Ω^e so that the expression

$$\begin{aligned} \Pi &= \int_{\Omega} \left[\frac{v}{2} \left(\frac{\partial u}{\partial x} \right)^2 + \frac{v}{2} \left(\frac{\partial u}{\partial y} \right)^2 - f_v u \right] d\Omega - \int_{\Gamma_n} q u d\Gamma = \\ &= \sum_{e=1}^N \left\{ \int_{\Omega^e} \left[\frac{v^e}{2} \left(\frac{\partial u^e}{\partial x} \right)^2 + \frac{v^e}{2} \left(\frac{\partial u^e}{\partial y} \right)^2 - f_v^e u^e \right] d\Omega - \int_{\Gamma_n^e} q^e u^e d\Gamma \right\}. \end{aligned} \quad (9.2.7)$$

2. Approximation of the unknown function u on each element as

$$u^e \approx \sum_i N_i^e u_i^e = [N_i^e] \mathbf{u}_i^e. \quad (9.2.8)$$

Here N_i are the interpolation functions of the e -th element, and $\mathbf{u}_i$ is the vector of the unknown interpolation coefficients (values of the unknown function at the interpolation nodes of the e -th element). Substitution of the approximation (9.2.8) into (9.2.7) results in

$$\begin{aligned} \Pi &\approx \sum_{e=1}^N \left\{ \int_{\Omega^e} \left[\frac{v^e}{2} \left(\sum_i \frac{\partial N_i^e}{\partial x} u_i^e \right)^2 + \frac{v^e}{2} \left(\sum_i \frac{\partial N_i^e}{\partial y} u_i^e \right)^2 - f_v^e \left(\sum_i N_i^e u_i^e \right) \right] d\Omega \right\} - \\ &- \sum_e \left\{ \int_{\Gamma_n^e} q^e \left(\sum_i N_i^e u_i^e \right) d\Gamma \right\}. \end{aligned} \quad (9.2.9)$$

The condition under which Π , given in this form, is stationary according to equation (9.1.16) is $\partial \Pi / \partial u_i^e = 0$, and it results in a system of equations for all unknown interpolation coefficients u_i^e . Differentiation of (9.2.9) with respect to the unknown coefficient u_j^e yields:

$$\begin{aligned} \frac{\partial \Pi}{\partial u_j^e} &= \sum_e \left\{ \int_{\Omega^e} \left[v^e \left(\sum_i \frac{\partial N_i^e}{\partial x} u_i^e \right) \frac{\partial N_j^e}{\partial x} + v^e \left(\sum_i \frac{\partial N_i^e}{\partial y} u_i^e \right) \frac{\partial N_j^e}{\partial y} \right] dx dy \right\} - \\ &- \sum_e \left\{ \int_{\Omega^e} f_v^e N_j^e dx dy - \int_{\Gamma_n^e} q^e N_j^e d\Gamma \right\} = \sum_e W^e. \end{aligned} \quad (9.2.10)$$

The system of equations for unknown coefficients u_i^e has the form $[K]\mathbf{u} = \mathbf{F}$. Terms of the matrix of the system are

$$K_{ij} = K_{ji} = \sum_e \left\{ \int_{\Omega^e} \left[v^e \left(\frac{\partial N_i^e}{\partial x} \frac{\partial N_j^e}{\partial x} + \frac{\partial N_i^e}{\partial y} \frac{\partial N_j^e}{\partial y} \right) \right] dx dy \right\}, \quad (9.2.11)$$

and terms of the vector on the right-hand side are

$$F_j = \sum_e \left\{ \int_{\Omega^e} N_j^e f_v^e dx dy + \int_{\Gamma_n^e} N_j^e q^e d\Gamma \right\}. \quad (9.2.12)$$

Accordingly, in order to assemble the system we firstly have to choose the geometrical shape of the elements and the interpolation functions, and afterwards use them to approximate the geometry of the domain Ω and the unknown function u .

9.2.3. An approximation of the geometry of the problem's domain

The approximation of a problem's domain Ω is performed by subdividing it into a finite number of elements Ω^e . From equation (9.2.7) it follows that certain rules should be respected in subdivision; namely:

- two distinct finite elements may possess common points only on their common boundary. Overlapping is not allowed (Figure 9.2.1a),
- the assembled finite elements should not leave any holes within the domain (Figure 9.2.1b),
- the constitutive parameters of a material v^e , as well as the sources f_v^e , are constant inside any element.

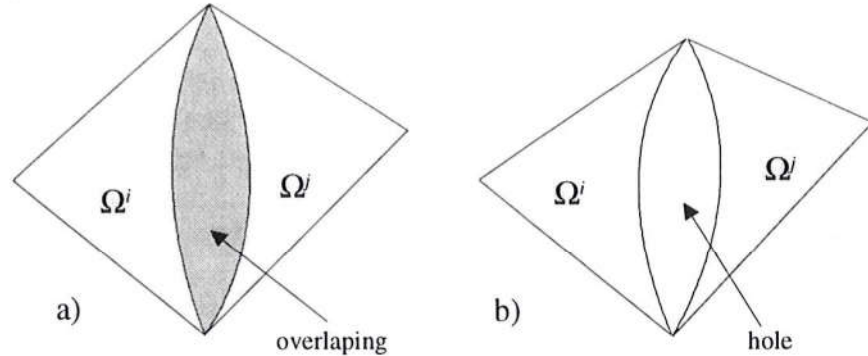


Figure 9.2.1. Forbidden finite-element sub-divisions

These rules can be obeyed if the geometry of the element is defined *analytically*. The points defining the geometry of the finite element are called geometrical nodes. Each element must be uniquely defined by geometrical nodes belonging to the element or to its boundary.

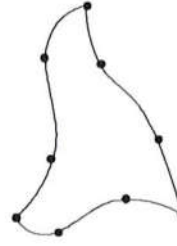
The shapes and geometrical nodes of commonly-used two-dimensional elements are shown in Figure 9.2.2. For the sake of simplicity, we shall develop here an application using only linear triangular elements. The same approach is used in applications using elements with more complex geometries.

Triangular elements

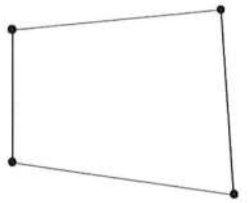
Linear (3)



Quadratic (6)



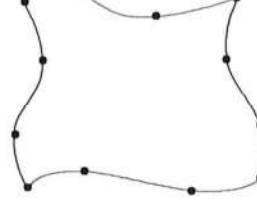
Cubic (9)

Quadrilateral elements

Linear (4)



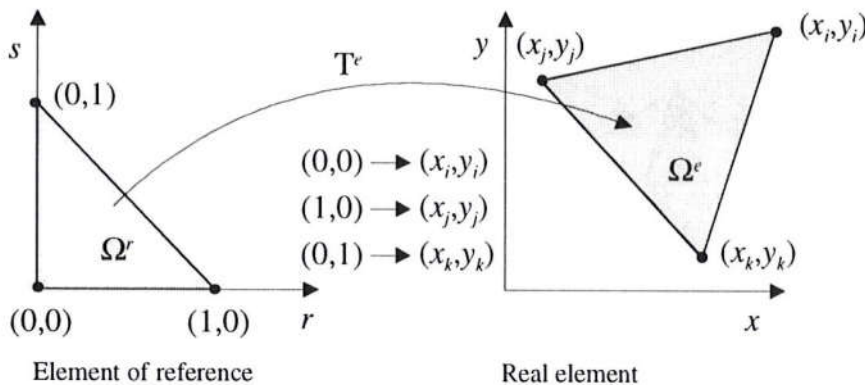
Quadratic (8)



Cubic (12)

Figure 9.2.2. Classic shapes of two-dimensional finite elements

We obtain the simplest analytical expressions by introducing the *elements of reference*. Such elements are defined as very simple geometrical shapes in a *non-dimensional abstract coordinate system* with coordinates (r,s) . Their geometry is mapped into the geometry of the real elements in (x,y) space by usage of geometrical transformations T^r . Reference and real linear triangular elements are shown in Figure 9.2.3.

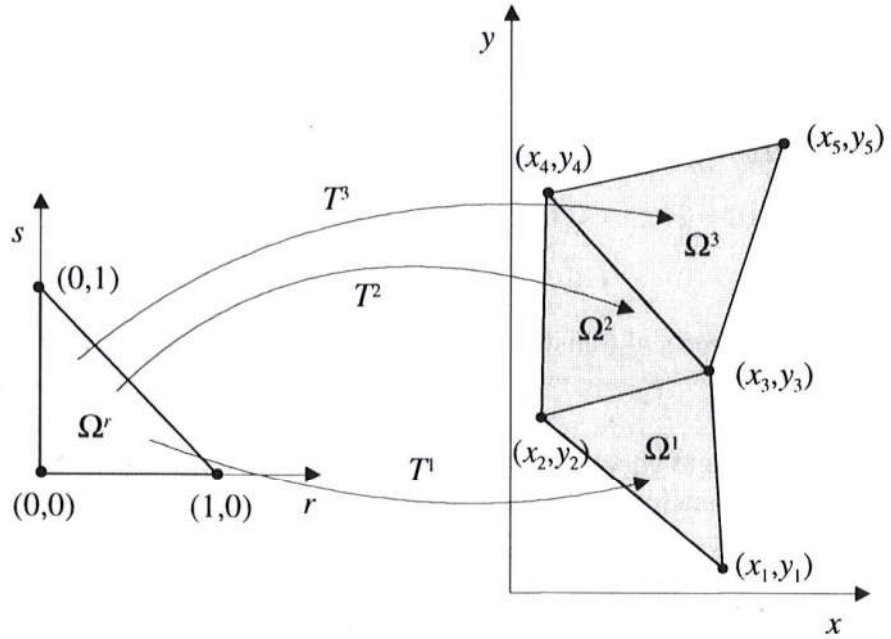
**Figure 9.2.3.** Element of reference and a real element

The coordinates of each point of a real element Ω^e are defined by the transformation T^r in terms of abstract coordinates (r,s) . The transformation T^r should satisfy the following requirements:

- this should involve one to one mapping. Accordingly, there is one and only one point of the real element Ω^e that corresponds to each point of the element of reference Ω^r ,
- the geometrical nodes of the element of reference have to be mapped onto the geometrical nodes of the real element Ω^e ,
- every part of the boundary of the element of reference Ω^r defined by the geometrical nodes on that boundary should be mapped onto the part of boundary of the real element Ω^e defined by the corresponding mapped geometrical nodes.

One element of reference of a particular type is mapped onto all the real elements of the same type by usage of different transformations T^i as shown in Figure 9.2.4.

Figure 9.2.4. Mapping of the element of reference onto real elements



Therefore,

$$\begin{aligned}\Omega^1 \rightarrow T^1: (r,s) \rightarrow (x,y)^1 &= [x(r,s,x_1,y_1,x_2,y_2,x_3,y_3), y(r,s,x_1,y_1,x_2,y_2,x_3,y_3)]^1, \\ \Omega^2 \rightarrow T^2: (r,s) \rightarrow (x,y)^2 &= [x(r,s,x_2,y_2,x_3,y_3,x_4,y_4), y(r,s,x_2,y_2,x_3,y_3,x_4,y_4)]^2, \\ \Omega^3 \rightarrow T^3: (r,s) \rightarrow (x,y)^3 &= [x(r,s,x_3,y_3,x_4,y_4,x_5,y_5), y(r,s,x_3,y_3,x_4,y_4,x_5,y_5)]^3.\end{aligned}\quad (9.2.13)$$

If we use identical geometrical transformations for both coordinates, we can write

$$\begin{aligned}x(r,s) &= [G_1(r,s) \quad G_2(r,s) \quad G_3(r,s)] \begin{bmatrix} x_i \\ x_j \\ x_k \end{bmatrix}, \\ y(r,s) &= [G_1(r,s) \quad G_2(r,s) \quad G_3(r,s)] \begin{bmatrix} y_i \\ y_j \\ y_k \end{bmatrix}.\end{aligned}\quad (9.2.14a,b)$$

Functions G_i are called *geometrical transformation functions* and they are usually polynomials. All linear triangular elements can be obtained by mapping the linear triangular element of reference using linear transformations. Thus, the functions G_i are of the form

$$G_i(r,s) = a_{i0} + a_{ir}r + a_{is}s \quad ; \quad i = 1, 2, 3. \quad (9.2.15)$$

From the conditions

$$\begin{aligned} x(0,0) &= x_i & x(1,0) &= x_j & x(0,1) &= x_k, \\ y(0,0) &= y_i & y(1,0) &= y_j & y(0,1) &= y_k, \end{aligned} \quad (9.2.16)$$

it follows that

$$\begin{aligned} G_1(0,0) &= 1 & G_2(0,0) &= 0 & G_3(0,0) &= 0, \\ G_1(1,0) &= 0 & G_2(1,0) &= 1 & G_3(1,0) &= 0, \\ G_1(0,1) &= 0 & G_2(0,1) &= 0 & G_3(0,1) &= 1, \end{aligned} \quad (9.2.17)$$

and the geometrical transformation functions for linear triangular elements are

$$G_1 = 1 - r - s, \quad G_2 = r, \quad G_3 = s. \quad (9.2.18)$$

Now we have to check if this transformation satisfies all the previously mentioned conditions. The transformation is a one to one transformation if the Jacobian matrix of the transformation J is non-singular. For the introduced transformation the following is true:

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix} = \begin{bmatrix} (x_j - x_i) & (y_j - y_i) \\ (x_k - x_i) & (y_k - y_i) \end{bmatrix}. \quad (9.2.19)$$

The determinant of the Jacobian matrix of this transformation is

$$\det[J] = (x_j - x_i)(y_k - y_i) - (x_k - x_i)(y_j - y_i) = 2S_\Delta, \quad (9.2.20)$$

and it is equal to twice the area of the triangle S_Δ defined by the vertices (x_i, y_i) , (x_j, y_j) , (x_k, y_k) . Thus, the transformation given by (9.2.14,18) is not one to one only if these vertices are on the same straight line. The part of the boundary defined by the geometrical nodes (1,0) and (0,1) of the element of reference is given by

$$1 - r - s = 0 \quad ; \quad 0 \leq r, s \leq 1. \quad (9.2.21)$$

We map it onto the real element by

$$x = [0 \quad r \quad 1-r] \begin{bmatrix} x_i \\ x_j \\ x_k \end{bmatrix} = x_j + (1-r)x_k, \quad y = [0 \quad r \quad 1-r] \begin{bmatrix} y_i \\ y_j \\ y_k \end{bmatrix} = y_j + (1-r)y_k.$$

These equations are parametric equations of a straight line between the points (x_j, y_j) and (x_k, y_k) . We can prove, in a similar way, that the boundaries of the element of reference are mapped onto corresponding parts of the real element in accordance with the previously mentioned requirement. It is important to note that the surfaces of all the finite elements possess orientations. Thus, all elements have to be connected to the geometrical nodes in

the same order, clockwise or counterclockwise with respect to an outward unit normal vector.

9.2.3.1. Organization of data on subdivision into elements

Let us number all the geometrical nodes of domain Ω sequentially from 1 to n and define all their coordinates in a common coordinate system (x,y) . Data on geometrical nodes can be grouped in an XYNODE array:

geometrical nodes

	1	2	3	...	n
x	x_1	x_2	x_3	...	x_n
y	y_1	y_2	y_3	...	y_n

array XYNODE

If all elements of the Ω domain are numbered sequentially from 1 to m , and if we list all the geometrical nodes (vertices of triangles) belonging to the individual element, we can group data on the connection between elements and geometrical nodes into an ELTCONN array:

elements

	1	2	3	...	m
i	i_1	i_2	i_3	...	i_m
j	j_1	j_2	j_3	...	j_m
k	k_1	k_2	k_3	...	k_m

array ELTCONN

The two arrays XYNODE and ELTCONN are sufficient enough to completely define the transformation T^e for each element. The coordinates in the direction of the x - and y -axis are mapped by

$$x^e(r,s) = [1-r-s \quad r \quad s] \begin{bmatrix} x_i^e \\ x_j^e \\ x_k^e \end{bmatrix}, \quad y^e(r,s) = [1-r-s \quad r \quad s] \begin{bmatrix} y_i^e \\ y_j^e \\ y_k^e \end{bmatrix}. \quad (9.2.22)$$

Example 9.2.3.1. Subdivision of a rectangular domain into four elements

A rectangular domain in an (x,y) coordinate system and its subdivision into four elements are shown in Figure 9.2.5.

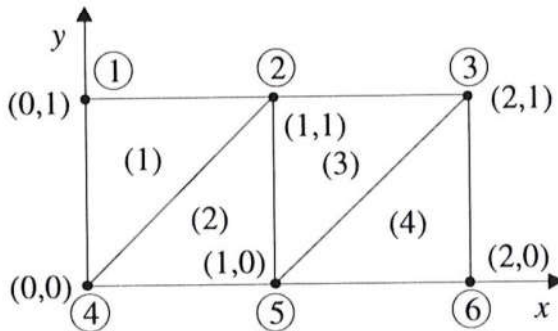


Figure 9.2.5. Subdivision of a rectangular domain

There are six geometrical nodes in this subdivision and the corresponding XYNODE array is

	1	2	3	4	5	6
x	0	1	2	0	1	2
y	1	1	1	0	0	0

The number of elements is equal to four, and the ELTCONN array is

	1	2	3	4
i	1	2	2	3
j	2	5	3	6
k	4	4	5	5

9.2.4. An approximation of the function on the element

The fundamental property of FEM is the *dependence of the approximation over an element only on the values of the function at the interpolation nodes of that element*. The form of this approximation is

$$u^e(x,y) \approx \sum_i N_i^e(x,y) u_i^e(x_i, y_i) . \quad (9.2.23)$$

Here, N_i are the interpolation functions, and u_i are the unknown coefficients representing the values of the function at chosen points of the element, which are called interpolation nodes. The approximation in equation (9.2.23) is defined on the real element. In this case, the interpolation functions depend on the geometry of the real element, and we replace this approximation with that on the reference element

$$u(r,s) \approx \sum_i N_i(r,s) u_i(r_i, s_i) , \quad (9.2.24)$$

together with the coordinate transformation

$$T:(r,s) \rightarrow [x(r,s) \ y(r,s)] = [G_1(r,s) \ G_2(r,s) \ G_3(r,s)] \begin{bmatrix} x_i & y_i \\ x_j & y_j \\ x_k & y_k \end{bmatrix} . \quad (9.2.25)$$

The interpolation functions in this approximation are independent of the geometry of the real element, and we use *a unique set of interpolation functions* N_i for elements having the same element of reference.

In the case of linear triangular elements, we choose interpolation nodes identical to the geometrical nodes. The unknown coefficients u_i , which are also called *nodal variables*, are values of the unknown functions at the vertices of the triangle. The approximation in this case is of the form

$$u(r,s) \approx [N_1(r,s) \ N_2(r,s) \ N_3(r,s)] \begin{bmatrix} u_i \\ u_j \\ u_k \end{bmatrix} . \quad (9.2.26)$$

If we assume a linear distribution of unknown function u over the whole element, the interpolation functions N_i are of the form

$$N_i(r,s) = b_{i0} + b_{ir}r + b_{is}s \quad ; \quad i = 1, 2, 3 . \quad (9.2.27)$$

From the conditions

$$u(0,0) = u_i \quad ; \quad u(1,0) = u_j \quad ; \quad u(0,1) = u_k , \quad (9.2.28)$$

it follows that

$$\begin{aligned} N_1(0,0) &= 1 \quad ; \quad N_2(0,0) = 0 \quad ; \quad N_3(0,0) = 0 , \\ N_1(1,0) &= 0 \quad ; \quad N_2(1,0) = 1 \quad ; \quad N_3(1,0) = 0 , \\ N_1(0,1) &= 0 \quad ; \quad N_2(0,1) = 0 \quad ; \quad N_3(0,1) = 1 , \end{aligned} \quad (9.2.29)$$

and the interpolation functions for linear triangular elements are

$$N_1 = 1 - r - s \equiv G_1 \quad , \quad N_2 = r \equiv G_2 \quad , \quad N_3 = s \equiv G_3 \quad (9.2.30)$$

They are identical to the geometrical transformation functions. The elements for which $N_i \equiv G_i$ are called *isoparametric elements*. The functions N_1 , N_2 and N_3 are graphically presented in Figure 9.2.6.

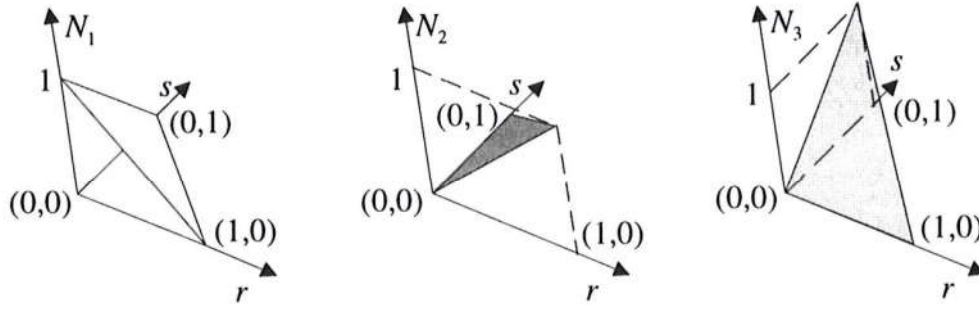


Figure 9.2.6. Interpolation functions of linear triangular elements

The equations that we want to solve are given in the real domain. They involve the unknown function u and its derivatives and integrals. Since we have used an approximation on the element of reference, we have to transform the operators of differentiation and integration into those for the space of the element of reference. For the first derivative, we get

$$\begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial s} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix}, \quad (9.2.31)$$

or written in short notation as

$$[\partial_{r,s}] = [J][\partial_{x,y}], \quad (9.2.32)$$

where $[J]$ is the Jacobian matrix of the geometrical transformation. Similarly, we can write

$$[\partial_{x,y}] = [j][\partial_{r,s}]. \quad (9.2.33)$$

Substitution of expression (9.2.33) into (9.2.32) results in $[j] = [J]^{-1}$. We have shown that the geometrical transformation is valid when $\det[J] \neq 0$, $[j]$ always exists and that we determine it from $[J]$. In our case this is

$$[J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}; \quad [J]^{-1} = [j] = \frac{1}{J_{11}J_{22} - J_{12}J_{21}} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix}. \quad (9.2.34)$$

We determine the terms of the Jacobian matrix directly by differentiation of the expression for the geometrical transformation of the element of reference

$$[x \ y] = \begin{bmatrix} G_1 & G_2 & G_3 \\ G_1 & G_2 & G_3 \end{bmatrix} \begin{bmatrix} x_i & y_i \\ x_j & y_j \\ x_k & y_k \end{bmatrix}, \quad (9.2.35)$$

and we finally obtain

$$[J] = \begin{bmatrix} \frac{\partial G_1}{\partial r} & \frac{\partial G_2}{\partial r} & \frac{\partial G_3}{\partial r} \\ \frac{\partial G_1}{\partial s} & \frac{\partial G_2}{\partial s} & \frac{\partial G_3}{\partial s} \end{bmatrix} \begin{bmatrix} x_i & y_i \\ x_j & y_j \\ x_k & y_k \end{bmatrix} = \begin{bmatrix} (x_j - x_i) & (y_j - y_i) \\ (x_k - x_i) & (y_k - y_i) \end{bmatrix}. \quad (9.2.36)$$

The change of variables defined by expression (9.2.14) allows the use of a simpler integration over the element of reference

$$\int_{\Omega^e} f(x,y) dx dy = \int_{\Omega^r} f[x(r,s), y(r,s)] \det[J'] dr ds, \quad (9.2.37)$$

instead of an integration over the real element.

9.2.5. Element integral forms of linear triangular elements used in solving Poisson's equation

The general form of the element integral forms used in solving Poisson's equation is determined by equations (9.2.11,12). For a linear triangular element, we have that

$$u^e = [N_1^e \ N_2^e \ N_3^e] \begin{bmatrix} u_i^e \\ u_j^e \\ u_k^e \end{bmatrix}, \quad (9.2.38)$$

and the element integral form of the e -th element is

$$W^e = [k^e] u^e + f^e = \begin{bmatrix} k_{11}^e & k_{12}^e & k_{13}^e \\ k_{21}^e & k_{22}^e & k_{23}^e \\ k_{31}^e & k_{32}^e & k_{33}^e \end{bmatrix} \begin{bmatrix} u_i^e \\ u_j^e \\ u_k^e \end{bmatrix} + \begin{bmatrix} f_1^e \\ f_2^e \\ f_3^e \end{bmatrix}. \quad (9.2.39)$$

From equation (9.2.11) it follows that we may calculate the terms of the matrix $[k^e]$ by

$$k_{ij}^e = \int_{\Omega^e} v^e \left(\frac{\partial N_i^e}{\partial x} \frac{\partial N_j^e}{\partial x} + \frac{\partial N_i^e}{\partial y} \frac{\partial N_j^e}{\partial y} \right) dx dy, \quad (9.2.40)$$

and therefore matrix $[k^e]$ can be written as

$$[k^e] = \int_{\Omega^e} v^e [N_{xy}^e]^T [N_{xy}^e] dx dy, \quad (9.2.41)$$

where $[N_{xy}^e]$ is the matrix

$$[N_{xy}^e] = \begin{bmatrix} \frac{\partial N_1^e}{\partial x} & \frac{\partial N_2^e}{\partial x} & \frac{\partial N_3^e}{\partial x} \\ \frac{\partial N_1^e}{\partial y} & \frac{\partial N_2^e}{\partial y} & \frac{\partial N_3^e}{\partial y} \end{bmatrix}, \quad (9.2.42)$$

and $[N_{xy}^e]^T$ is the transpose of matrix $[N_{xy}^e]$. When we use an approximation on the element of reference we obtain

$$[N_{xy}^e] = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial s}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial s}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial N_1}{\partial r} & \frac{\partial N_2}{\partial r} & \frac{\partial N_3}{\partial r} \\ \frac{\partial N_1}{\partial s} & \frac{\partial N_2}{\partial s} & \frac{\partial N_3}{\partial s} \end{bmatrix} = [j^e][N_{rs}], \quad (9.2.43)$$

and the transpose of matrix $[N_{xy}^e]^T$ for the approximation on the element of reference is

$$[N_{xy}^e]^T = [N_{rs}]^T [j^e]^T. \quad (9.2.44)$$

By usage of equation (9.2.37) a transformation into an integration over the element of reference results in

$$\begin{aligned} [k^e] &= \int_{\Omega^e} v^e [N_{xy}^e]^T [N_{xy}^e] dx dy = \\ &= \int_0^1 \int_0^{1-r} v^e [N_{rs}]^T [j^e]^T [j^e] [N_{rs}] \det[J^e] dr ds. \end{aligned} \quad (9.2.45)$$

The constitutive parameter of a material v^e is constant over the whole element, the matrices $[N_{rs}]$ and $[j^e]$ are constant as well, and $\det[J^e] = 2S_\Delta^e$. Thus,

$$\begin{aligned} [k^e] &= 2v^e S_\Delta^e [N_{rs}]^T [j^e]^T [j^e] [N_{rs}] \int_0^1 \int_0^{1-r} ds dr = \\ &= v^e S_\Delta^e [N_{rs}]^T [j^e]^T [j^e] [N_{rs}]. \end{aligned} \quad (9.2.46)$$

If we consider that

$$[N_{rs}] = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}; \quad [j^e] = \frac{1}{2S_\Delta^e} \begin{bmatrix} (y_k^e - y_i^e) & -(y_j^e - y_i^e) \\ -(x_k^e - x_i^e) & (x_j^e - x_i^e) \end{bmatrix}, \quad (9.2.47)$$

we then obtain the terms of matrix $[k^e]$ as

$$[k^e] = \frac{v^e}{2[(x_j^e - x_i^e)(y_k^e - y_i^e) - (x_k^e - x_i^e)(y_j^e - y_i^e)]} \begin{bmatrix} k_{11}^e & k_{12}^e & k_{13}^e \\ k_{21}^e & k_{22}^e & k_{23}^e \\ k_{31}^e & k_{32}^e & k_{33}^e \end{bmatrix}. \quad (9.2.48)$$

The matrix terms k_{ij}^e in this equation are

$$\begin{aligned} k_{11}^e &= (y_k^e - y_j^e)^2 + (x_k^e - x_j^e)^2, \\ k_{12}^e &= k_{21}^e = (y_k^e - y_j^e)(y_i^e - y_k^e) + (x_k^e - x_j^e)(x_i^e - x_k^e), \\ k_{13}^e &= k_{31}^e = (y_j^e - y_i^e)(y_k^e - y_j^e) + (x_j^e - x_i^e)(x_k^e - x_j^e), \\ k_{22}^e &= (y_k^e - y_i^e)^2 + (x_k^e - x_i^e)^2, \\ k_{23}^e &= k_{32}^e = (y_i^e - y_k^e)(y_j^e - y_i^e) + (x_i^e - x_k^e)(x_j^e - x_i^e), \\ k_{33}^e &= (y_j^e - y_i^e)^2 + (x_j^e - x_i^e)^2. \end{aligned} \quad (9.2.49)$$

From equation (9.2.12) it follows that

$$f^e = \int_{\Omega^e} f_v^e \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \end{bmatrix} dx dy + \int_{\Gamma^e} q^e \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \end{bmatrix} d\Gamma. \quad (9.2.50)$$

In solving a static electric or magnetic field problem we are most frequently able to determine the boundaries where

$$\frac{\partial \varphi}{\partial n} = 0 \quad \text{or} \quad \frac{\partial A_z}{\partial n} = 0. \quad (9.2.51)$$

Equipotentials of a static electric field, and field lines of a static magnetic field are perpendicular to such boundaries. Thus, in calculation of the element vector f^e , we introduce the assumption $q^e = 0$ and transform equation (9.2.50) into

$$f^e = f_v^e \int_{\Omega^e} \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \end{bmatrix} dx dy = f_v^e \int_0^1 \int_0^{1-r} \begin{bmatrix} 1-r-s \\ r \\ s \end{bmatrix} \det[J^e] ds dr = \frac{f_v^e S_{\Delta}^e}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad (9.2.52)$$

since f_v is constant over the whole element.

9.2.6. Assembly of a system of equations

In previous sections, we described the two initial steps in the application of FEM: subdivision of the domain into elements followed by approximation of the unknown function on the element. After these two steps, we know the element integral form $w^e = [k^e]u^e = f^e$. The vector u^e contains the unknown coefficients (nodal variables) of the e -th element, which is different for each element.

The next step in the application of FEM is the assembly of a system of equations for the unknown nodal variables of the *complete domain* in the form $[K]U + F = 0$. The global vector U contains all the nodal variables of the complete domain, and all the vectors u^e are thus contained within it. We obtain the global matrix $[K]$ and vector F by summation of the element integral forms for all elements

$$K_{ij} = \sum_e K_{ij}^e \quad ; \quad F_j = \sum_e F_j^e .$$

Summation of the previously defined element integral forms is impossible because the vectors u^e differ for each element. Therefore, we have to rewrite them in terms of the global vector for *all nodal variables* U . We obtain this by the expansion of $[k^e]$ and f^e into $[K^e]$ and F^e which are of the same dimensions as $[K]$ and F . All terms of $[K^e]$ and F^e , except the ones contained in $[k^e]$ and $[f^e]$, are equal to zero. The element integral forms for linear triangular elements are

$$w^e = \begin{bmatrix} k_{11}^e & k_{12}^e & k_{13}^e \\ k_{21}^e & k_{22}^e & k_{23}^e \\ k_{31}^e & k_{32}^e & k_{33}^e \end{bmatrix} \begin{bmatrix} u_i^e \\ u_j^e \\ u_k^e \end{bmatrix} + \begin{bmatrix} f_1^e \\ f_2^e \\ f_3^e \end{bmatrix} . \quad (9.2.53)$$

If the Ω domain contains n nodes numbered sequentially from 1 to n , the global vector for the nodal variables is $U^T = [u_1 \ u_2 \ \dots \ u_i \ \dots \ u_j \ \dots \ u_k \ \dots \ u_n]$. The expanded element integral forms are of the form

$$W^e = \begin{bmatrix} . & . & . & . & . & . & . & . & . \\ 0 & . & k_{11}^e & . & k_{12}^e & . & k_{13}^e & . & 0 \\ . & . & . & . & . & . & . & . & . \\ 0 & . & k_{12}^e & . & k_{22}^e & . & k_{23}^e & . & 0 \\ . & . & . & . & . & . & . & . & . \\ 0 & . & k_{13}^e & . & k_{23}^e & . & k_{33}^e & . & 0 \\ . & . & . & . & . & . & . & . & . \end{bmatrix} \begin{bmatrix} . \\ u_i \\ . \\ u_j \\ . \\ u_k \\ . \end{bmatrix} + \begin{bmatrix} . \\ f_1^e \\ . \\ f_2^e \\ . \\ f_3^e \\ . \end{bmatrix} . \quad (9.2.54)$$

After expansion of the element integral forms, we obtain a system of equations for the

unknown nodal variables using

$$W = \sum_{e=1}^m W^e = \left\{ \sum_{e=1}^m [K^e] \right\} U + \left\{ \sum_{e=1}^m F^e \right\} = [K] U + F, \quad (9.2.55)$$

where m is the total number of elements into which the domain of the problem has been subdivided.

Example 9.2.6.1. The assembly of a system for the domain of Example 9.2.3.1.

The expanded integral form of the first element is

$$W^1 = \begin{bmatrix} K_{11}^1 & K_{12}^1 & 0 & K_{14}^1 & 0 & 0 \\ K_{21}^1 & K_{22}^1 & 0 & K_{24}^1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ K_{41}^1 & K_{42}^1 & 0 & K_{44}^1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} + \begin{bmatrix} F_1^1 \\ F_2^1 \\ 0 \\ F_4^1 \\ 0 \\ 0 \end{bmatrix}. \quad (9.2.56)$$

We expand the integral forms of other elements in the same way. The final form of the system is

$$W = W^1 + W^2 + W^3 + W^4 = \begin{bmatrix} K_{11} & K_{12} & 0 & K_{14} & 0 & 0 \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & 0 \\ 0 & K_{32} & K_{33} & 0 & K_{35} & K_{36} \\ K_{41} & K_{42} & 0 & K_{44} & K_{45} & 0 \\ 0 & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} \\ 0 & 0 & K_{63} & 0 & K_{65} & K_{66} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} + \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{bmatrix}. \quad (9.2.57)$$

9.2.6.1. The algorithm for the assembly of the system

We assemble the system (9.2.55) by an algorithm ASSSYS that uses data from the ELTCONN array, the matrix $[k^e]$ and the vector f^e

```
DO e = 1,m
  DO u = 1,3
    i = ELTCONN(u,e)
    DO v = 1,3
      j = ELTCONN(v,e)
```



```

      K(i,j) = K(i,j) + ke(u,v)
    ENDDO
      F(i) = F(i) + fe(u)
    ENDDO
  ENDDO

```

We now have to introduce the Dirichlet boundary conditions in the form $u = u_0$ on the part of the boundary $\Gamma = \Gamma_0$ into the assembled system. After subdivision into finite elements, the Dirichlet boundary conditions transform to

$$u_i(x_i, y_i) = u_{i0} \quad ; \quad (x_i, y_i) \in \Gamma_0 . \quad (9.2.58)$$

The simplest way of introducing them into the system is by using a large number in the diagonal term. The matrix $[K]$ and the vector $\mathbf{F}$ are assembled without any need for attention to the Dirichlet boundary conditions. Then we introduce the condition $u_i = u_{i0}$ replacing the diagonal term K_{ii} by $K_{ii} + \alpha$, where α is a very large number compared to all other terms K_{ij} and $K_{ii} + \alpha \approx \alpha$ (for example, in a six significant digit computer it is $1.23456 + 10^{15} = 1.00000 \cdot 10^{15}$), and the term F_i is replaced by αu_{i0} . This enforces the system solution $u_i = u_{i0}$. This method is simple to code and retains the symmetry of the global system matrix.

9.2.7. The solution of systems of linear algebraic equations in FEM

The solution of the system $[K]\mathbf{U} = \mathbf{F}$ is the next step in the application of FEM. The accuracy and range of application of FEM are in essence limited only by the *dimension of the system* that can be solved by an available computer. We classify methods of solutions into two groups: *iterative and direct*. Both groups are widely applied. We shall describe here *Gaussian elimination*, which is the fundamental direct method for the solution of a system of linear algebraic equations. It is performed in two basic steps: triangularization and back-substitution.

9.2.7.1. Triangularization

Within this step, we reduce the original system into an upper-triangular form

$$\begin{bmatrix} K_{11}' & K_{12}' & K_{13}' & \dots & K_{1,n-1}' & K_{1n}' \\ 0 & K_{22}' & K_{23}' & \dots & K_{2,n-1}' & K_{2n}' \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ 0 & 0 & 0 & \dots & K_{n-1,n-1}' & K_{n-1,n}' \\ 0 & 0 & 0 & \dots & 0 & K_{nn}' \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ u_{n-1} \\ u_n \end{bmatrix} = \begin{bmatrix} F_1' \\ F_2' \\ \cdot \\ \cdot \\ F_{n-1}' \\ F_n' \end{bmatrix}, \quad (9.2.59)$$

using a systematic series of linear transformations. The unknowns u_s , where $s = 1, 2, \dots, n-1$, are eliminated from the equations numbered $s+1, s+2, \dots, n$. The elimination of u_s is accomplished in two steps:

1. Multiplication of the equation number s by the coefficient $-K_{is}/K_{ss}$.
 2. Summation of this multiplied equation number s with the equation number i .
- We do these two steps for values $s = 1, 2, \dots, n-1$ and $i = s+1, s+2, \dots, n$. We eliminate the unknown u_s from the equations numbered $s+1, s+2, \dots, n$ in step number s , and thus modify the terms

$$\left. \begin{aligned} K_{ij}^{(s)} &= K_{ij}^{(s-1)} - \frac{K_{is}^{(s-1)}}{K_{ss}^{(s-1)}} K_{sj}^{(s-1)} \\ F_i^{(s)} &= F_i^{(s-1)} - \frac{K_{is}^{(s-1)}}{K_{ss}^{(s-1)}} F_s^{(s-1)} \end{aligned} \right\} \begin{aligned} &s = 1, 2, \dots, n-1, \\ &i, j = s+1, s+2, \dots, n. \end{aligned} \quad (9.2.60)$$

In the last step we eliminate the unknown u_{n-1} from equation number n and the system is transformed into an upper-triangular form. Using the algorithm

```
DO s = 1, n-1
  DO i = s+1, n
    a = K(i,s)/K(s,s)
    F(i) = F(i) - a*F(s)
    DO j = s+1, n
      K(i,j) = K(i,j) - a*K(s,j)
    ENDDO
  ENDDO
ENDDO
```

we apply the triangularization in the space occupied by the initial matrix.

9.2.7.2. Back-substitution

We solve the system in upper-triangular form starting from the last equation involving only one unknown working towards the first equation using

$$u_k = \frac{1}{K_{kk}^{(s-1)}} \left(F_k^{(s-1)} - \sum_{j=k+1}^n K_{kj}^{(s-1)} u_j \right) \quad ; \quad k = n, n-1, \dots, 1. \quad (9.2.61)$$

If the upper-triangular matrix is stored in the space of the original matrix, we use the algorithm

```
u(n) = F(n)/K(n,n)
DO k = n-1, 1, -1
  a = 0
```

```

DO j = k + 1, n
  a = a + K(k, j) * u(j)
ENDDO
u(k) = (F(k) - a) / K(k, k)
ENDDO

```

for back-substitution. The fundamental properties of matrices of systems obtained by FEM are

- very big dimensions,
- the matrix of the system in FEM is obtained by summation of very sparse expanded element matrices and a great number of its terms are equal to zero.
- they are most frequently symmetrical.

Because of these properties, storage of the complete matrix in a computer memory and direct usage of Gaussian elimination is inappropriate in FEM. Therefore, we use decomposition of the matrix $[K]$ into an upper-triangular matrix $[G]$ and a lower-triangular matrix $[D]$ in such a way that $[K] = [D][G]$ is valid. Here the matrices $[D]$ and $[G]$ are of the forms

$$[D] = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ D_{21} & 1 & 0 & \dots & 0 \\ D_{31} & D_{32} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ D_{n1} & D_{n2} & D_{n3} & \dots & 1 \end{bmatrix} ; \quad [G] = \begin{bmatrix} G_{11} & G_{12} & G_{13} & \dots & G_{1n} \\ 0 & G_{22} & G_{23} & \dots & G_{2n} \\ 0 & 0 & G_{33} & \dots & G_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & G_{nn} \end{bmatrix} \quad (9.2.62)$$

The easiest way to derive the algorithm for decomposition is to compare terms of the product $[D][G]$ with terms of the matrix $[K]$. We compare these terms in regions $P(p)$ defined by the index p

$$\left[\begin{array}{c|c|c|c|c} p=1 & p=2 & \dots & p=s & \\ \hline K_{11} & K_{12} & \dots & K_{1s} & \dots K_{1n} \\ \hline - & \lrcorner & & & \\ \hline K_{21} & K_{22} & \dots & K_{2s} & \dots K_{2n} \\ \hline - & - & \lrcorner & & \\ \hline \vdots & \vdots & \dots & \vdots & \dots \vdots \\ \hline - & - & - & - & \lrcorner \\ \hline K_{s1} & K_{s2} & \dots & K_{ss} & \dots K_{sn} \\ \hline - & - & - & - & - \\ \hline \vdots & \vdots & \dots & \vdots & \dots \vdots \\ \hline K_{n1} & K_{n2} & \dots & K_{ns} & \dots K_{nn} \end{array} \right] \quad (9.2.63)$$

For the region $p = 1$, we have that

$$1G_{11} = K_{11} \Rightarrow G_{11} = K_{11} , \quad (9.2.64)$$

and for the region $p = 2$

$$\begin{aligned} D_{21}G_{11} &= K_{21} \Rightarrow D_{21} = \frac{K_{21}}{G_{11}} , \\ 1G_{12} &= K_{12} \Rightarrow G_{12} = K_{12} , \\ D_{21}G_{12} + 1G_{22} &= K_{22} \Rightarrow G_{22} = K_{22} - D_{21}G_{12} , \end{aligned} \quad (9.2.65)$$

and in the region $p = s$, the terms of the matrices $[D]$ and $[G]$ are calculated as

$$\left. \begin{aligned} D_{si} &= \frac{1}{G_{ii}} \left(K_{si} - \sum_{m=1}^{i-1} D_{sm}G_{mi} \right) \\ G_{is} &= K_{is} - \sum_{m=1}^{i-1} D_{im}G_{ms} \end{aligned} \right\} i = 1, 2, \dots, s-1 , \quad (9.2.66)$$

$$D_{ss} = 1 ,$$

$$G_{ss} = K_{ss} - \sum_{m=1}^{s-1} D_{sm}G_{ms} .$$

We can see from this expression that the terms of the matrices $[D]$ and $[G]$ for the region $p = s$ are calculated using the terms of the matrix $[K]$ from this region and the previously calculated terms of the matrices $[D]$ and $[G]$ from the regions $p = 1, 2, \dots, s-1$. Therefore, we can organize the decomposition in the area where the matrix $[K]$ is stored using the algorithm DGDECO

```
DO s = 2,n
  DO i = 1,s - 1
    DO m = 1,i - 1
      K(s,i) = K(s,i) - K(s,m)*K(m,i)
      K(i,s) = K(i,s) - K(i,m)*K(m,s)
    ENDDO
    K(s,i) = K(s,i)/K(i,i)
  ENDDO
  DO m = 1,s - 1
    K(s,s) = K(s,s) - K(s,m)*K(m,s)
  ENDDO
ENDDO
```

When this algorithm is accomplished, the terms of the initial matrix $[K]$ are destroyed, and the terms of the matrices $[D]$ and $[G]$ are stored in their places. If the matrix $[K]$ is symmetric, then $K_{ij} = K_{ji}$, and from expressions (9.2.66) it follows that $G_{ij} = D_{ji}G_{ii}$. In such a case, we do not have to calculate and store the matrix $[D]$. We need only the upper trian-

gle of the matrix $[K]$, and we calculate and store the terms of the matrix $[G]$ within it. Substitution of $G_{ij} = D_{ji}G_{ii}$ into (9.2.66) results in an expression for calculating terms of the matrix $[G]$ for a symmetrical matrix $[K]$ in the region $p = s$

$$\begin{aligned} G_{is} &= K_{is} - \sum_{m=1}^{i-1} G_{mi} G_{ms} \quad ; \quad i = 1, 2, \dots, s-1, \\ G_{ss} &= K_{ss} - \sum_{m=1}^{s-1} \frac{G_{ms}^2}{G_{mm}}. \end{aligned} \quad (9.2.67)$$

The algorithm SYDGDECO for calculating terms of the upper-triangular matrix $[G]$ of the $[D][G]$ decomposition of the symmetrical matrix $[K]$ is

```
DO s = 2,n
  DO i = 1,s - 1
    DO m = 1,i - 1
      K(i,s) = K(i,s) - K(m,i)*K(m,s)/K(m,m)
    ENDDO
  ENDDO
  DO m = 1,s - 1
    K(s,s) = K(s,s) - K(m,s)^2/K(m,m)
  ENDDO
ENDDO
```

The matrix $[K]$ possesses a great number of zero terms. We differentiate between two areas:

- an area of zeros in which all the terms in a row or column are equal to zero, starting from the first term,
- the rest that contain non-zero terms.

A *skyline profile* separates the two. This is a line above which there are no non-zero terms in the upper triangle of the matrix $[K]$, and a line below which there are no non-zero terms in the lower triangle of the matrix $[K]$. In the expression

$$\begin{bmatrix} K_{11} & K_{12} & \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} \\ K_{21} & K_{22} & \begin{bmatrix} K_{23} & K_{24} \end{bmatrix} & \begin{bmatrix} 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 \end{bmatrix} & K_{32} & K_{33} & 0 & \begin{bmatrix} 0 \end{bmatrix} & \begin{bmatrix} K_{36} \end{bmatrix} \\ 0 & \begin{bmatrix} K_{42} & 0 & K_{44} & K_{45} & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & K_{54} & K_{55} & K_{56} \\ 0 & 0 & \begin{bmatrix} K_{63} & 0 & K_{65} & K_{66} \end{bmatrix} \end{bmatrix},$$

we have shown a skyline profile for one matrix. The matrices resulting from application of the rules for assembling the system have symmetrical skyline profiles of the upper and lower triangular parts of $[K]$. An essential property of matrices $[D]$ and $[G]$ in this case is that their skyline profiles are identical to the skyline profile of the matrix $[K]$. This follows from equations (9.2.66). If for the region $p = s$:

$$\left. \begin{array}{l} K_{su} = 0 \\ K_{us} = 0 \end{array} \right\} u = 1, 2, \dots, l \quad ; \quad l \leq s-1, \quad (9.2.68)$$

then

$$\begin{aligned} D_{s1} &= \frac{K_{s1}}{G_{11}} = 0 & (K_{s1} = 0) \\ D_{s2} &= \frac{1}{G_{22}} (K_{s2} - D_{s1} G_{11}) = 0 & (K_{s2} = 0, D_{s1} = 0) \\ & \vdots \\ D_{sl} &= \frac{1}{G_{ll}} \left(K_{sl} - \sum_{i=1}^{l-1} D_{si} G_{il} \right) = 0 & (K_{sl} = 0, D_{sk} = 0 \quad ; \quad k = 1, \dots, l) \\ G_{1s} &= K_{1s} = 0 & (K_{1s} = 0) \\ G_{2s} &= K_{2s} - D_{11} G_{1s} & (K_{2s} = 0, G_{1s} = 0) \\ & \vdots \\ G_{ls} &= K_{ls} - \sum_{i=1}^{l-1} D_{li} G_{is} = 0 & (K_{ls} = 0, G_{us} = 0 \quad ; \quad u = 1, \dots, l-1) \end{aligned} \quad (9.2.69)$$

This property enables a considerable saving of computer memory since for the $[D][G]$ decomposition of the matrix $[K]$ we need only know the terms of the matrix $[K]$ inside the skyline profile. If the matrix $[K]$ is symmetrical, we have to store only terms of the upper triangle below the skyline profile. After the $[D][G]$ decomposition of the matrix $[K]$, the system is transformed into $[D][G]U = F$ and we solve it in two steps:

1. Elimination

$$[D]F' = F. \quad (9.2.70)$$

This is the lower-triangular system and we solve it by

$$\begin{aligned} F_1' &= F_1, \\ F_i' &= F_i - \sum_{j=1}^{i-1} D_{ij} F_j' \quad ; \quad i = 2, 3, \dots, n. \end{aligned} \quad (9.2.71)$$

2. Back-substitution

$$[G]U' = F' \quad (9.2.72)$$

This is the upper-triangular system and we solve it starting from the last equation

$$u_n = \frac{F'_n}{G_{nn}}, \quad (9.2.73)$$

$$u'_i = \frac{1}{G_{ii}} \left(F'_i - \sum_{j=i+1}^n G_{ij} u'_j \right) \quad ; \quad i = n-1, n-2, \dots, 1.$$

If we store the matrices $[D]$ and $[G]$ in the space of the matrix $[K]$, the algorithm for solving the system (9.2.73) is

```
DO i = 2,n
  a = 0
  DO j = 1, i - 1
    a = a + K(i,j)*F(j)
  ENDDO
  F(i) = F(i) - a
ENDDO
F(n) = F(n)/K(n,n)
DO i = n - 1, 1, - 1
  a = 0
  DO j = i + 1, n
    a = a + K(i,j)*F(j)
  ENDDO
  F(i) = (F(i) - a)/K(i,i)
ENDDO
```

When this algorithm has been carried out, the vector F contains the solution U . If $[K]$ is a symmetrical matrix, the elimination step can be done using terms of the matrix $[G]$ from the expression $G_{ij} = D_{ji}G_{ii}$ which are

$$F'_1 = F_1,$$

$$F'_i = F_i - \sum_{j=1}^{i-1} \frac{G_{ji}}{G_{jj}} F_j \quad ; \quad i = 2, 3, \dots, n. \quad (9.2.74)$$

By the algorithm SYMSOL

```
DO i = 2,n
  a = 0
  DO j = 1, i - 1
    a = a + K(j,i)*F(j)/K(j,j)
```

```

      ENDDO
      F(i) = F(i) - a
    ENDDO
    F(n) = F(n)/K(n,n)
    DO i = n - 1, 1, - 1
      a = 0
      DO j = i + 1, n
        a = a + K(i,j)*F(j)
      ENDDO
      F(i) = (F(i) - a)/K(i,i)
    ENDDO

```

we can solve a symmetrical system after the $[D][G]$ decomposition using the algorithm SYMDGDECO.

9.2.8. Organization of data on the equation system

We adopt the organization of data on the equation system according to the advantages of the $[D][G]$ decomposition. We have to store only terms of the matrix $[K]$ below the skyline profile, and because of the symmetricalness of the matrix $[K]$ we calculate and store only the upper-triangular matrix $[G]$ in space utilised for storage of the upper triangle of the matrix $[K]$. The organization is based on knowledge of the skyline profile of the matrix $[K]$. We determine it using data from the array ELTCONN in which we have stored data on the connectivity between the elements and the nodal variables.

We obtain the system matrix $[K]$ by summation of expanded matrices of the element integral forms $[K^e]$. The non-zero terms of the matrix $[K^e]$ are

$$K_{ij}^e ; \begin{cases} i = \text{ELTPOV}(u, e) & ; \quad u = 1, 2, 3 \\ j = \text{ELTPOV}(v, e) & ; \quad v = 1, 2, 3 \end{cases} \quad (9.2.75)$$

Due to the symmetricalness of the matrix $[K^e]$, we have to analyse only its upper-triangular part. For this part $j > i$, and the height above the diagonal of every non-zero term of the matrix $[K_{ij}^e]$ is $h_{ij} = j - i$. Accordingly, the height of the column number j above the diagonal to the skyline profile for the e -th term is

$$h_j^e = \max_i \{ h_{ij} \} , \quad (9.2.76)$$

and the height of the column j above the diagonal to the skyline profile for the system matrix $[K]$ is

$$h_j = \max_e \{ h_j^e \} . \quad (9.2.77)$$

We apply these expressions for column heights of the matrix $[K]$ above the diagonal to the skyline profile to the HEIGCOL algorithm:

```

DO e = 1,m
  DO u = 1,3
    i = ELTCONN(u,e)
    DO v = 1,3
      j = ELTCONN(v,e)
      IF j > i
        THEN
          hh = j - i
          IF hh > h(j)
            THEN
              h(j) = hh
            ENDIF
          ENDIF
        ENDDO
      ENDDO
    ENDDO
  ENDDO

```

The diagonal term is not included into the column height. After determination of the heights of all columns, we store the terms of the matrix $[K]$ in the one-dimensional PROFIL array, column by column. In doing this, we use the pointer for the diagonal term of each column POIDIA defined by

```

POIDIA(1) = 1,
POIDIA(i) = POIDIA(i - 1) + h(i) + 1 ; i = 1,2,...n.

```

Accordingly, POIDIA(n) is the dimension of the PROFIL array needed for storage of all elements of the upper triangle of the matrix $[K]$. The term K_{ij} is in the array PROFIL at the position POIDIA(j) - $j + i$. We generate the pointer POIDIA using the algorithm POINTER

```

POIDIA(1) = 1
DO j = 2,n
  POIDIA(j) = POIDIA(j - 1) + h(j) + 1
ENDDO

```

We generate the system matrix in the PROFIL array and vector F using the algorithm GENSYS:

```

DO e = 1,m
  DO u = 1,3
    i = ELTCONN(u,e)
    DO v = 1,3
      j = ELTCONN(v,e)
      IF j ≥ i
        THEN

```



```

      l = POIDIA(j) - j + i
      PROFIL(l) = PROFIL(l) + ke(u,v)
    ENDIF
    F(i) = F(i) + fe(v)
  ENDDO
ENDDO

```

Example 9.2.8.1. Storage of the system from Example 9.2.6.1.

The matrix of the system is

$$[K] = \begin{bmatrix} K_{11} & K_{12} & 0 & K_{14} & 0 & 0 \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & 0 \\ 0 & K_{32} & K_{33} & 0 & K_{35} & K_{36} \\ K_{41} & K_{42} & 0 & K_{44} & K_{45} & 0 \\ 0 & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} \\ 0 & 0 & K_{63} & 0 & K_{65} & K_{66} \end{bmatrix}. \quad (9.2.78)$$

The vector of column heights for this numeration is $\mathbf{h}_j^T = [0 \ 1 \ 1 \ 3 \ 3 \ 3]$ and the pointer is $\text{POIDIA}^T = [1 \ 3 \ 5 \ 9 \ 13 \ 17]$. Thus, the PROFIL array must have 17 terms

$$[K_{11} \ K_{12} \ K_{22} \ K_{23} \ K_{33} \ K_{14} \ K_{24} \ 0 \ K_{44} \ K_{25} \ K_{35} \ K_{45} \ K_{55} \ K_{36} \ 0 \ K_{56} \ K_{66}]$$

The number of non-zero terms of the matrix $[K]$ is independent of the method of node numeration. A useful way of lowering the number of zero terms under the skyline profile is to lower the biggest difference between ordinal numbers of nodes belonging to the same element as much as possible. Let us consider the renumumerated nodes from Example 9.2.6.1. as shown in Figure 9.2.7.

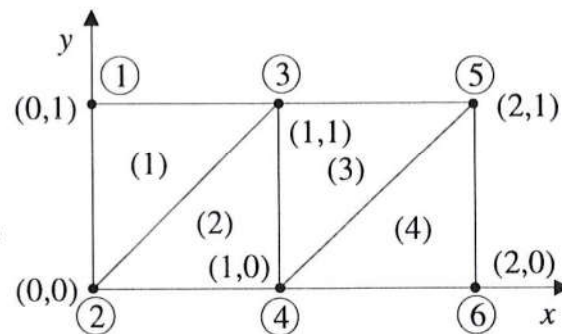


Figure 9.2.7. Renumerated nodes from Example 9.2.6.1.

The connection between the elements and the nodes is now

$$\text{ELTCONN} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 4 & 6 \\ 3 & 3 & 5 & 5 \end{bmatrix}, \quad (9.2.79)$$

and the system matrix takes the form

$$[K] = \begin{bmatrix} K_{11} & K_{12} & K_{13} & 0 & 0 & 0 \\ K_{21} & K_{22} & K_{23} & K_{24} & 0 & 0 \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & 0 \\ 0 & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} \\ 0 & 0 & K_{53} & K_{54} & K_{55} & K_{56} \\ 0 & 0 & 0 & K_{64} & K_{65} & K_{66} \end{bmatrix}. \quad (9.2.80)$$

The vector of column heights for this numeration is $\mathbf{h}_j^T = [0 \ 1 \ 2 \ 2 \ 2 \ 2]$ and the pointer is $\text{POIDIA}^T = [1 \ 3 \ 6 \ 9 \ 12 \ 15]$. Therefore, the array PROFIL must now have 15 terms

$$[K_{11} \ K_{12} \ K_{22} \ K_{13} \ K_{23} \ K_{33} \ K_{24} \ K_{34} \ K_{44} \ K_{35} \ K_{45} \ K_{55} \ K_{46} \ K_{56} \ K_{66}]$$

By such renumbering of nodes, we achieve an optimal solution since there are no zero terms in the PROFIL array.

Storing the matrix into a PROFIL array considerably lowers the amount of computer memory needed. In order to utilize this kind of storage we need to adjust the algorithms for the $[D][G]$ decomposition and for the solution of the system by considering operations with zero terms above the skyline profile and symmetry of the matrix $[K]$. The SYULDECO algorithm is modified to the algorithm SKYSYDECO for calculation of the upper-triangular matrix $[G]$ of the symmetric matrix $[K]$ stored in the PROFIL array:

```
DO s = 2,n
  is = s - (POIDIA(s) - POIDIA(s - 1) - 1)
  DO i = is,s - 1
    ii = i - (POIDIA(i) - POIDIA(i - 1) - 1)
    mp = is
    IF ii > is
      THEN
        mp = ii
      ENDIF
    kis = POIDIA(s) - s + i
    DO m = mp,i - 1
      kmi = POIDIA(i) - i + m
      kms = POIDIA(s) - s + m
      kmm = POIDIA(m)
      PROFIL(kis) = PROFIL(kis) - PROFIL(kmi)*PROFIL(kms)/PROFIL(kmm)
    ENDDO
  ENDDO
DO m = is,s - 1
  kss = POIDIA(s)
  kms = POIDIA(s) - s + m
```

```

      kmm = POIDIA(m)
      PROFIL(kss) = PROFIL(kss) - PROFIL(kms)2/PROFIL(kmm)
    ENDDO
  ENDDO

```

The algorithm SYMSOL is modified into the algorithm SKYSYSOL for the solution of a symmetric system after determination of the upper-triangular matrix $[G]$ by the algorithm SKYSYDECO:

```

DO i = 2,n
  is = i - (POIDIA(i) - POIDIA(i - 1) - 1)
  DO j = is,i - 1
    kji = POIDIA(i) - i + j
    kjj = POIDIA(j)
    F(i) = F(i) - PROFIL(kji)*F(j)/PROFIL(kjj)
  ENDDO
ENDDO
knn = POIDIA(n)
F(n) = F(n)/PROFIL(knn)
DO i = n - 1,1, - 1
  kii = POIDIA(i)
  a = 0
  DO j = i + 1,n
    hj = POIDIA(j) - POIDIA(j - 1) - 1
    IF hj ≥ j - i
      THEN
        kij = POIDIA(j) - j + i
        a = a + PROFIL(kij)*F(j)
      ENDIF
    ENDDO
    F(i) = (F(i) - a)/PROFIL(kii)
  ENDDO

```

9.2.9. Calculation of electric and magnetic field strength

After the solution of the system $[K]U = F$, we obtain an approximation for the potential φ or A_z in the complete domain Ω . In the analysis of electromagnetic devices, we are often interested in local stresses and critical values of field vectors. Thus, in the practical application of FEM we have to calculate the field vectors $\mathbf{E}$ and $\mathbf{D}$, or $\mathbf{B}$ and $\mathbf{H}$.

9.2.9.1. Calculation of the electric field vectors E and D

The vectors of a static electric field are related to the scalar electric potential by $E = -\nabla\phi$ and $D = \epsilon E$. In two-dimensional problems this is

$$E = -\frac{\partial\phi}{\partial x} \mathbf{a}_x - \frac{\partial\phi}{\partial y} \mathbf{a}_y = E_x \mathbf{a}_x + E_y \mathbf{a}_y . \quad (9.2.81)$$

By usage of the expression for differentiation of the approximation on the element of reference (9.2.33), we obtain

$$\begin{aligned} \begin{bmatrix} \frac{\partial\phi}{\partial x} \\ \frac{\partial\phi}{\partial y} \end{bmatrix} &= [j^e] \begin{bmatrix} \frac{\partial G_1}{\partial r} & \frac{\partial G_2}{\partial r} & \frac{\partial G_3}{\partial r} \\ \frac{\partial G_1}{\partial s} & \frac{\partial G_2}{\partial s} & \frac{\partial G_3}{\partial s} \end{bmatrix} \begin{bmatrix} \phi_i^e \\ \phi_j^e \\ \phi_k^e \end{bmatrix} = \\ &= \frac{1}{2S_\Delta^e} \begin{bmatrix} (y_k^e - y_i^e) & -(y_j^e - y_i^e) \\ -(x_k^e - x_i^e) & (x_j^e - x_i^e) \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \phi_i^e \\ \phi_j^e \\ \phi_k^e \end{bmatrix} = \\ &= \frac{1}{2S_\Delta^e} \begin{bmatrix} (y_j^e - y_k^e) & (y_k^e - y_i^e) & (y_i^e - y_j^e) \\ (x_k^e - x_j^e) & (x_i^e - x_k^e) & (x_j^e - x_i^e) \end{bmatrix} \begin{bmatrix} \phi_i^e \\ \phi_j^e \\ \phi_k^e \end{bmatrix} , \end{aligned} \quad (9.2.82)$$

and by the expression

$$\begin{aligned} E_x^e &= -\frac{1}{2S_\Delta^e} [(y_j^e - y_k^e) \phi_i^e + (y_k^e - y_i^e) \phi_j^e + (y_i^e - y_j^e) \phi_k^e] , \\ E_y^e &= -\frac{1}{2S_\Delta^e} [(x_k^e - x_j^e) \phi_i^e + (x_i^e - x_k^e) \phi_j^e + (x_j^e - x_i^e) \phi_k^e] , \\ D_x^e &= \epsilon^e E_x^e , \\ D_y^e &= \epsilon^e E_y^e , \end{aligned} \quad (9.2.83a - d)$$

we can calculate the static electric field vectors on the e -th element.

9.2.9.2. Calculation of the magnetic field vectors B and H

The relationship between static magnetic field vectors and the magnetic vector potential is $\mathbf{B} = \nabla \times \mathbf{A}_z$ and $\mathbf{H} = \mathbf{B}/\mu$. In two-dimensional problems this is

$$\mathbf{B} = \frac{\partial A_z}{\partial y} \mathbf{a}_x - \frac{\partial A_z}{\partial x} \mathbf{a}_y = B_x \mathbf{a}_x + B_y \mathbf{a}_y . \quad (9.2.84)$$

We obtain the expression for the magnetic field vectors using the expression for differentiation of the approximation on the element of reference (9.2.33) on the e -th element as

$$\begin{aligned} B_x^e &= \frac{1}{2S_\Delta^e} [(x_k^e - x_j^e) A_i^e + (x_i^e - x_k^e) A_j^e + (x_j^e - x_i^e) A_k^e] , \\ B_y^e &= \frac{1}{2S_\Delta^e} [(y_j^e - y_k^e) A_i^e + (y_k^e - y_i^e) A_j^e + (y_i^e - y_j^e) A_k^e] , \\ H_x^e &= \frac{B_x^e}{\mu^e} , \\ H_y^e &= \frac{B_y^e}{\mu^e} . \end{aligned} \quad (9.2.85)$$

The expressions (9.2.83) and (9.2.85) show that the field vectors are constant over the whole of an element. They follow from the linear approximation of potentials on the element.

9.2.10. Visual representation of results of calculation by lines of constant potential

It is very hard to analyse the results of calculations in numerical form. One has to study a sequence of very similar numbers, and it is very tedious and discouraging for the user. Therefore, it is necessary to present the results in graphical form. We most frequently visualise two-dimensional fields using lines of constant potential as well as field lines. The equation for field lines in a two-dimensional magnetic field is

$$\frac{dy}{dx} = \frac{B_y}{B_x} \quad \Rightarrow \quad B_x dy - B_y dx = 0 . \quad (9.2.88)$$

By substitution of the expression for the x - and y -components of the vector $\mathbf{B}$ into this equation, we obtain

$$\frac{\partial A_z}{\partial x} dx + \frac{\partial A_z}{\partial y} dy = dA_z = 0 \Rightarrow A = \text{const.} \quad (9.2.89)$$

Thus, in a two-dimensional magnetic field, lines of constant magnetic vector potential A_z are in fact field lines.

9.2.10.1. Calculation of equipotential lines

The approximation of the potential function u on the element of reference is

$$u^r(r,s) = \begin{bmatrix} 1-r-s & r & s \end{bmatrix} \begin{bmatrix} u_i^e \\ u_j^e \\ u_k^e \end{bmatrix}, \quad (9.2.90)$$

The equipotential line of potential u_0 is given by $u = u_0$, and is a line segment on the element of reference

$$r(u_j^e - u_i^e) + s(u_k^e - u_i^e) + u_i^e - u_0 = 0. \quad (9.2.91)$$

The element of reference is defined by

$$s \geq 0 \quad ; \quad r \geq 0 \quad ; \quad 1 - r - s \geq 0, \quad (9.2.92)$$

and its boundaries are

$$\begin{aligned} (i-j) &\rightarrow s=0 \quad ; \quad 0 \leq r \leq 1, \\ (j-k) &\rightarrow s=1-r \quad ; \quad 0 \leq r \leq 1, \\ (k-i) &\rightarrow r=0 \quad ; \quad 0 \leq s \leq 1. \end{aligned} \quad (9.2.93)$$

The conditions for calculating equipotential lines follow from the definition of the element of reference (9.2.92,93) and from the equation of the equipotential line on the e -th element (9.2.91), they are:

1. An equipotential line of potential u_0 passes over the e -th element if

$$\min\{u_i^e, u_j^e, u_k^e\} \leq u_0 \leq \max\{u_i^e, u_j^e, u_k^e\}. \quad (9.2.94)$$

2. An equipotential line of potential u_0 intersects the boundary $(i-j)$ at the point

$$[x_{ij}^e \quad y_{ij}^e] = \begin{bmatrix} 1 - \frac{u_0 - u_i^e}{u_j^e - u_i^e} & \frac{u_0 - u_i^e}{u_j^e - u_i^e} & 0 \end{bmatrix} \begin{bmatrix} x_i^e & y_i^e \\ x_j^e & y_j^e \\ x_k^e & y_k^e \end{bmatrix}, \quad (9.2.95)$$

if

$$(u_j^e > u_i^e \text{ and } u_i^e \leq u_0 \leq u_j^e) \text{ or } (u_j^e < u_i^e \text{ and } u_i^e \geq u_0 \geq u_j^e) . \quad (9.2.96)$$

3. The boundary $(i - j)$ is the equipotential line of potential u_0 if $u_i^e = u_j^e = u_0$.
4. An equipotential line of potential u_0 intersects the boundary $(j - k)$ at the point

$$[x_{jk}^e \ y_{jk}^e] = \begin{bmatrix} 0 & \frac{u_0 - u_k^e}{u_j^e - u_k^e} & \frac{u_j^e - u_0}{u_j^e - u_k^e} \end{bmatrix} \begin{bmatrix} x_i^e & y_i^e \\ x_j^e & y_j^e \\ x_k^e & y_k^e \end{bmatrix}, \quad (9.2.97)$$

if

$$(u_j^e > u_k^e \text{ and } u_k^e \leq u_0 \leq u_j^e) \text{ or } (u_j^e < u_k^e \text{ and } u_k^e \geq u_0 \geq u_j^e) . \quad (9.2.98)$$

5. The boundary $(j - k)$ is the equipotential line of the potential u_0 if $u_j^e = u_k^e = u_0$.
6. An equipotential line of potential u_0 intersects the boundary $(k - i)$ at the point

$$[x_{ki}^e \ y_{ki}^e] = \begin{bmatrix} 1 - \frac{u_0 - u_i^e}{u_k^e - u_i^e} & 0 & \frac{u_0 - u_i^e}{u_k^e - u_i^e} \end{bmatrix} \begin{bmatrix} x_i^e & y_i^e \\ x_j^e & y_j^e \\ x_k^e & y_k^e \end{bmatrix}, \quad (9.2.99)$$

if

$$(u_k^e > u_i^e \text{ and } u_i^e \leq u_0 \leq u_k^e) \text{ or } (u_k^e < u_i^e \text{ and } u_i^e \geq u_0 \geq u_k^e) . \quad (9.2.100)$$

7. The boundary $(i - j)$ is the equipotential line of potential u_0 if $u_k^e = u_i^e = u_0$.

9.2.11. An example of calculation

We prepare the field problem for solving by FEM using the following steps:

- analysis of the geometry of the problem and the setting up of a model that is an appropriate approximation of the real problem,
- generation of a finite element mesh,
- identification of the properties of the materials involved and their association to the finite element mesh,
- definition of boundary conditions and their introduction into the system of FEM equations.

This complete procedure is usually called *preprocessing*, and the part of FEM programming packages that support it is called the *preprocessor*.

The solution phase for the system of FEM equations follows the preparation phase. It is the central task in solving the problem, and the part of FEM programming packages that support it is called the *kernel*. The results of this phase are values of the unknown function at the nodes defining the mathematical approximation.

The next step is to calculate quantities acceptable to an engineer using a mathematical approximation. Most frequently, this comprises a graphical presentation of the results, a calculation of the maximum local stresses, and determination of the integral parameters of the problem. This phase is called the *postprocessing*, and the part of FEM programming packages that support this is called the *postprocessor*.

As an example of a calculation of a two-dimensional static electric and magnetic field, we have chosen a three-phase-shielded busbar. This is a three-dimensional problem, but the ratio of its longitudinal dimension to its cross-sectional dimensions justifies a two-dimensional modelling. The cross-section of the busbar is shown in Figure 9.2.8.

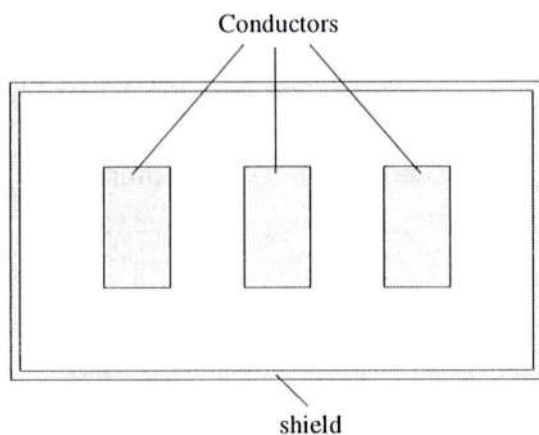


Figure 9.2.8. Cross-section of a shielded busbar

Example 9.2.11.1. The static electric field

The mathematical model of the busbar and the finite element mesh for calculating its static electric field is shown in Figure 9.2.9.

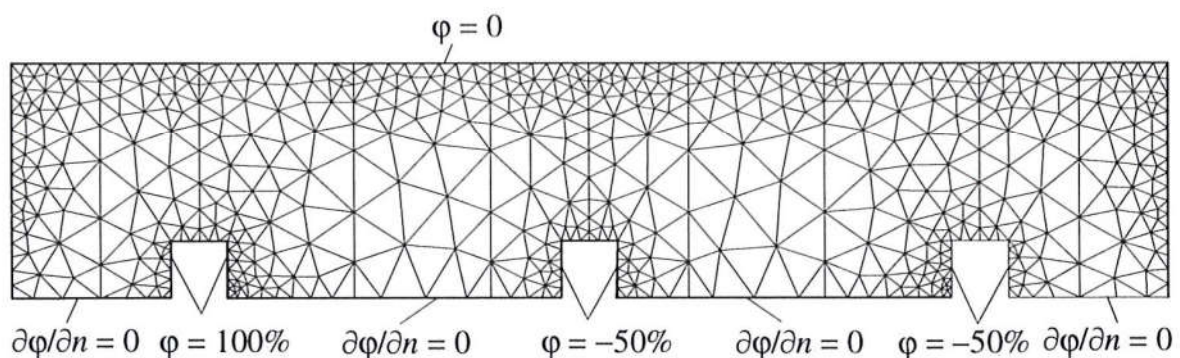


Figure 9.2.9. The finite element mesh for calculating static electric field of a busbar

We have used here Dirichlet boundary conditions on the boundaries of the conductive parts because their potentials are known. Neumann boundary conditions are applied on the line of symmetry of the problem.

At a typical instant, the potential of the shield is equal to zero, the potential of the conductor on the left side is 100% of the maximum phase voltage and the potential of the central and right conductors are - 50% of the maximum phase voltage. $\partial\phi/\partial n = 0$ on the line of symmetry. Equipotential lines for the chosen boundary conditions are shown in Figure 9.2.10.

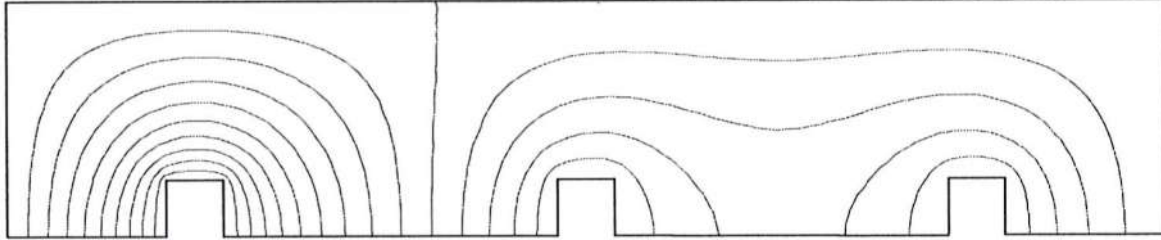


Figure 9.2.10. Equipotential lines from the FEM solution of the static electric field of the busbar, the potential difference between equipotential lines is 10% of the maximum phase voltage

By a different choice of boundary conditions, one can simulate various instants of normal or faulty operation.

Example 9.2.11.2. The static magnetic field of a busbar

The setting up of the model for solving this problem is more complex since the magnetic field occupies the complete space. In the modelling of such a problem in FEM, we usually introduce artificial distant boundaries whose influence on the accuracy of the solution in the region of interest is negligible.

A practical question is immediately posed: how far should the artificial distant boundaries be set? In solving three-dimensional problems with a given distribution of current by a multipole expansion, we proved that at a sufficient distance from the current distribution, one can approximate them using a magnetic dipole.

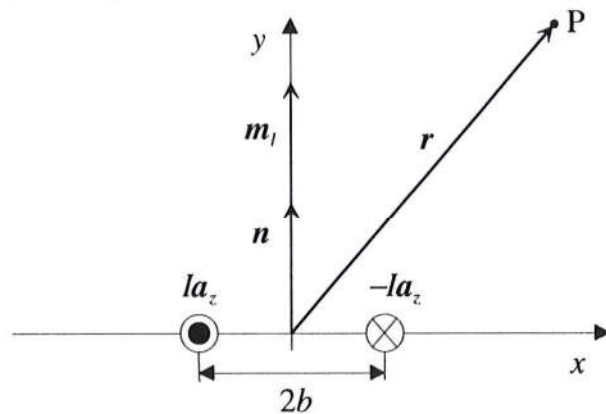


Figure 9.2.11. Magnetic line dipole

Analogously, we can prove that a given two-dimensional distribution of current may also be approximated by a magnetic line dipole. This consists of two line currents in opposite directions as shown in Figure 9.2.11. The observation distance is much larger than the distance between the line currents. The magnetic vector potential of the magnetic line dipole decreases reciprocally with distance

$$A = \frac{\mu}{2\pi} \frac{1}{r^2} \mathbf{m}_l \times \mathbf{r} . \quad (9.2.101)$$

Here $\mathbf{m}_l$ is the magnetic dipole moment per unit length of the magnetic line dipole as defined by the product of the area of a unit length and the current in the lines

$$\mathbf{m}_l = 2bI\mathbf{n} . \quad (9.2.102)$$

If distance away the point of observation's is four to six times greater than the dimensions of the currents, we can approximate the distribution by a magnetic line dipole as the contributions of other terms, decreasing by r^{-2} , r^{-3} etc., may be neglected with respect to the first term, which is decreasing by r^{-1} . Therefore, we can set the artificial boundaries at a distance four to six times larger than the dimensions of the current distribution.

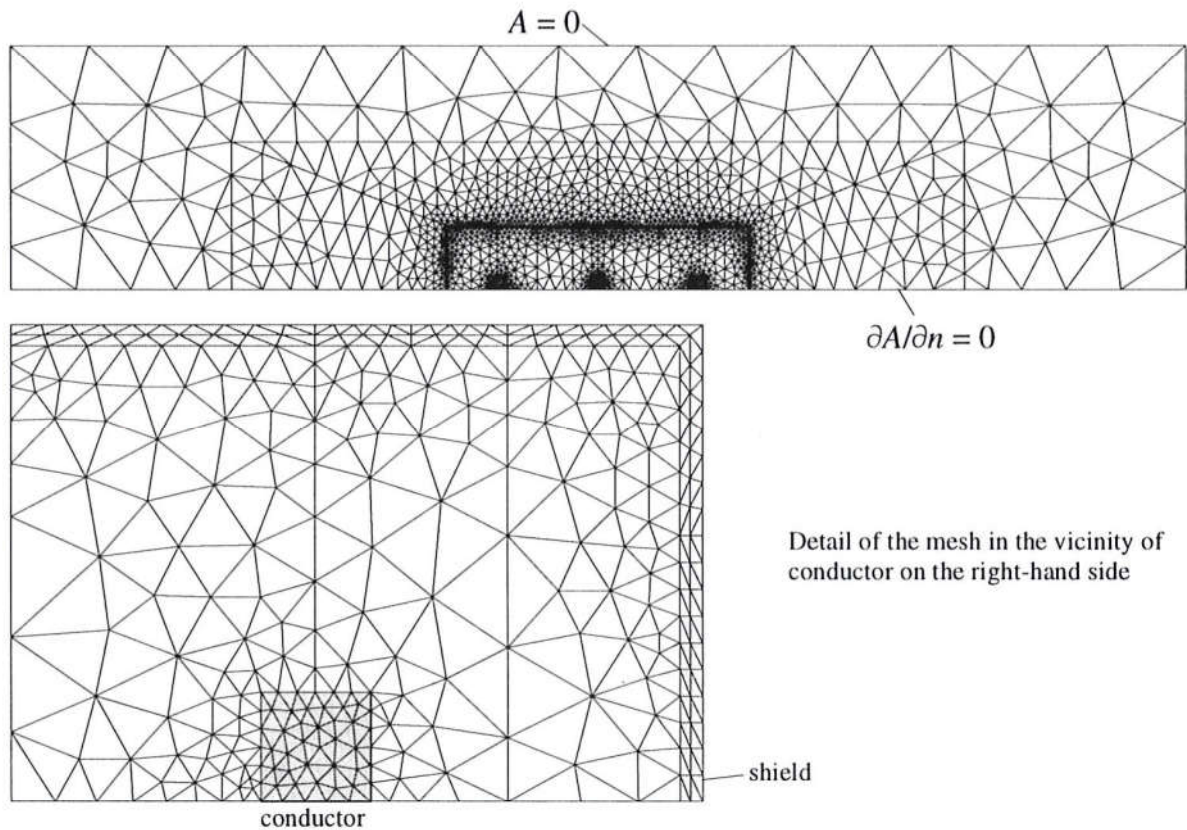


Figure 9.2.12. Boundaries and the finite element mesh for calculating the magnetic field of a busbar

The mathematical model for calculating the static magnetic field and the finite element mesh is shown in Figure 9.2.12.

Here we have applied Dirichlet boundary conditions $A_z = 0$ on the artificial distant borders, and Neumann boundary conditions $\partial A_z / \partial n = 0$ on the line of symmetry of the problem. The conductor on the left carries 100% of the maximum line current, and the central and right conductors carry - 50% of the maximum line current. The field lines of the static magnetic field in the shielded busbar are shown in Figure 9.2.13.

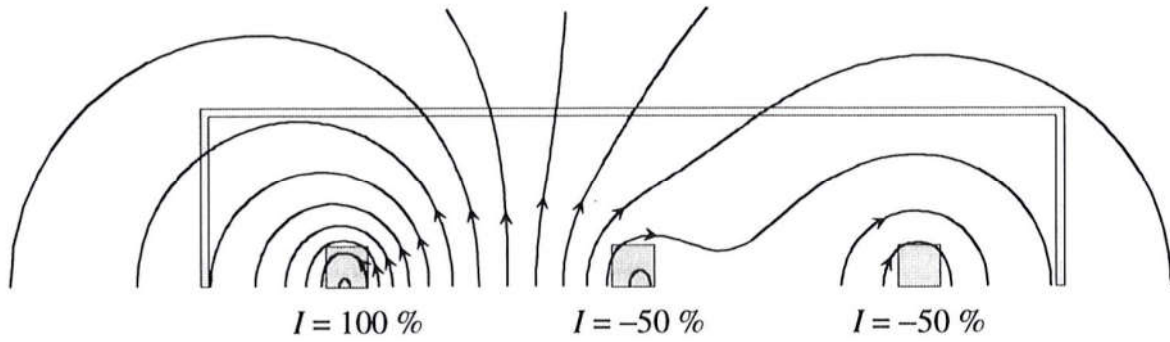


Figure 9.2.13. Field lines of the static magnetic field in busbar determined by FEM

9.3. NON-LINEAR PROBLEMS

Non-linear problems are usually experienced when solving a magnetic field problem involving ferromagnetic materials because of their extremely non-linear constitutive parameters. The global system in such a problem is of the form $[K(U)]U = F$.

This is a non-linear system that cannot be solved by any direct method. We apply iterative procedures, the most widely used of which is the *Newton-Raphson method* because of its fast convergence. In order to present an application of this method, we introduce the vector of residual functions $O(U) = [K(U)]U - F$, which is a non-linear function of the vector for all unknowns.

Let us denote the solution in the i -th step as U^i and suppose its accuracy to be unsatisfactory. In such a case,

$$O(U^i) = [K(U^i)]U^i - F \neq 0. \quad (9.3.1)$$

In the $(i + 1)$ -th step we can expand the vector function $O(U^{i+1})$ into a Taylor series. If we consider only derivatives of first order, we obtain

$$\begin{bmatrix} O_1^{i+1} \\ O_2^{i+1} \\ \vdots \\ O_n^{i+1} \end{bmatrix} - \begin{bmatrix} O_1^i \\ O_2^i \\ \vdots \\ O_n^i \end{bmatrix} = \begin{bmatrix} \frac{\partial O_1}{\partial U_1} & \frac{\partial O_1}{\partial U_2} & \cdots & \frac{\partial O_1}{\partial U_n} \\ \frac{\partial O_2}{\partial U_1} & \frac{\partial O_2}{\partial U_2} & \cdots & \frac{\partial O_2}{\partial U_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial O_n}{\partial U_1} & \frac{\partial O_n}{\partial U_2} & \cdots & \frac{\partial O_n}{\partial U_n} \end{bmatrix}^i \begin{bmatrix} U_1^{i+1} - U_1^i \\ U_2^{i+1} - U_2^i \\ \vdots \\ U_n^{i+1} - U_n^i \end{bmatrix}. \quad (9.3.2)$$

Our aim is to obtain U^{i+1} converging to an exact solution. Then the residual is $O(U^{i+1}) = 0$, and from the previous equation it follows that:

$$-\begin{bmatrix} O_1^i \\ O_2^i \\ \vdots \\ O_n^i \end{bmatrix} = \begin{bmatrix} \frac{\partial O_1}{\partial U_1} \big|_i & \frac{\partial O_1}{\partial U_2} \big|_i & \cdots & \frac{\partial O_1}{\partial U_n} \big|_i \\ \frac{\partial O_2}{\partial U_1} \big|_i & \frac{\partial O_2}{\partial U_2} \big|_i & \cdots & \frac{\partial O_2}{\partial U_n} \big|_i \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial O_n}{\partial U_1} \big|_i & \frac{\partial O_n}{\partial U_2} \big|_i & \cdots & \frac{\partial O_n}{\partial U_n} \big|_i \end{bmatrix} \left\{ \begin{bmatrix} U_1^{i+1} \\ U_2^{i+1} \\ \vdots \\ U_n^{i+1} \end{bmatrix} - \begin{bmatrix} U_1^i \\ U_2^i \\ \vdots \\ U_n^i \end{bmatrix} \right\}. \quad (9.3.3)$$

The matrix of coefficients in the system of equations (9.3.3) is a Jacobian matrix $[J]$ of the vector function $\mathbf{O}(\mathbf{U})$ calculated in the i -th step. If we introduce a correction vector $\Delta \mathbf{U}^i = \mathbf{U}^{i+1} - \mathbf{U}^i$, and use equations (9.3.1,3), we can write

$$\mathbf{U}^{i+1} = \mathbf{U}^i + \Delta \mathbf{U}^i = \mathbf{U}^i - [J]^{-1} \{ [K^i] \mathbf{U}^i - \mathbf{F} \}. \quad (9.3.4)$$

This expression defines the iterative procedure needed to solve the non-linear system of equations. In the i -th step we determine the correction vector from the *linear* system $[J] \Delta \mathbf{U}^i = [K^i] \mathbf{U}^i - \mathbf{F}$. Afterwards, we determine the solution for the step $i+1$ as $\mathbf{U}^{i+1} = \mathbf{U}^i + \Delta \mathbf{U}^i$. We repeat this procedure until the correction vector $\Delta \mathbf{U}^i$ is not equal to zero, or until it becomes negligible with respect to the vector $\mathbf{U}^i$.

The core application of this method is to determine the element integral forms to be used in the iterative procedure. The global system of equations is assembled and solved in the same way as in linear problems. Here, we are describing only an application of linear triangular elements, and the contribution of an individual element $\mathbf{O}^e(\mathbf{U}^e)^k$ to the vector $\mathbf{O}(\mathbf{U})^k$ used in solving non-linear static magnetic fields is

$$\mathbf{O}^e(\mathbf{A}^e)^k = \begin{bmatrix} O_1^e \\ O_2^e \\ O_3^e \end{bmatrix}^k = \frac{\mathbf{v}^{e^k}}{4S_\Delta} \begin{bmatrix} k_{11}^e & k_{12}^e & k_{13}^e \\ k_{21}^e & k_{22}^e & k_{23}^e \\ k_{31}^e & k_{32}^e & k_{33}^e \end{bmatrix} \begin{bmatrix} A_1^e \\ A_2^e \\ A_3^e \end{bmatrix}^k - \begin{bmatrix} f_1^e \\ f_2^e \\ f_3^e \end{bmatrix} = \mathbf{v}^{e^k} [\mathbf{G}^e] \mathbf{A}^{e^k} - \mathbf{f}^e. \quad (9.3.5)$$

We shall omit the notation of iteration number from now on since all expressions include quantities from the same iteration. Since the terms g_{ij} of the matrix $[G]$ and f_i of the vector $\mathbf{f}$ in equation (9.3.5) depend only on the geometry of the element, terms of the Jacobian matrix for the vector function $\mathbf{o}^e(\mathbf{A}^e)$ are of the form

$$J_{ij}^e = \mathbf{v}^e g_{ij}^e + \frac{\partial \mathbf{v}^e}{\partial A_j^e} \left(\sum_{k=1}^3 g_{ik} A_k \right). \quad (9.3.6)$$

Therefore, in calculating contribution of each element to the Jacobian matrix, we need to know the derivatives $\partial \mathbf{v} / \partial A_j$. We must bear in mind the dependence of the reluctance $\mathbf{v}$ on

the absolute value of the magnetic flux density B , and we thus determine these derivatives using

$$\frac{\partial v}{\partial A_i} = \frac{\partial v}{\partial B} \frac{\partial B_i}{\partial A_i} . \quad (9.3.7)$$

By usage of expressions (9.2.83,85), we may easily show that the final element integral form of the iterative solution of a non-linear magnetic field is

$$\left\{ v^e [g^e] + \frac{\partial v^e}{\partial B^e} \frac{1}{S_{\Delta}^e B^e} [g^e] A^e A^{eT} [g^e]^T \right\} \Delta A^e = -v^e [g^e] A^e + f^e . \quad (9.3.8)$$

The i -th step in the iterative procedure consists of:

- determination of v^e and $\partial v^e / \partial B^e$ with respect to $v(B)$. In the first step, we take v^e from the linear part of the dependence $v(B)$,
- calculation of the correction vector ΔU from the system of linear algebraic equations assembled from the element integral forms (9.3.8),
- testing of the criterion for ending the procedure. Most frequently this is the condition that the Euclidian norm of the correction vector $\Delta U^T \Delta U$ should be smaller than a prior given value ϵ .

The term $\partial v^e / \partial B^e$ is strongly dependent on the method of approximation for the function $v(B)$, and this essentially influences the convergence of the procedure. We most frequently use approximation in two regions:

- the unsaturated linear part in which dependence between B and H is linear, and we approximate it using a straight line,
- the knee and the saturated part of the $B - H$ dependence, which we approximate using polynomials, most frequently splines.

Example 9.3.1. The static magnetic field of an electromagnet

We shall present the non-linear calculation of a static magnetic field by calculating the magnetic field of an electromagnet.

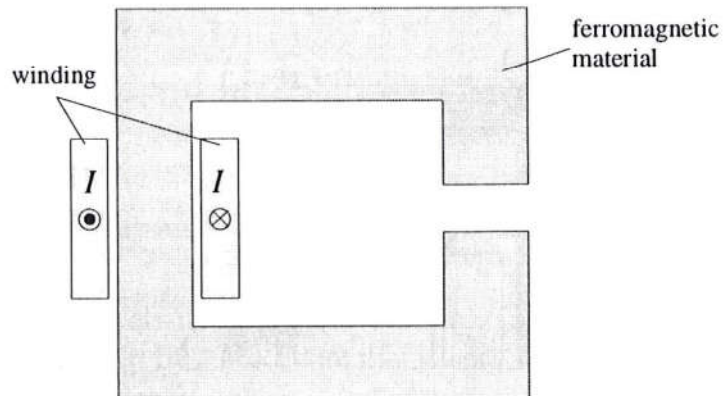


Figure 9.3.1. Electromagnet

A two-dimensional model of the electromagnet is shown in Figure 9.3.1. and the finite element mesh is shown in Figures 9.3.2,3.

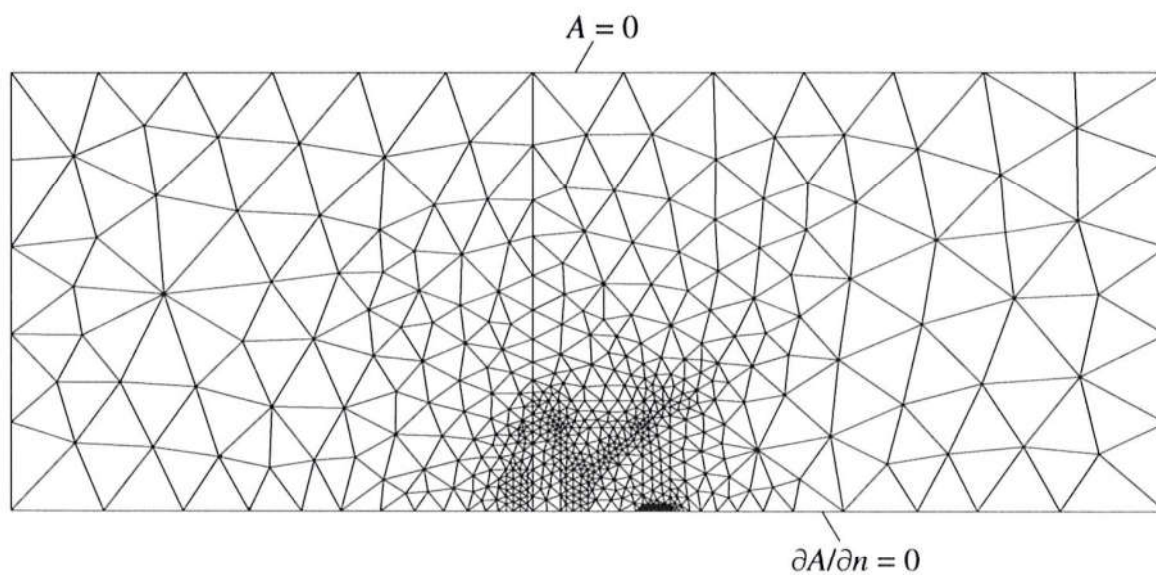


Figure 9.3.2. Boundaries and the finite element mesh for a two-dimensional calculation of static magnetic field of an electromagnet

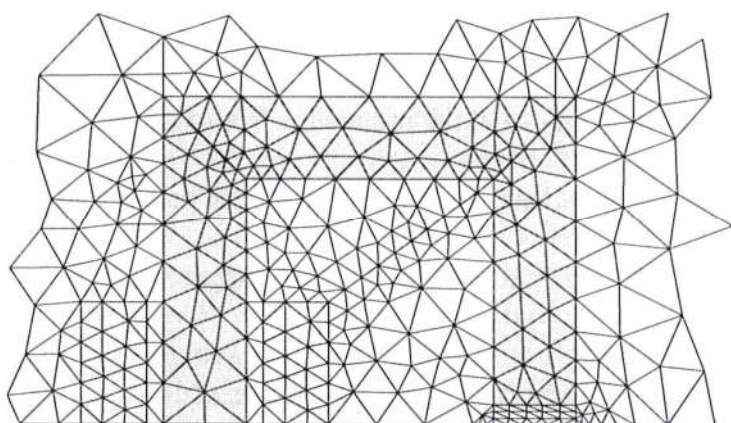


Figure 9.3.3. The mesh in the vicinity of an electromagnet in detail

On the line of symmetry, symmetry, we set the Neumann boundary condition $\partial A_z / \partial n = 0$, and on the artificial boundaries, we set the Dirichlet boundary conditions $A_z = 0$.

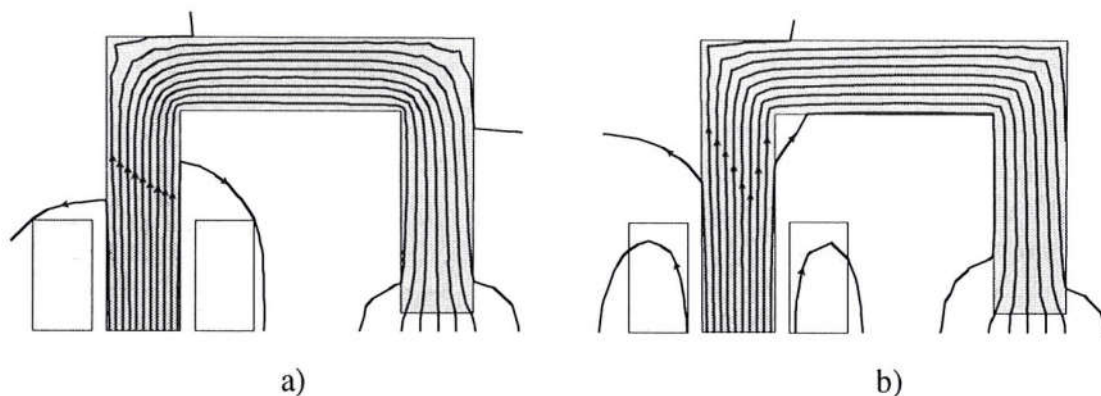


Figure 9.3.4. Results of FEM calculation for an electromagnet; a) unsaturated state, b) saturated state

The results of calculation for an unsaturated state of the magnet are shown in Figure 9.3.3.a) and the results for the saturated state of a part of the magnet are shown in Figure 9.3.3.b).

9.4. A QUASI-STATIC FIELD

9.4.1. Sinusoidal fields

We solve sinusoidal fields in a phasor domain. The magnetic vector potential in problems involving linear materials satisfies the vector diffusion equation in a phasor domain $\Delta \underline{A} - j\omega\mu\kappa \underline{A} = -\mu \underline{J}$.

For the sake of simplicity we shall analyse only two-dimensional fields. The magnetic vector potential and the current density in such a field possesses only a component in the direction of the z -axis, and the vector diffusion equation is transformed to

$$\frac{\partial^2 \underline{A}}{\partial x^2} + \frac{\partial^2 \underline{A}}{\partial y^2} - j\omega\mu\kappa \underline{A} + \mu \underline{J}_s = 0 . \quad (9.4.1)$$

The first step in the application of FEM is determination of the integral form that is equivalent to this equation. Using the weighted residual method, we obtain

$$\int_S W_i \left(\frac{\partial^2 \underline{A}}{\partial x^2} + \frac{\partial^2 \underline{A}}{\partial y^2} - j\omega\mu\kappa \underline{A} + \mu \underline{J}_s \right) dS = 0 , \quad (9.4.2)$$

and by application of Green's theorem, we can transform this equation to

$$\int_S \left(\frac{\partial W_i}{\partial x} \frac{\partial \underline{A}}{\partial x} + \frac{\partial W_i}{\partial y} \frac{\partial \underline{A}}{\partial y} \right) dS - \oint_{\Gamma} W_i \frac{\partial \underline{A}}{\partial n} dl - j\omega\mu\kappa \int_S W_i \underline{A} dS = -\mu \int_S W_i \underline{J}_s dS . \quad (9.4.3)$$

After introducing the Dirichlet and Neumann boundary conditions into this expression, an application of Galerkin's method results in

$$\begin{aligned} \sum_j \underline{A}_j \int_S \left(\frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \right) dS - j\omega \sum_j \mu\kappa \underline{A}_j \int_S N_j dS = \\ = - \sum_j \mu \underline{J}_{sj} \int_S N_j dS + \sum_j \frac{\partial \underline{A}_j}{\partial n} \int_{\Gamma_n} N_j d\Gamma . \end{aligned} \quad (9.4.4)$$

We shall apply the linear triangular elements introduced in Section 9.2.3. The element integral form is *complex* and we determine it by a procedure that is very similar to that in Section 9.2.5. The final element integral form is

$$W^e = \left\{ \frac{1}{4S_\Delta^e} \begin{bmatrix} k_{11}^e & k_{12}^e & k_{13}^e \\ k_{21}^e & k_{22}^e & k_{23}^e \\ k_{31}^e & k_{32}^e & k_{33}^e \end{bmatrix} + j\omega\mu^e\kappa^e \frac{S_\Delta^e}{12} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \right\} \begin{bmatrix} \Delta_i^e \\ \Delta_j^e \\ \Delta_k^e \end{bmatrix} + \frac{\mu_e J_s^e S_\Delta^e}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \quad (9.4.5)$$

The real terms of the element integral form matrix are equal to those for a static field. Other steps in application of FEM are the same as those in Unit 9.2, but we must take into account that the element integral forms are complex.

Example 9.4.1.1. The sinusoidal quasi-static field of a busbar

We have defined the geometry of a busbar in Section 9.2.11. and the finite element mesh is thus equal to that from Example 9.2.11.2. In solving the sinusoidal quasi-static field in a shielded busbar, we choose a case where it carries a symmetrical three-phase current system. The conductor on the left carries a current $I\cos(\omega t)$, the central conductor carries a current $I\cos(\omega t - 2\pi/3)$, and the conductors on the right carry a current $I\cos(\omega t - 4\pi/3)$ at a frequency of 50 Hz.

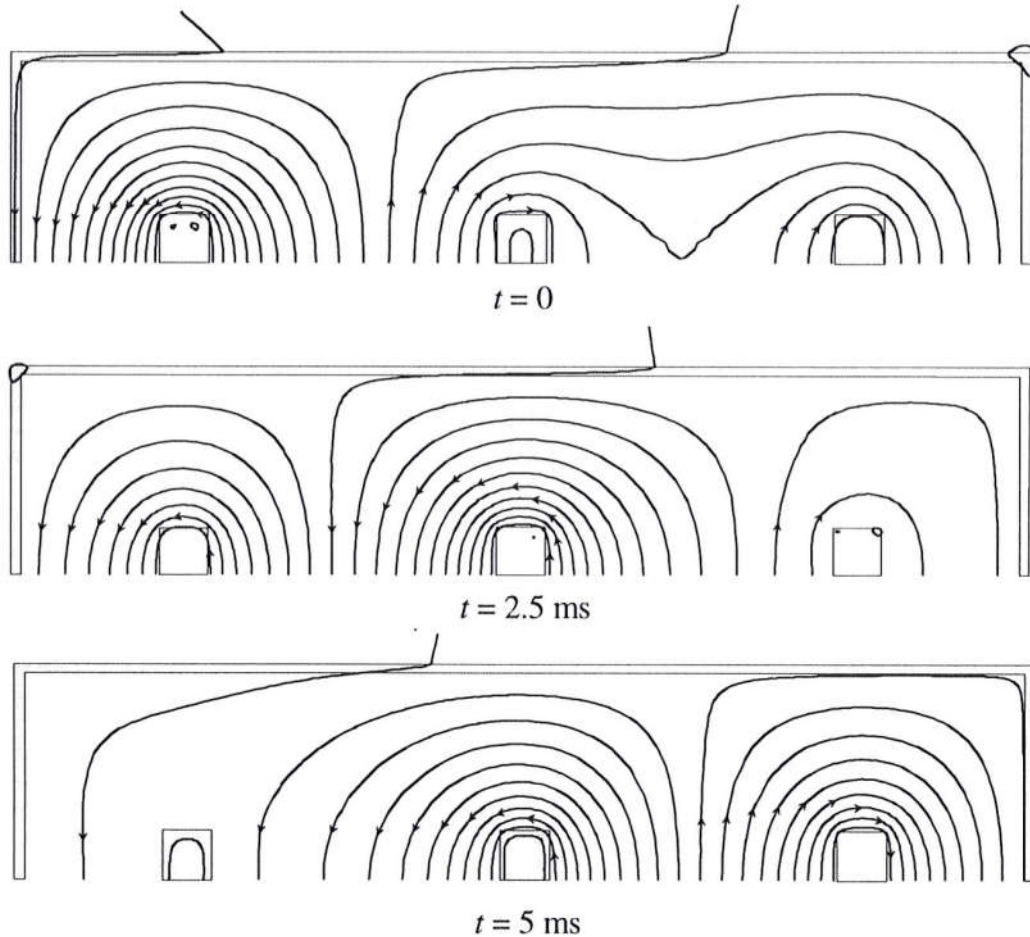


Figure 9.4.1. Field lines of magnetic field strength of a sinusoidal quasi-static field in a busbar at various moments in time

The magnetic field visualised by the field lines at various moments in time is shown in Figure 9.4.1. The skin effect at this frequency for the chosen geometry of the busbar is illustrated by the distribution of magnitude of current density phasor along the symmetry line of the central conductor.

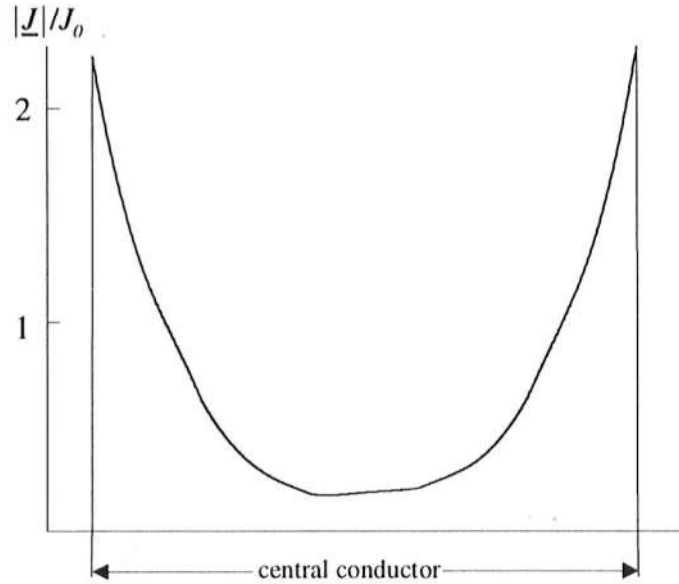


Figure 9.4.2. The distribution of the normalized magnitude of the current density phasor $\underline{J}$. $J_0 = I/S$ here, and S is the area of the conductor.

9.4.2. Non-sinusoidal fields

The magnetic vector potential in a linear two-dimensional problem involving fields that are non-sinusoidal in time satisfies the equation

$$\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} - \mu \kappa \frac{\partial A}{\partial t} + \mu J_s = 0. \quad (9.4.6)$$

Application of Galerkin's method in a spatial domain (x,y) , analogous to the previous section, results in the element integral form

$$W^e = \frac{1}{4S_\Delta^e} \begin{bmatrix} k_{11}^e & k_{12}^e & k_{13}^e \\ k_{21}^e & k_{22}^e & k_{23}^e \\ k_{31}^e & k_{32}^e & k_{33}^e \end{bmatrix} \begin{bmatrix} A_i^e \\ A_j^e \\ A_k^e \end{bmatrix} + \mu^e \kappa^e \frac{S_\Delta^e}{12} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} A_i^e \\ A_j^e \\ A_k^e \end{bmatrix} + \frac{\mu^e J_s^e S_\Delta^e}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \quad (9.4.7)$$

The system of equations for unknown coefficients after assembly takes the form

$$[K]A + [S] \frac{\partial A}{\partial t} + F = 0. \quad (9.4.8)$$

We solve this equation by application of the weighted residual method in a time domain T

$$\int_T \mathbf{w}_t^T \left\{ [K] \mathbf{A} + [S] \frac{\partial \mathbf{A}}{\partial t} + \mathbf{F} \right\} dt = 0 . \quad (9.4.9)$$

Here, $\mathbf{w}_t$ are the weighting functions in a time domain. Next, we use the usual procedure for approximation in a time domain $\mathbf{A} \approx \sum N_i(t) \mathbf{A}_i$.

$\mathbf{A}_i$ is the vector of nodal variables at a moment i . We take the same interpolating function $N_i(t)$ for each component of vector $\mathbf{A}_i$, and therefore it is scalar. The polynomials of lowest order that we may choose for N_i are of first order since only the first derivatives with respect to time are included in equation (9.4.9). Let us now consider a typical time element of length Δt connecting the moments $i + 1$ and i , as shown in Figure 9.4.3.

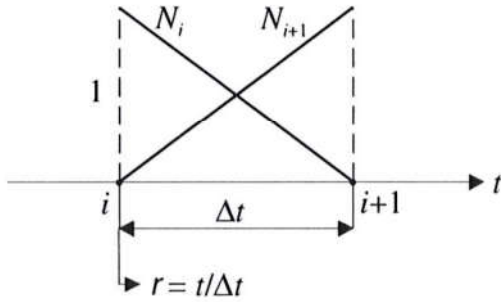


Figure 9.4.3. Linear interpolating functions on a time element

We prescribe the linear interpolating functions as being

$$\left. \begin{aligned} N_i &= 1 - r & ; & \quad \frac{\partial N_i}{\partial t} = -\frac{1}{\Delta t} \\ N_{i+1} &= r & ; & \quad \frac{\partial N_{i+1}}{\partial t} = \frac{1}{\Delta t} \end{aligned} \right\} \quad 0 \leq r \leq 1 \quad ; \quad r = \frac{t}{\Delta t} , \quad (9.4.10)$$

and they satisfy all the conditions set in Section 9.2.3. If we suppose that the time domain is identical to one time element, equation (9.4.9) becomes

$$\int_0^1 \mathbf{w}_t^T \left\{ [K] (N_i \mathbf{A}_i + N_{i+1} \mathbf{A}_{i+1}) + [S] \left(\frac{\partial N_i}{\partial t} \mathbf{A}_i + \frac{\partial N_{i+1}}{\partial t} \mathbf{A}_{i+1} \right) + \mathbf{F} \right\} dr = 0 . \quad (9.4.11)$$

This expression describes the initial value problem since we have assumed that the nodal variable vector at a moment i , $\mathbf{A}_i$ is known, and we want to determine the nodal variable vector $\mathbf{A}_{i+1}$ at a moment $i + 1$. Substitution of the interpolating functions (9.4.10) into expression (9.4.11) results in an expression relating $\mathbf{A}_i$ and $\mathbf{A}_{i+1}$

$$\left([K] \int_0^1 W_t r dr + [S] \int_0^1 W_t \frac{dr}{\Delta t} \right) \mathbf{A}_{i+1} + \left([K] \int_0^1 W_t (1-r) dr - [S] \int_0^1 W_t \frac{dr}{\Delta t} \right) \mathbf{A}_i + \int_0^1 W_t \mathbf{F} dr = 0 .$$

We may introduce various weighting functions W_t into this expression. In general, we can write

$$\left(\frac{1}{\Delta t} [S] + \theta [K] \right) \mathbf{A}_{i+1} + \left(-\frac{1}{\Delta t} [S] + (1 - \theta) [K] \right) \mathbf{A}_i + \theta \mathbf{F}_{i+1} + (1 - \theta) \mathbf{F}_i = 0 \quad (9.4.12)$$

Here

$$\theta = \frac{\int_0^1 w_i r dr}{\int_0^1 w_i dr} \quad (9.4.13)$$

If we want to apply Galerkin's method in a time domain as well, we then choose the weighting function $w_i = r$. In such a case $\theta = 2/3$.

Calculation of the time dependence of the magnetic vector potential is iterative. We determine the vector $\mathbf{A}_{i+1}$, which is the initial value for the next step, from the known value $\mathbf{A}_i$ at a moment i using expression (9.4.12).

Example 9.4.2.1. Switching on a coaxial cable

A coaxial cable and its finite element mesh are shown in Figure 9.4.4. At a moment in time $t = 0$, we switch on the current $i(t) = I \sin(\omega t)$ in the inner conductor of the cable.

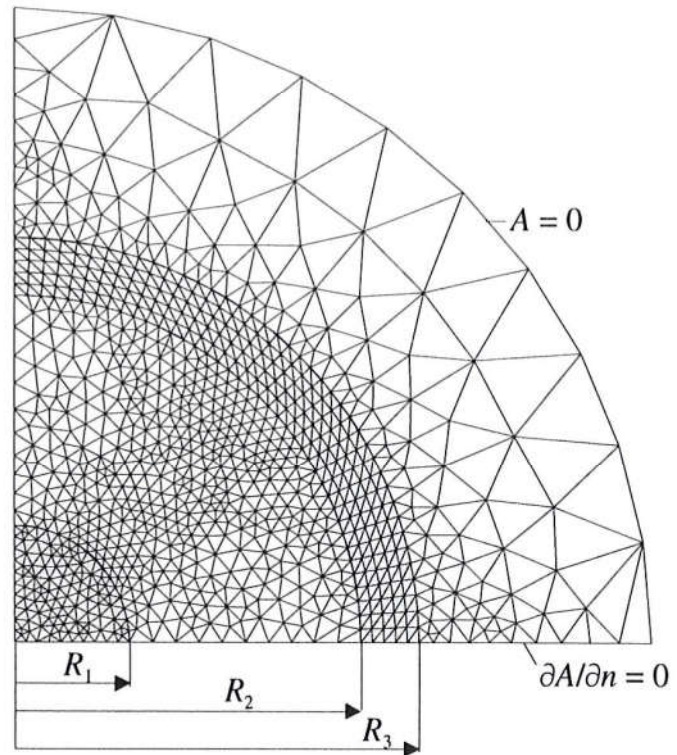


Figure 9.4.4. A finite element mesh and geometry of a coaxial cable

The distribution of current density in the inner conductor and in the shield are shown in Figure 9.4.5.

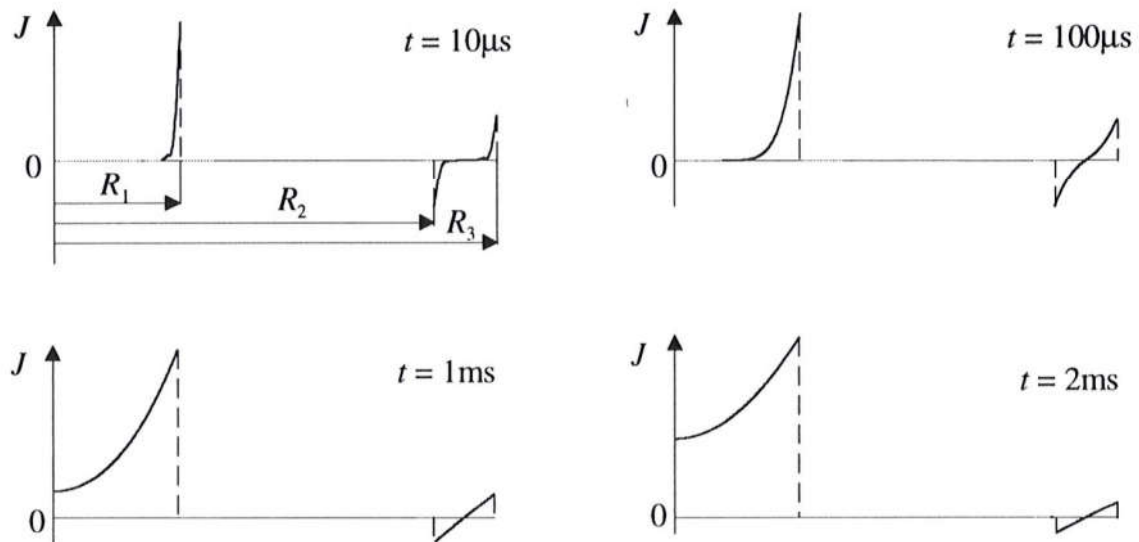


Figure 9.4.5. The distribution of current density in the inner conductor and in the shield of the coaxial cable during the switching on of a current $i(t) = I \sin(\omega t)$ at a frequency $f = 50$ Hz.

PROBLEMS

- 9.1. Write the routine for the calculation of equipotential lines for a two-dimensional static field!
- 9.2. Derive the algorithm for the calculation of field lines in a two-dimensional static electric field!
Hint: use Euler's method for the numerical solving of ordinary differential equations.
- 9.3. Prove expression (9.3.9) for the calculation of the element integral form in the solution of a non-linear static magnetic field!
- 9.4. Prove expression (9.4.5) for the calculation of the element integral form in the solution of a quasi-static electromagnetic field!
- 9.5. Write the routine for the time-stepped solving of a non-sinusoidal quasi-static electromagnetic field!
- 9.6. Prove that choice of weighting functions $w_i = \delta(i)$ in expression (9.4.12) results in the well-known Euler's method for solving ordinary differential equations!

10. THE METHOD OF MOMENTS

In this chapter, we shall describe methods for transforming linear integral equations into a system of linear algebraic equations. We may give this methods its common name, the *method of moments* (in continuation MOM). We shall focus our attention only on problems involving linear materials. In such problems, MOM is most effective and reduces problems to that of determining unknown sources on the boundaries between materials. Therefore, we frequently refer to MOM as the boundary element method.

First of all, we shall define MOM and introduce the most frequently used procedures for reducing integral equations to a system of algebraic equations. Afterwards, we shall describe the application of MOM to the solution of three-dimensional static electric field problems. Scattering of electromagnetic waves and the solution of scattering problems using MOM will be outlined as well. At the end of this chapter, we shall describe thin-wire structures, introduce Pocklington's equation and finally show the application of MOM to the solution of radiation problems.

10.1. DEFINITION OF MOM

We can write an integral equation in symbolical form as

$$L(f) = g, \quad (10.1.1)$$

where L is an integral operator, g the prescribed function and f the unknown function we are going to determine. The first step in the reduction of expression (10.1.1) to a system of linear algebraic equations is the expansion of f in terms of prescribed functions f_i , $i = 1, \dots, N$ in the domain of the function f

$$f = \sum_{i=1}^N \alpha_i f_i, \quad (10.1.2)$$

where α_i are the unknown coefficients of the expansion. The functions f_i are called *expansion functions* or *basis functions*. To find the exact solution in most cases, we have to find the sum of the infinite series from (10.1.2). So we analyse approximate solutions which lead to a sum of a finite number of terms from (10.1.2). Substitution of equation (10.1.2) into (10.1.1) and utilization of the linearity of operator L results in

$$\sum_{i=1}^N \alpha_i L(f_i) = g. \quad (10.1.3)$$

We use the method of weighted residuals to obtain a system of linear algebraic equations for unknown coefficients α_i . Equation (10.1.3) is valid at every point of the Ω domain of

function f_i . Therefore, for any set of functions v_i , $i = 1, \dots, N$, we get that

$$\int_{\Omega} v_j \left\{ \sum_{i=1}^N \alpha_i L(f_i) - g \right\} d\Omega = 0 \quad ; \quad j = 1, \dots, N. \quad (10.1.4)$$

In seeking an approximate solution using MOM, we firstly choose the expansion functions f_i and N , and afterwards set the *weighting functions* v_i , $i = 1, \dots, N$ in the domain of the expansion functions Ω . Then expression (10.1.4) becomes the weighted integral of the residual due to the approximation. This is a system of linear algebraic equations for unknown coefficients α_i . The matrix form of this system is

$$\begin{bmatrix} \int_{\Omega} v_1 L(f_1) d\Omega & \int_{\Omega} v_1 L(f_2) d\Omega & \dots & \int_{\Omega} v_1 L(f_N) d\Omega \\ \int_{\Omega} v_2 L(f_1) d\Omega & \int_{\Omega} v_2 L(f_2) d\Omega & \dots & \int_{\Omega} v_2 L(f_N) d\Omega \\ \vdots & \vdots & \ddots & \vdots \\ \int_{\Omega} v_N L(f_1) d\Omega & \int_{\Omega} v_N L(f_2) d\Omega & \dots & \int_{\Omega} v_N L(f_N) d\Omega \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{bmatrix} = \begin{bmatrix} \int_{\Omega} v_1 g d\Omega \\ \int_{\Omega} v_2 g d\Omega \\ \vdots \\ \int_{\Omega} v_N g d\Omega \end{bmatrix}. \quad (10.1.5)$$

When the matrix of this system is non-singular, we obtain a solution for α_i by the method for solving a linear algebraic equation system described in Chapter 9.

The matrix of the system resulting from the application of MOM is *full*, as opposed to the sparse matrix resulting from the application of FEM. But the dimensions of the MOM matrix compared to the FEM matrix for the same problem are substantially smaller because the integral operators connect field sources positioned only on the boundaries of the different materials. Conversely, differential operators correlate the whole space occupied by the field. From a number of possible choices of weighting functions, most frequently we use two:

1. Galerkin's method where we choose $v_i = f_i$. The main difficulty in utilization of this method is a complex and often long-lasting computation of the matrix elements by multiple numerical integrations.
2. Point matching. In this method, we choose the Dirac δ -function for the weighting function. Due to the property of the Dirac δ -function

$$\int_{\Omega} \delta(\mathbf{r} - \mathbf{r}_0) f(\mathbf{r}) d\Omega = f(\mathbf{r}_0), \quad (10.1.6)$$

we lower the number of multiple numerical integrations by one. This choice enforces an exact solution at discrete points of the Ω domain determined beforehand. This is the origin of the title of this method. The accuracy of the solution depends upon the number and position of these points.

10.2. THREE-DIMENSIONAL STATIC ELECTRIC FIELDS

10.2.1. An integral formulation of the problem

We shall derive integral equations governing a problem where a conductive region has a known potential and two dielectric regions have different permittivities, as shown in Figure 10.2.1.

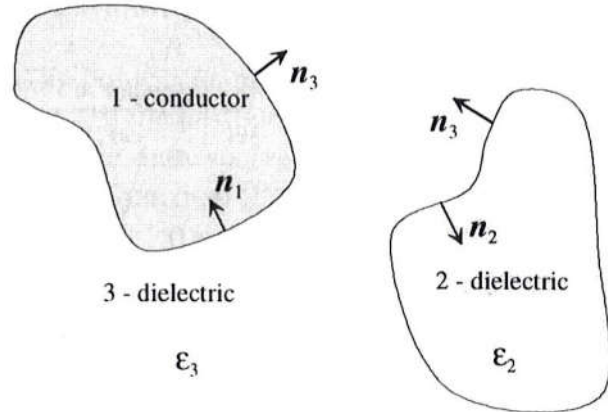


Figure 10.2.1. Problem of a static electric field

Here, $\mathbf{n}_i$ are the unit vectors normal to the boundary surface S_i of the volume V_i directed into the i -th region. The potential φ_i in the i -th region satisfies Laplace's equation $\Delta\varphi_i = 0$, and the boundary conditions are

$$\begin{aligned} \varphi_1(\mathbf{r}) = \varphi_3(\mathbf{r}) = \varphi_{10} ; \mathbf{r} \in S_1, \quad \varphi_3(\mathbf{r}) = \varphi_2(\mathbf{r}) ; \mathbf{r} \in S_2, \\ \epsilon_2 \frac{\partial \varphi_2}{\partial \mathbf{n}_2} = -\epsilon_3 \frac{\partial \varphi_3}{\partial \mathbf{n}_3} ; \mathbf{r} \in S_2, \quad \frac{\partial \varphi_1}{\partial \mathbf{n}_1} = 0 ; \mathbf{r} \in S_1. \end{aligned} \quad (10.2.1a - d)$$

In order to solve Laplace's equation, we introduce Green's function

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi} \frac{1}{|\mathbf{r} - \mathbf{r}'|}, \quad (10.2.2)$$

which satisfies $-\Delta G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}')$. Here δ is a Dirac δ -function. With $\mathbf{r}'$ we have denoted points at which the field sources are positioned. If we multiply $\Delta\varphi_i = 0$ by G and $-\Delta G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}')$ by φ_i and afterwards add these equations together, we obtain

$$G(\mathbf{r}, \mathbf{r}') \Delta\varphi_i(\mathbf{r}) - \varphi_i \Delta G(\mathbf{r}, \mathbf{r}') = \varphi_i(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}'). \quad (10.2.3)$$

Integration of this equation over the volume of the i -th region V_i followed by application of Green's theorem results in

$$\int_{S_i} \left\{ -G(\mathbf{r}, \mathbf{r}') \frac{\partial \varphi_i(\mathbf{r}')}{\partial \mathbf{n}_i} + \varphi_i(\mathbf{r}) \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial \mathbf{n}_i} \right\} dS_i = \begin{cases} \varphi_i(\mathbf{r}) & ; \mathbf{r} \in V_i, \\ 0 & ; \mathbf{r} \notin V_i. \end{cases} \quad (10.2.4)$$

If we apply this expression to all regions of the problem, add the resulting equations and

introduce the boundary conditions, we get

$$\int_{S_1} -G(\mathbf{r}, \mathbf{r}') \frac{\partial \varphi_3(\mathbf{r}')}{\partial \mathbf{n}_1} dS_1 - \left(1 - \frac{\epsilon_3}{\epsilon_2}\right) \int_{S_2} G(\mathbf{r}, \mathbf{r}') \frac{\partial \varphi_3(\mathbf{r}')}{\partial \mathbf{n}_3} dS_2 = \begin{cases} \varphi_1(\mathbf{r}) ; \mathbf{r} \in V_1 , \\ \varphi_2(\mathbf{r}) ; \mathbf{r} \in V_2 , \\ \varphi_3(\mathbf{r}) ; \mathbf{r} \in V_3 . \end{cases} \quad (10.2.5)$$

The term $\epsilon_0(\partial \varphi_3)/(\partial \mathbf{n}_1)$ is equal to the total surface charge σ_1 on the conductor-dielectric boundary, and the term $\epsilon_0(1 - \epsilon_3/\epsilon_2)(\partial \varphi_3)/(\partial \mathbf{n}_3)$ is equal to the polarization surface charge σ_2 on the boundary between the two dielectrics. We derive the second integral equation for quantities involved in (10.2.5) from the boundary condition (10.2.1c). This is

$$\begin{aligned} & -\frac{\epsilon_2 - \epsilon_3}{2\epsilon_2} \frac{\partial \varphi(\mathbf{r}_P)}{\partial \mathbf{n}_3} - \left(1 - \frac{\epsilon_3}{\epsilon_2}\right) \int_{S_2} \frac{\partial \varphi(\mathbf{r}')}{\partial \mathbf{n}_3} \frac{\partial G(\mathbf{r}_P, \mathbf{r}')}{\partial \mathbf{n}_3} dS_2 + \\ & + \int_{S_1} \frac{\partial \varphi_3(\mathbf{r}')}{\partial \mathbf{n}_1} \frac{\partial G(\mathbf{r}_P, \mathbf{r}')}{\partial \mathbf{n}_3} dS_1 = 0 . \end{aligned} \quad (10.2.6)$$

Here

$$\mathbf{r}_P \in S_2 \quad \text{and} \quad \frac{\partial G(\mathbf{r}_P, \mathbf{r}')}{\partial \mathbf{n}_3} = \nabla G(\mathbf{r}_P, \mathbf{r}') \cdot \mathbf{n}_3 . \quad (10.2.7)$$

If we apply equation (10.2.5) to the surface S_1 with known potential, and equation (10.2.6) to the surface S_2 , we get the integral equations

$$\begin{aligned} \varphi_{01}(\mathbf{r}) - \sum_{i=1}^2 \int_{S_i} \sigma_i(\mathbf{r}') G(\mathbf{r}, \mathbf{r}') dS_i &= 0 \quad ; \quad \mathbf{r} \in S_1 , \\ \sigma_2(\mathbf{r}) - 2 \frac{\epsilon_3 - \epsilon_2}{\epsilon_3 + \epsilon_2} \sum_{i=1}^2 \int_{S_i} \sigma_i(\mathbf{r}') \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial \mathbf{n}_3} dS_i &= 0 \quad ; \quad \mathbf{r} \in S_2 . \end{aligned} \quad (10.2.8a,b)$$

We may apply expressions (10.2.8a,b) to problems with more conductive and dielectric regions. Then S_1 is the union of all conductor-dielectric surfaces, and S_2 is the union of all dielectric-dielectric surfaces. When applying (10.2.8b), we have to use a pair of permittivities corresponding to each individual boundary.

10.2.2. Application of MOM

The first step in the application of MOM is the choice of expansion function. We most frequently divide the domain of the unknown field source distribution into sub-domains and define an approximation on each individual sub-domain. We can also define an approximation on the entire domain. We shall describe here the simplest procedure for approximating on a sub-domain. The domains of the unknown surface charge are in fact

the boundary surfaces S_1 and S_2 . We divide them into sub-domains ΔS_{ij} , where $i = 1, 2$ and $j = 1, \dots, N$. Afterwards we choose constants on the sub-domain for the expansion functions

$$f_{ij}(\mathbf{r}) = \begin{cases} 1 & ; \quad \mathbf{r} \in \Delta S_{ij} \\ 0 & ; \quad \mathbf{r} \notin \Delta S_{ij} \end{cases} ; \quad i=1,2 \quad ; \quad j=1, \dots, N_i . \quad (10.2.9)$$

Now we can approximate the unknown distribution of surface charge using constants on the sub-domains

$$\sigma_i(\mathbf{r}) = \sum_{j=1}^{N_i} \sigma_{ij} f_{ij} \quad ; \quad i=1,2 \quad ; \quad j=1, \dots, N_i . \quad (10.2.10)$$

We determine the unknown coefficients σ_{ij} by point matching. The weighting functions are $v_{ij}(\mathbf{r}) = \delta(\mathbf{r}_{C,ij})$, $i = 1, 2$ and $j = 1, \dots, N_i$. Here $\mathbf{r}_{C,ij}$ is the position vector of the central point of the j -th sub-domain belonging to the i -th surface. Introducing expression (10.2.13) and the weighting functions into (10.2.10) results in a system of linear algebraic equations for unknown coefficients σ_{ij} .

$$\sum_{i=1}^2 \sum_{j=1}^{N_i} \sigma_{ij} \int_{\Delta S_{ij}} G(\mathbf{r}_{C,1k}, \mathbf{r}') dS = \varphi_{01}(\mathbf{r}_{C,1k}) \quad ; \quad k=1, \dots, N_1 , \quad (10.2.11a,b)$$

$$\sigma_2(\mathbf{r}_{C,2l}) - 2 \frac{\epsilon_3 - \epsilon_2}{\epsilon_3 + \epsilon_2} \sum_{i=1}^2 \sum_{j=1}^{N_i} \sigma_{ij} \int_{\Delta S_{ij}} \frac{\partial G(\mathbf{r}_{C,2l}, \mathbf{r}')}{\partial n_3} dS = 0 \quad ; \quad l=1, \dots, N_2 .$$

Example 10.2.1. A plate condenser

The geometry of the problem is shown in Figure 10.2.2. We need to determine an approximation of the unknown surface charge distribution on the plates. The potential difference between the plates is φ_0 .

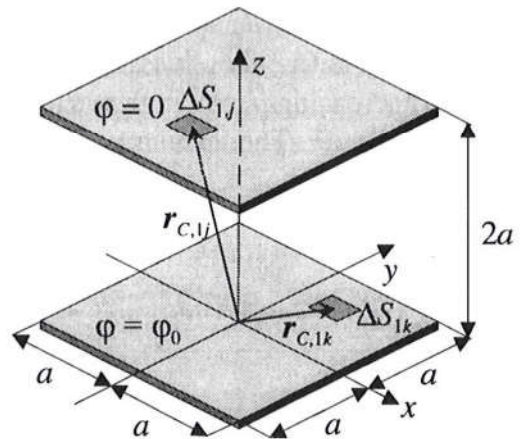


Figure 10.2.2. A plate condenser

The boundary surfaces in this problem are the square S_{11} ($2a \times 2a$) in the plane $z = 0$ and the square S_{12} ($2a \times 2a$) in the plane $z = 2a$. In order to apply MOM in the previously described way, we introduce $S_1 = S_{11} + S_{12}$.

We divide the boundary surfaces into rectangles on which we prescribe a uniform distribution of surface charge. In this case, we can write (10.2.11a) in matrix form as $[P][\sigma_1] = [\varphi_0]$. The terms P_{kj} of the matrix $[P]$ are the contributions of the rectangle ΔS_{1j} to the potential at a point $\mathbf{r}_{C,1k}$. The terms of the vector $[\sigma_1]$ are the unknown coefficients of the approximation of surface charge distribution at the points $\mathbf{r}_{C,1k}$, and the terms of the vector $[\varphi_0]$ are the known potentials at these points.

We derived the expression for the potential of a uniformly charged rectangle in Section 4.7.3, and we shall use it here to calculate the terms of $[P]$. If we divide both plates into an equal number of rectangles $2N$ in the x - and y -directions, then the number of unknowns is $4N^2$.

The method used for choosing the position of the points $\mathbf{r}_{C,1k}$, or in other words, the way of dividing the plate into rectangles, strongly influences the accuracy of our calculation. The rule of thumb is to prescribe a dense division in areas where we expect rapid changes of surface charge distribution. As the surface charge is singular at the edges we prescribe the algorithm for non-uniform division into rectangles $\Delta S_{1mn} : (x, y, z_0) ; x d_m \leq x \leq x g_m ; y d_n \leq y \leq y g_n ; m = 1, \dots, 2N ; n = 1, \dots, 2N$. The coordinates $x d_m$ and $x g_m$ are calculated by

$$x d_m = \begin{cases} a \left[-1 + \left(\frac{i}{N} \right)^p \right] & ; \quad i = 0, \dots, N \quad ; \quad m = i + 1 , \\ a \left[1 - \left(1 - \frac{i}{N} \right)^p \right] & ; \quad i = 1, \dots, N-1 \quad ; \quad m = N + i + 1 , \end{cases} \quad (10.2.12)$$

$$x g_m = \begin{cases} a \left[-1 + \left(\frac{i}{N} \right)^p \right] & ; \quad i = 1, \dots, N \quad ; \quad m = i , \\ a \left[1 - \left(1 - \frac{i}{N} \right)^p \right] & ; \quad i = 1, \dots, N \quad ; \quad m = N + i , \end{cases}$$

The coordinates $y d_m$ and $y g_m$ are calculated using the same algorithm. Changing the parameter p influences this division. For $p = 1$, all rectangles are of the same dimensions, and by increasing p , rectangles near the edges become smaller while ones inside the plate become greater. The influence of p on accuracy is shown in Figure 10.2.3.

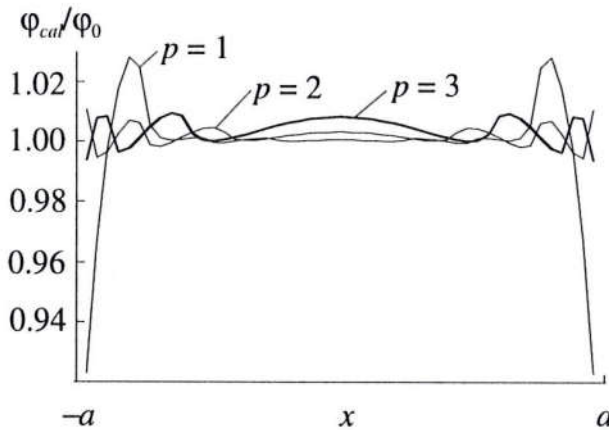


Figure 10.2.3. The ratio between calculated φ_{cal} and prescribed potential φ_0 on the plate at $z = 0$ along the x -axis

We may see that, by good division, accuracy can be highly improved without increasing the dimensions of the system and the time needed for calculation. The normalized approximations of charge density distribution on the plate in the plane $z = 0$ for $p = 1$ and $p = 3$ are shown in Figure 10.2.4.

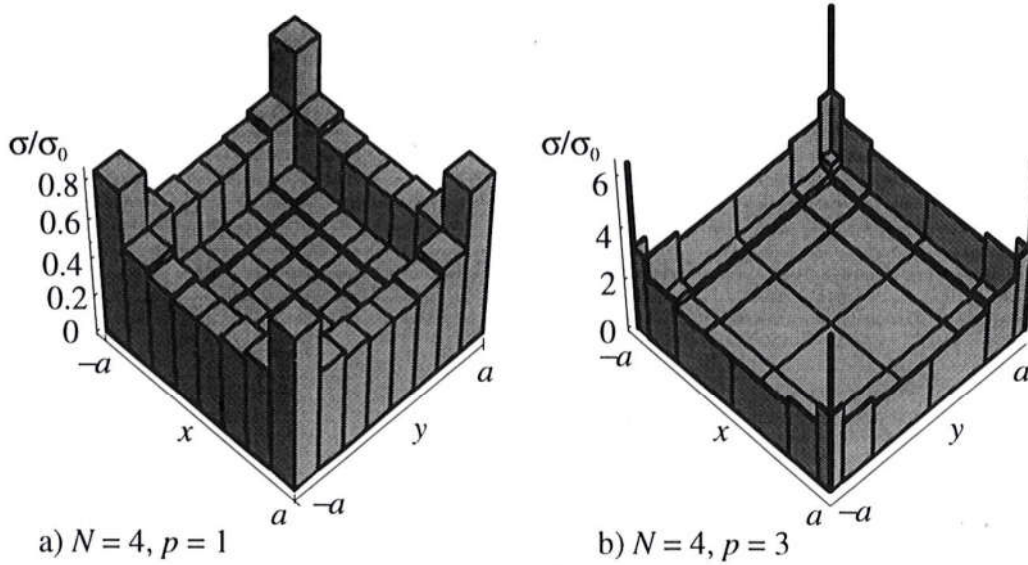


Figure 10.2.4. Normalized approximations of charge density distribution on the plate for $N = 4$, $\sigma_0 = \epsilon_0 \phi_0 / (2a)$; a) uniform division ($p = 1$), b) non-uniform division ($p = 3$)

10.3. SCATTERING OF ELECTROMAGNETIC WAVES

Problems where a body is immersed in a known external field are called scattering problems. The sources of the external field are excluded from the problem.

To avoid unnecessary complications, we shall focus our attention on two-dimensional sinusoidal problems. In such problems, we prescribe that the field quantities are independent of the z -coordinate in a Cartesian coordinate system. Therefore, we can write the field vectors as $\underline{E} = \underline{E}_x(x,y)\underline{a}_x + \underline{E}_y(x,y)\underline{a}_y + \underline{E}_z(x,y)\underline{a}_z$, $\underline{H} = \underline{H}_x(x,y)\underline{a}_x + \underline{H}_y(x,y)\underline{a}_y + \underline{H}_z(x,y)\underline{a}_z$ and $\underline{J} = \underline{J}_x(x,y)\underline{a}_x + \underline{J}_y(x,y)\underline{a}_y + \underline{J}_z(x,y)\underline{a}_z$. The two main Maxwell's equations $\nabla \times \underline{E} = -j\omega\mu\underline{H}$ and $\nabla \times \underline{H} = \underline{J} + j\omega\epsilon\underline{E}$ are then transformed to

$$\begin{aligned} \frac{\partial \underline{E}_z}{\partial y} \underline{a}_x - \frac{\partial \underline{E}_z}{\partial x} \underline{a}_y + \left(\frac{\partial \underline{E}_y}{\partial x} - \frac{\partial \underline{E}_x}{\partial y} \right) \underline{a}_z &= -j\omega\mu (\underline{H}_x \underline{a}_x + \underline{H}_y \underline{a}_y + \underline{H}_z \underline{a}_z), \\ \frac{\partial \underline{H}_z}{\partial y} \underline{a}_x - \frac{\partial \underline{H}_z}{\partial x} \underline{a}_y + \left(\frac{\partial \underline{H}_y}{\partial x} - \frac{\partial \underline{H}_x}{\partial y} \right) \underline{a}_z &= (j\omega\epsilon \underline{E}_x + \underline{J}_x) \underline{a}_x + (j\omega\epsilon \underline{E}_y + \underline{J}_y) \underline{a}_y + \\ &+ (j\omega\epsilon \underline{E}_z + \underline{J}_z) \underline{a}_z. \end{aligned} \quad (10.3.1a,b)$$

These equations show that the mutually connected quantities are $\underline{J}_z$, $\underline{E}_z$, $\underline{H}_x$ and $\underline{H}_y$ forming the transverse magnetic part (in continuation TM) with respect to the z axis of the

electromagnetic field. The quantities $\underline{H}_z, \underline{J}_x, \underline{J}_y, \underline{E}_x, \underline{E}_y$ form the transverse electric part (in continuation TE) of the electromagnetic field.

We can resolve every two-dimensional electromagnetic field into a sum of the TM and TE parts. Since the TM and TE part are independent, we may analyse them separately and reduce the vector problem to that of a scalar one.

10.3.1. Integral form of the equations

10.3.1.1. TM field

In a two-dimensional TM field, we get that $\underline{E} = \underline{E}_z(x,y)\underline{a}_z$, $\underline{H} = \underline{H}_x(x,y)\underline{a}_x + \underline{H}_y(x,y)\underline{a}_y$ and $\underline{J} = \underline{J}_z(x,y)\underline{a}_z$ and thus we reduce the system (10.3.1a,b) to

$$\begin{aligned} \frac{\partial \underline{E}_z}{\partial y} \underline{a}_x - \frac{\partial \underline{E}_z}{\partial x} \underline{a}_y &= -j\omega\mu(\underline{H}_x \underline{a}_x + \underline{H}_y \underline{a}_y) , \\ \left(\frac{\partial \underline{H}_y}{\partial x} - \frac{\partial \underline{H}_x}{\partial y} \right) \underline{a}_z &= (j\omega \underline{E}_z + \underline{J}_z) \underline{a}_z . \end{aligned} \quad (10.3.2a,b)$$

From these equations, we derive the scalar equation $\Delta \underline{E}_z - k^2 \underline{E}_z = j\omega\mu \underline{J}_z$ where $k^2 = \omega^2\epsilon\mu = 2\pi/\lambda$ is the wave number. This equation possesses the form of a two-dimensional Helmholtz equation.

In Section 2.4.2, we showed that, in order to find its integral solution, we have to determine Green's function. Here, Green's function represents the field of a unit line current. Due to axial symmetry, we solve this problem in a cylindrical coordinate system. The electric field strength phasor in the space around the line current satisfies a Bessel's differential equation of zero order

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{d \underline{E}_z}{dr} \right) + k^2 \underline{E}_z = 0 . \quad (10.3.3)$$

The solutions of this equation are zero-order Bessel functions of the first and second kind and zero-order Hankel functions of the first and second kind. The solutions that satisfy the condition of vanishing at large distances from the source for complex k are zero-order Hankel functions of the second kind

$$H_0^{(2)}(kr) = J_0(kr) - jN_0(kr) , \quad (10.3.4)$$

which represent outward travelling waves. We assume a solution in the form

$$\underline{E}_z = -jI\omega\mu C H_0^{(2)}(kr) . \quad (10.3.5)$$

The constant C is determined from the condition

$$\lim_{r \rightarrow 0} \oint \underline{H} r d\alpha = I, \quad (10.3.6)$$

since the line current is equal to I , and, due to axial symmetry, $\underline{H}$ is independent of the α coordinate. We determine the phasor of the magnetic field strength $\underline{H}$ from equation $\nabla \times \underline{E} = -j\omega\mu\underline{H}$ in a cylindrical coordinate system as being

$$\underline{H} = -\frac{1}{-j\omega\mu} \frac{\partial \underline{E}_z}{\partial r} = -\frac{jI\omega\mu C}{j\omega\mu} \frac{\partial}{\partial r} H_0^{(2)}(kr). \quad (10.3.7)$$

The small argument approximation of the zero-order Hankel function of the second kind is

$$H_0^{(2)}(kr) = 1 - j\frac{2}{\pi} \left(\ln \frac{kr}{2} + \gamma \right), \quad (10.3.8)$$

where $\gamma = 0.5772\dots$ which is Euler's constant. Therefore we get that

$$\lim_{r \rightarrow 0} \left[-IC \frac{\partial}{\partial r} H_0^{(2)}(kr) \right] = j \frac{2IC}{\pi r}, \quad (10.3.9)$$

and from (10.3.6) we consequently obtain $C = 1/4j$. If the source of the field is positioned at a point $\mathbf{r}'$, we have to translate the coordinate system for that vector, and so the subsequent Green's function for the two-dimensional Helmholtz equation is

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4j} H_0^{(2)}(k|\mathbf{r} - \mathbf{r}'|). \quad (10.3.10)$$

The final expression for electric field strength at a point $\mathbf{r}$ caused by unit line current at a point $\mathbf{r}'$ is

$$E_z(\mathbf{r}) = -\frac{kZ'}{4} H_0^{(2)}(k|\mathbf{r} - \mathbf{r}'|), \quad (10.3.11)$$

where $Z' = (\mu/\epsilon)^{1/2}$. The general solution for the prescribed current distribution is thus

$$\underline{E}_z(\mathbf{r}) = -\frac{kZ'}{4} \int_S \underline{J}_z(\mathbf{r}') H_0^{(2)}(k|\mathbf{r} - \mathbf{r}'|) dS. \quad (10.3.12)$$

Let us analyse a perfectly conductive body immersed in an external TM field with respect to the z -axis, as shown in Figure 10.3.1.

The external field induces surface currents $\underline{J}_z$ on the surface of the body which, in turn, causes the scattered field $\underline{E}_z^s$. We determine this field using equation (10.3.21), which we apply only to the surface of the body. On a perfect conductor, the tangential component of electric field strength is equal to zero, and so on the curve C , we have that $\underline{E}_z = \underline{E}_z^I + \underline{E}_z^s = 0$.

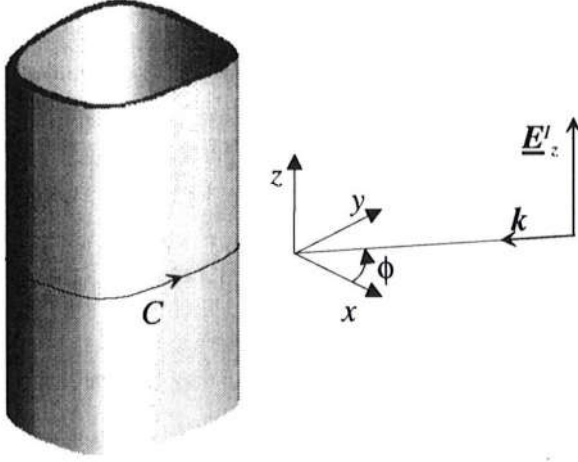


Figure 10.3.1. A perfectly conducting body immersed in an external TM field

Using expression (10.3.12), we derive the integral equation as being

$$\underline{E}_z^I(\mathbf{r}) = -\frac{kZ'}{4} \int_C \underline{J}_z(\mathbf{r}') H_0^{(2)}(k|\mathbf{r}-\mathbf{r}'|) dl \quad ; \quad \mathbf{r}' \in C, \quad (10.3.13)$$

where we know the external field $\underline{E}^I$, whereas the density of the surface current $\underline{J}_z$ is unknown and thus needs to be determined.

10.3.1.2. TE field

In problems involving dielectrics and perfect conductors ($\kappa \rightarrow \infty$), currents can only flow on surfaces of the perfect conductors S , and the integral solution for the magnetic vector potential is thus:

$$\underline{A}(\mathbf{r}) = \mu \int_S \underline{J}(\mathbf{r}') G(\mathbf{r}, \mathbf{r}') dS', \quad (10.3.14)$$

where G is the Green's function of the problem. We calculate the magnetic field strength using

$$\underline{H} = \frac{1}{\mu} \nabla \times \underline{A} = \nabla \times \int_S \underline{J}(\mathbf{r}') G(\mathbf{r}, \mathbf{r}') dS'. \quad (10.3.15)$$

Utilization of the identity $\nabla \times (\underline{J}G) = \nabla G \times \underline{J} + G(\nabla \times \underline{J}) = -\underline{J} \times \nabla G$ results in

$$\underline{H}(\mathbf{r}) = - \int_S \underline{J}(\mathbf{r}') \times \nabla G(\mathbf{r}, \mathbf{r}') dS'. \quad (10.3.16)$$

Here, we are analysing a scattering problem where we calculate the total magnetic field by $\underline{H} = \underline{H}^I + \underline{H}^s$, where $\underline{H}^I$ is the prescribed external field, and $\underline{H}^s$ is the scattered field caused by currents flowing on the conductor surfaces. We calculate this field by expression (10.3.16). On the conductor-dielectric boundary we have to satisfy the boundary condition for the tangential component of magnetic field strength, which is

$$\underline{J}(\mathbf{r}) = \mathbf{n} \times \underline{H} = \mathbf{n} \times \underline{H}^I(\mathbf{r}) + \mathbf{n} \times \underline{H}^S(\mathbf{J}, \mathbf{r}) \quad ; \quad \mathbf{r} \in S . \quad (10.3.17)$$

The tangential magnetic field is discontinuous at surface S . At the exterior of the surface S , the tangential component of $\underline{H}$ is finite, whereas inside S it is equal to zero, and on S it is equal to the surface current density. Therefore, we have to evaluate the limit of $\underline{H}^S$ as we approach S from the exterior

$$\underline{J}(\mathbf{r}) + \lim_{\mathbf{r} \rightarrow S} \left[\mathbf{n} \times \int_S \underline{J}(\mathbf{r}') \times \nabla G(\mathbf{r}, \mathbf{r}') dS' \right] = \mathbf{n} \times \underline{H}^I(\mathbf{r}) \quad ; \quad \mathbf{r} \in S . \quad (10.3.18)$$

In the two-dimensional sinusoidal TE field shown in Figure 10.3.2, we get that $\underline{H} = [\underline{H}_z^I(x, y) + \underline{H}_z^S(x, y)]\mathbf{a}_z$, $\underline{E} = \underline{E}_x(x, y)\mathbf{a}_x + \underline{E}_y(x, y)\mathbf{a}_y$ i $\underline{J} = \underline{J}_x(x, y)\mathbf{a}_x + \underline{J}_y(x, y)\mathbf{a}_y = \underline{J}\mathbf{a}_\rho$.

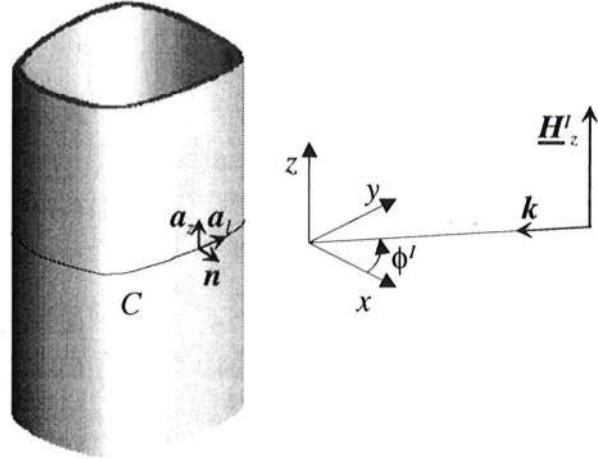


Figure 10.3.2. A perfectly conductive body in an external TE field

If we introduce these quantities into equation (10.3.18) and then perform a dot product with the unit vector $\mathbf{a}_\rho$, we obtain a *scalar* integral equation

$$\underline{J}(\mathbf{r}) + \lim_{\mathbf{r} \rightarrow C} \left[\mathbf{a}_\rho \cdot \mathbf{n} \times \int_C \underline{J}(\mathbf{r}') \mathbf{a}_\rho' \times \nabla G(\mathbf{r}, \mathbf{r}') dl' \right] = -\underline{H}_z^I(\mathbf{r}) \quad ; \quad \mathbf{r} \in C , \quad (10.3.19)$$

since $\mathbf{a}_\rho \cdot \mathbf{n} \times \mathbf{a}_z \underline{H}_z^I = \mathbf{a}_\rho' \cdot (-\mathbf{a}_\rho) \underline{H}_z^I = -\underline{H}_z^I$. We derived the two-dimensional Green's function in the previous section. By usage of the expression

$$\frac{dH_0^{(2)}(x)}{dx} = -H_1^{(2)}(x) , \quad (10.3.20)$$

and of the identity

$$\mathbf{a}_\rho \cdot \mathbf{n} \times \left(\mathbf{a}_\rho' \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} \right) = \mathbf{a}_\rho \cdot \mathbf{n} \times \left[(\mathbf{a}_z \times \mathbf{n}') \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} \right] = \mathbf{n}' \cdot \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} = \cos \alpha' ,$$

equation (10.3.19) is transformed to

$$\underline{J}(\mathbf{r}) - \lim_{\mathbf{r} \rightarrow C} \frac{k}{4j} \int_C \underline{J}(\mathbf{r}') \cos \alpha' H_1^{(2)}(k|\mathbf{r} - \mathbf{r}'|) dl' = -\underline{H}_z^I(\mathbf{r}) \quad ; \quad \mathbf{r} \in C . \quad (10.3.21)$$

To determine the limit where $\mathbf{r} \rightarrow C$, we extract a small section of the curve C in the

vicinity of the point $P(r)$ bounded by points M and N , as shown in Figure 10.3.3.

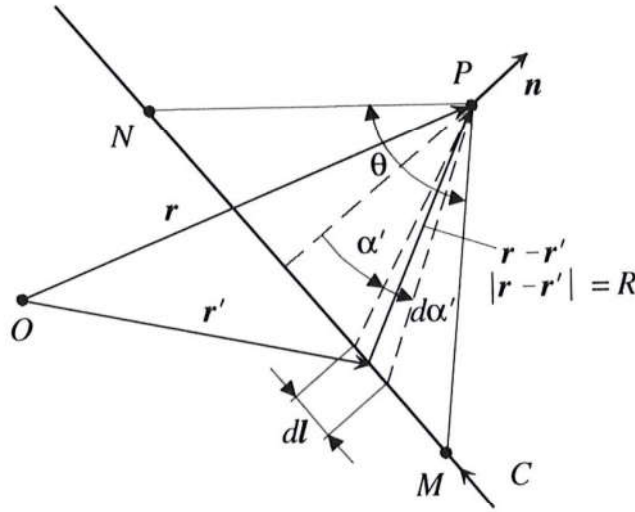


Figure 10.3.3. For deriving the integral equation for scattering a TE field

Equation (10.3.21) then becomes

$$\begin{aligned} \underline{J}(r) - \frac{k}{4j} \int_{C-MN} \underline{J}(r') \cos \alpha' H_1^{(2)}(k|r-r'|) dl' - \\ - \lim_{r \rightarrow MN} \frac{k}{4j} \int_{MN} \underline{J}(r') \cos \alpha' H_1^{(2)}(k|r-r'|) dl' = -\underline{H}_z^I(r) \quad ; \quad r \in C . \end{aligned} \quad (10.3.22)$$

On a small arc of the curve between the points M and N , we can assume that the current density $\underline{J}(r')$ is constant. In order to calculate the limiting value of the integral along the arc, we introduce the small argument approximation of $H_1^{(2)}$

$$H_1^{(2)}(k|r-r'|) \approx \frac{k|r-r'|}{2} + j \frac{2}{k\pi|r-r'|} \approx j \frac{2}{k\pi|r-r'|} . \quad (10.3.23)$$

The limiting value of the integral along the arc between the points M and N now becomes

$$\frac{\underline{J}(r)}{2\pi} \int_M^N \frac{\cos \alpha'}{|r-r'|} dl' = \frac{\underline{J}(r)}{2\pi} \int_M^N d\alpha' = \underline{J}(r) \frac{\theta}{2\pi} . \quad (10.3.24)$$

As the point P approaches the arc, θ approaches π . If we allow the distance MN to tend to zero, equation (10.3.21) reduces to

$$\frac{\underline{J}(r)}{2} - \frac{k}{4j} \int_C \underline{J}(r') \cos \alpha' H_1^{(2)}(k|r-r'|) dl' = -\underline{H}_z^I(r) \quad ; \quad r \in C , \quad (10.3.25)$$

and this is the integral equation for scattering problems in a TE field.

$$\underline{E}'_z(\mathbf{r}_{C,m}) = \sum_{i=1}^N \underline{J}_{z,i} \frac{kZ'}{4} \int_{\Delta C_i} H_0^{(2)}(k|\mathbf{r}_m - \mathbf{r}'_i|) dl \quad ; \quad m = 1, \dots, N. \quad (10.3.28)$$

This is the system of N linear algebraic equations for N unknowns that we can write in matrix form as $[Z]\underline{J} = \underline{E}'$. Here $\underline{J}$ is the vector of the unknown current coefficients, $\underline{E}'$ is the vector of the complex terms that are equal to the external field at the mid-points of the segments $\underline{E}'_i = \underline{E}_{z,i}(x_i, y_i)$, $i = 1, \dots, N$, and $[Z]$ is the matrix with complex terms

$$Z_{ij} = \frac{kZ'}{4} \int_{\Delta C_j} H_0^{(2)}[k\sqrt{(x_i - x')^2 + (y_i - y')^2}] dl' . \quad (10.3.29)$$

We cannot calculate them using any analytical procedure, and so we use numerical integration. We usually choose N so that we can analyse ΔC as a line segment. If this is the case, we use the parametric form of a straight line equation

$$\left. \begin{aligned} x_j'(t) &= x_{jp} + (x_{jk} - x_{jp})t \\ y_j'(t) &= y_{jp} + (y_{jk} - y_{jp})t \end{aligned} \right\} \quad 0 \leq t \leq 1 . \quad (10.3.30)$$

Here (x_{jp}, y_{jp}) is the first, and (x_{jk}, y_{jk}) is the last point of j -th segment. Substitution of the parametric form (10.3.30) into (10.3.29) transforms the calculation of Z_{ij} to

$$Z_{ij} = \frac{kZ'}{4} \Delta l_j \int_0^1 H_0^{(2)}[k\sqrt{(x_i - x_j'(t))^2 + (y_i - y_j'(t))^2}] dt' , \quad (10.3.31)$$

where Δl_j is the length of j -th segment. To calculate this integral, we use a one-dimensional numerical integration.

We now need to focus our attention on the calculation of the diagonal terms Z_{ii} . The point (x_i, y_i) in this case is on the segment l_i , and the Hankel function $H_0^{(2)}$ together with the integral in (10.3.31) become singular. We define the point (x_i, y_i) in terms of t_0 as

$$x_i(t_0) = x_{ip} + (x_{ik} - x_{ip})t_0 \quad , \quad y_i(t_0) = y_{ip} + (y_{ik} - y_{ip})t_0 . \quad (10.3.32)$$

We then analyse the vicinity of the point where the small argument approximation of the Hankel function $H_0^{(2)}$ is valid, and calculate the diagonal element Z_{ii} as being

$$Z_{ii} = \frac{kZ'}{4} \Delta l_i \left\{ \int_0^{t_0 - \zeta} H_0^{(2)}[k(t_0 - t)\Delta l_i] dt + S_{ii} + \int_{t_0 + \zeta}^1 H_0^{(2)}[k(t - t_0)\Delta l_i] dt \right\} . \quad (10.3.33)$$

The integrals in this expression are not singular and we can calculate them numerically. The extracted part S_{ii} is equal to

$$S_{ii} = \lim_{\epsilon \rightarrow 0} \left[\int_{t_0 - \zeta}^{t_0 - \epsilon} \left(1 - j \frac{2}{\pi} \left(\ln \frac{k(t_0 - t)\Delta l_i}{2} + \gamma \right) \right) dt + \int_{t_0 + \epsilon}^{t_0 + \zeta} \left(1 - j \frac{2}{\pi} \left(\ln \frac{k(t - t_0)\Delta l_i}{2} + \gamma \right) \right) dt \right] ,$$

and we may calculate it analytically. The final expression for calculating the diagonal terms Z_{ii} is

$$Z_{ii} = \frac{kZ'}{4} \Delta l_i \left\{ \int_0^{t_0 - \zeta} H_0^{(2)}[k(t_0 - t) \Delta l_i] dt + \int_{t_0 + \zeta}^1 H_0^{(2)}[k(t - t_0) \Delta l_i] dt + \right. \\ \left. + 2\zeta \left[1 - j \frac{2}{\pi} \left(\ln \frac{k\zeta \Delta l_i}{2} + 1 + \gamma \right) \right] \right\}. \quad (10.3.34)$$

We determine the current coefficients by solution of the system $\underline{I} = [Z]^{-1} \underline{E} = [Y] \underline{E}$. After determining current coefficients, we calculate the electric field strength in the surroundings of the conductive body by

$$\underline{E}_z(x, y) = \underline{E}^I - \frac{kZ'}{4} \sum_{j=1}^N I_j \Delta l_j \int_0^1 H_0^{(2)}[k \sqrt{(x - x_j'(t))^2 + (y - y_j'(t))^2}] dt'. \quad (10.3.35)$$

To calculate the far-field, we substitute the large argument approximation of $H_0^{(2)}$

$$H_0^{(2)}(k|\mathbf{r} - \mathbf{r}'|) \underset{r \gg r'}{\approx} \sqrt{\frac{2j}{\pi k|\mathbf{r} - \mathbf{r}'|}} e^{-jk|\mathbf{r} - \mathbf{r}'|}, \quad (10.3.36)$$

into expression (10.3.35) and use the approximations

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} \underset{r \gg r'}{\approx} \frac{1}{r}; \quad e^{j|\mathbf{r} - \mathbf{r}'|} \underset{r \gg r'}{\approx} e^{j\left(r - \frac{\mathbf{r} \cdot \mathbf{r}'}{r}\right)}, \quad (10.3.37)$$

thus avoiding any numerical integration.

Example 10.3.2.1. Scattering of a plane TM wave on a conductor of square cross-section

A conductor of square cross-section is placed in a field of a plane TM wave as shown in Figure 10.3.5.

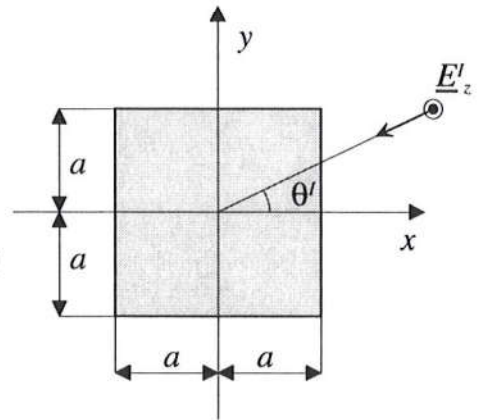


Figure 10.3.5. A conductor of square cross-section in an external TM field

The plane TM wave with respect to the z -axis is incident at an angle of θ' and governed by

$$\underline{E}_z^I = E_0^I e^{jk(x \cos \theta^I + y \sin \theta^I)} \quad (10.3.38)$$

Since the current is singular at the corners, we apply a non-uniform division here to segments by the algorithm described by expressions (10.2.12).

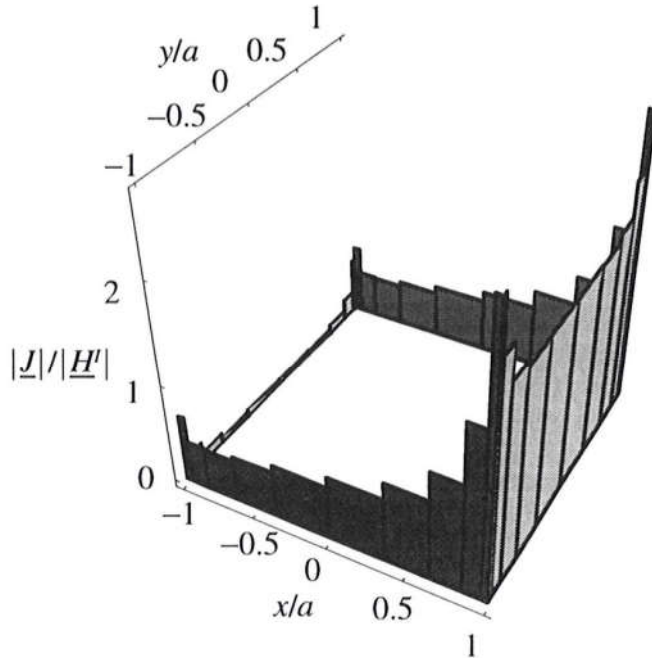


Figure 10.3.6. The normalized absolute value of the current coefficient phasors on the surface of the conductor, $|\underline{H}'| = |\underline{E}_z^I|/Z'$

An approximation of the surface current on the conductor for $N = 40$ segments with $p = 2$, $\theta^I = 0$ and $ka = 2$ is shown in Figure 10.3.6. We have calculated the electric field strength in the vicinity of the conductor applying expression (10.3.35) and have presented it in Figure 10.3.7.

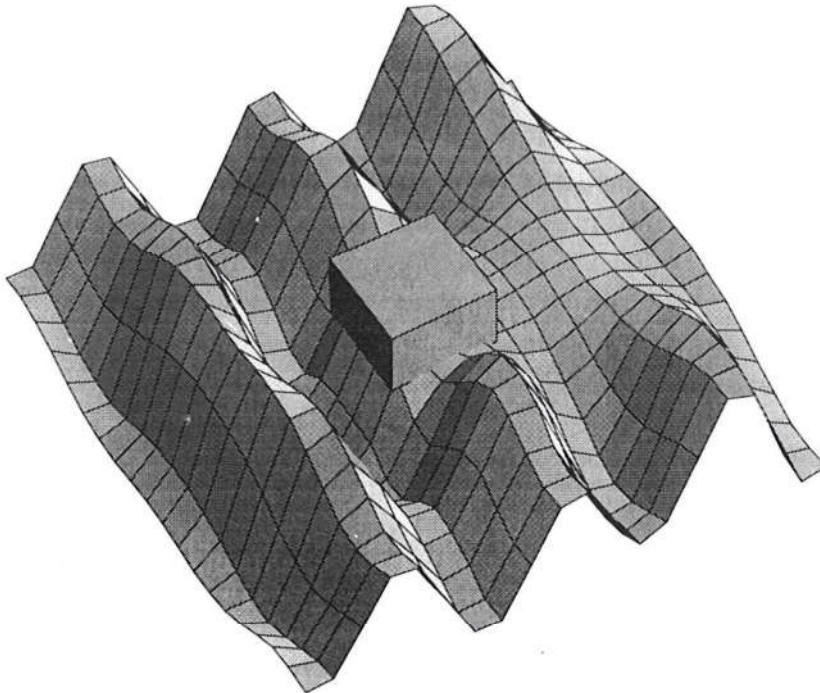


Figure 10.3.7. Distribution of the electric field strength in the vicinity of the conductor (the near field); a presentation in the time domain at a moment in time $t = 0$

We calculated electric field strength in the far-field zone by application of expression (10.3.35) and approximations (10.3.37,38). The results are shown in Figure 10.3.8.

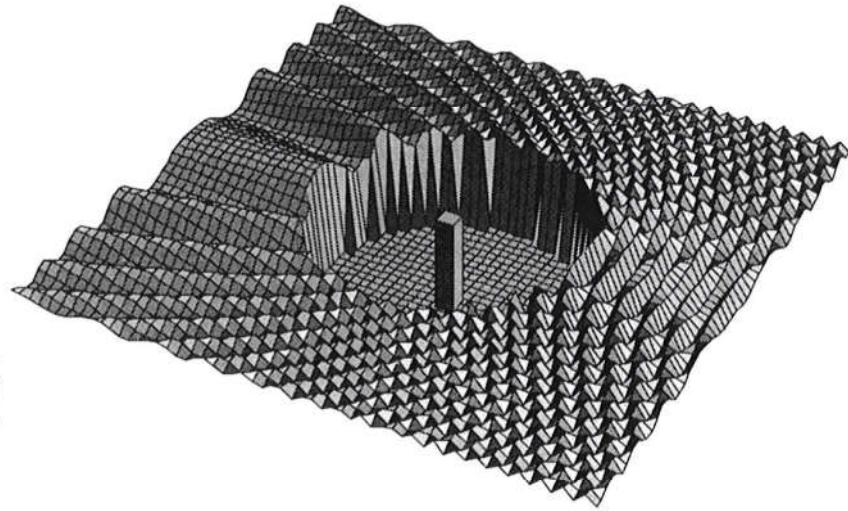


Figure 10.3.8. The distribution of the absolute value of electric field strength phasor in the far-field zone

10.3.2.2. Scattering in a TE field

In order to transform the integral equation (10.3.25) into a system of linear algebraic equations, we use point matching. The system of equations is thus:

$$\frac{J(r_{C,m})}{2} - \sum_{i=1}^N \underline{J}_i \frac{k}{4j} \int_{\Delta C_i} \cos \alpha' H_1^{(2)}(k|\mathbf{r}_{C,m} - \mathbf{r}'|) dl' = -\underline{H}_z^I(\mathbf{r}_{C,m}) ; m = 1, \dots, N . \quad (10.3.39)$$

Here, $\mathbf{r}_{C,m}$, $m = 1, \dots, N$ are the position vectors of the mid-points of the m -th segment, and $\underline{J}_i$ are complex coefficients of the approximation for unknown current. The matrix form of the system is $[\Gamma]\underline{J} = \underline{H}$. We calculate elements of the complex matrix $[\Gamma]$ using

$$\Gamma_{ij} = \frac{\delta_{ij}}{2} - \frac{k}{4j} \int_{\Delta C_j} H_1^{(2)}(k|\mathbf{r} - \mathbf{r}'|) \mathbf{n}' \cdot \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} dl' , \quad (10.3.40)$$

where δ_{ij} is the Kronecker delta

$$\delta_{ij} = \begin{cases} 1 ; & i = j , \\ 0 ; & i \neq j . \end{cases} \quad (10.3.41)$$

If ΔC_i are line segments, we calculate the line integral (10.3.40) by introducing the parametric form (10.3.30). The unit normal vector $\mathbf{n}$ is equal to the cross product of the unit tangential vector to ΔC and the unit vector in the z direction. On a line, it is constant

$$\mathbf{n}_i = \frac{(y_{ik} - y_{ip})\mathbf{a}_x - (x_{ik} - x_{ip})\mathbf{a}_y}{\sqrt{(x_{ik} - x_{ip})^2 + (y_{ik} - y_{ip})^2}} , \quad (10.3.42)$$

and we can transform expression (10.3.40) to

$$\Gamma_{ij} = \frac{\delta_{ij}}{2} - \frac{k}{4j} \frac{1}{\Delta l_j} \int_0^1 H_1^{(2)} [k \sqrt{(x_{C,i} - x'(t))^2 + (y_{C,i} - y'(t))^2} f(t) dt, \quad (10.3.43)$$

$$f(t) = \frac{(y_{jk} - y_{jp})(x_{C,i} - x'(t)) - (x_{jk} - x_{jp})(y_{C,i} - y'(t))}{\sqrt{(x_{C,i} - x'(t))^2 + (y_{C,i} - y'(t))^2}}.$$

We calculate the integral in this expression numerically by a one-dimensional integration. The integral contributions to the diagonal terms of the matrix $[\Gamma]$ representing the division of the curve C into line segments are equal to zero, since in this situation, $\cos \alpha' = 0$.

After determination of the unknown coefficients $\underline{J}_i$, we calculate the magnetic field strength in the surrounding area of the conductor by expression (10.3.16) as

$$\underline{H}(x,y) = \underline{H}^I - \frac{k}{4j} \sum_{i=1}^N \underline{J}_i \int_{\Delta C_i} \left(\underline{n}' \cdot \frac{\underline{r} - \underline{r}'}{|\underline{r} - \underline{r}'|} \right) H_1^{(2)}(k|\underline{r} - \underline{r}'|) d\ell'. \quad (10.3.44)$$

In order to avoid numerical integration in far-field computation, we substitute the large argument approximation of $H_1^{(2)}$

$$H_1^{(2)}(k|\underline{r} - \underline{r}'|) \approx \sqrt{\frac{2j}{\pi k|\underline{r} - \underline{r}'|}} j e^{-jk|\underline{r} - \underline{r}'|}, \quad r \gg r', \quad (10.3.45)$$

and the approximations

$$\frac{1}{|\underline{r} - \underline{r}'|} \approx \frac{1}{r} \quad ; \quad e^{j|\underline{r} - \underline{r}'|} \approx e^{j\left(r - \frac{\underline{r} \cdot \underline{r}'}{r}\right)} \quad ; \quad \frac{\underline{r} - \underline{r}'}{|\underline{r} - \underline{r}'|} \approx \frac{\underline{r}}{r}, \quad (10.3.46)$$

into expression (10.3.44)

Example 10.3.2.1. Scattering of a plane TE wave on a conductor of square cross-section

A conductor of square cross-section is placed in the external field of a plane TE wave.

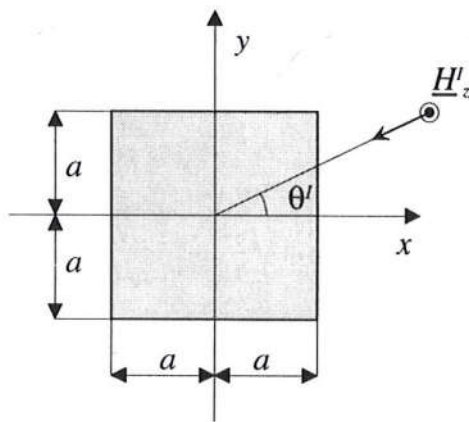


Figure 10.3.9. Conductor of square cross-section in an external TE field

The plane TE wave with respect to the z -axis incident at an angle of θ' is given by

$$\underline{H}_z^I = H_0^I e^{jk(x \cos \theta^I + y \sin \theta^I)}, \quad (10.3.47)$$

as shown in Figure 10.3.9. Since the current possesses no singularities at the corners, we have used a uniform segmentation here.

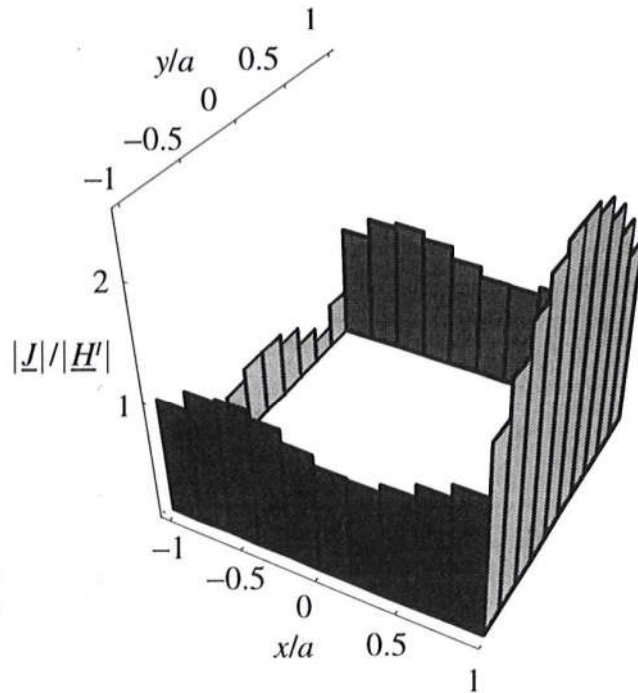


Figure 10.3.10. Normalized absolute value of the current coefficients phasor on the surface of the conductor

Approximation of the surface current for $N = 40$ segments with $p = 1$, $\theta^I = 0$ and $ka = 2$ is shown in Figure 10.3.10. We have calculated the magnetic field strength in the vicinity of the conductor using expression (10.3.44) and presented it in Figure 10.3.11.

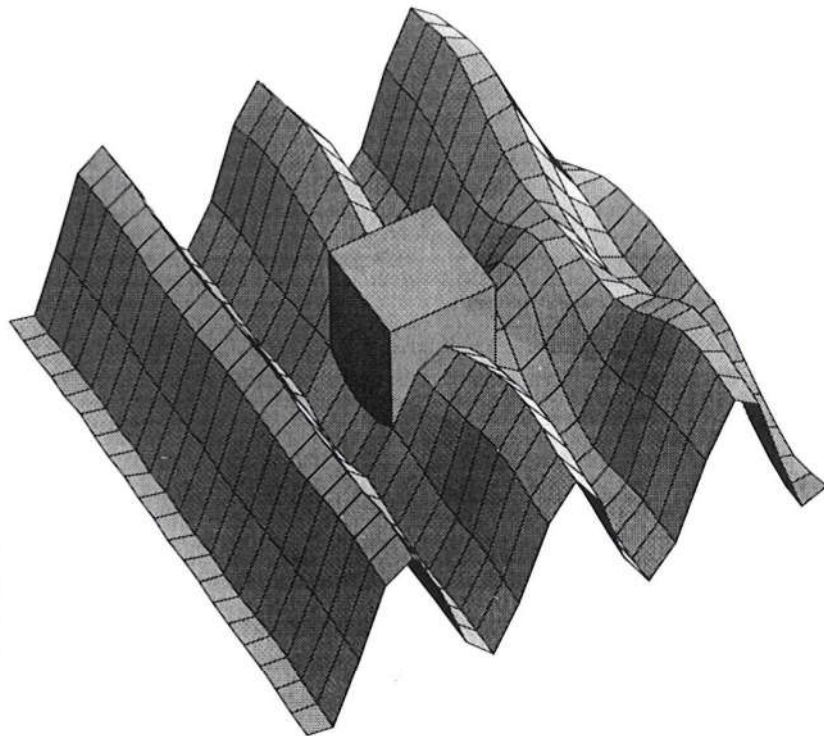


Figure 10.3.11. Magnetic field strength in the vicinity of the conductor (near field), presentation in a time domain at a moment in time $t = 0$

To calculate the magnetic field strength in the far-field zone, we have used expression (10.3.44) and the approximations (10.3.45,46). The results are shown in Fig. 10.3.12.

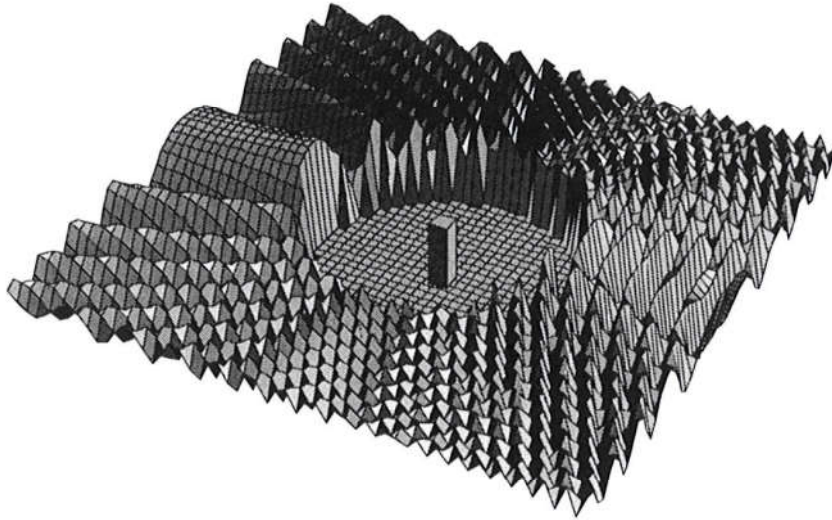


Figure 10.3.12. Absolute value of the magnetic field strength phasor in the far-field zone

10.4. THIN-WIRE STRUCTURES

In this unit, we shall treat radiation and scattering problems involving structures consisting of perfect cylindrical conductors in an unbounded homogeneous space. We assume that current flows only in the axial direction of a conductor and that there is no variation at the circumference. These requirements are very strict, but they are justified in numerous practical examples involving thin-wire antennae.

10.4.1. Equivalent sources

When solving radiation problems, we are often interested in a limited region of space. In this situation, we can replace regions of no interest with equivalent sources. We may either place these sources at regions of no interest or at the boundaries of the region of interest. The choice of equivalent sources is not unique, and we can thus construct them in various ways.

When constructing equivalent sources, we have to make sure that the original field and sources in the region of interest remain the same and that all boundary conditions are satisfied. If we have two different choices of equivalent sources resulting in the same solution within the region of interest, we call these problems equivalent.

In the first chapter, we showed that it is possible to introduce magnetic charge as a source of the field. To construct an equivalent source, we introduce magnetic currents $\underline{J}_m$ formed by magnetic charges to Faraday's law. Then the two main Maxwell's equations are transformed to $\nabla \times \underline{E} = -j\omega \underline{B} - \underline{J}_m$ and $\nabla \times \underline{H} = \underline{J} + j\omega \underline{D}$.

From these equations, we derive the boundary conditions for the tangential components of $\underline{E}$ and $\underline{H}$ between any two different materials. Usage of the procedure described in Section 1.5.1.2. results in the boundary conditions $\underline{n} \times (\underline{H}_1 - \underline{H}_2) = \underline{K}$ and

$\mathbf{n} \times (\underline{\mathbf{E}}_1 - \underline{\mathbf{E}}_2) = \underline{\mathbf{K}}_m$. Here, $\underline{\mathbf{K}}_m$ is the phasor of the magnetic current sheet that enables a discontinuity of the tangential component of electric field strength.

Therefore, we may substitute part of the space with known $\mathbf{n} \times \underline{\mathbf{E}}$ on the boundary of the region of interest using a magnetic current sheet flowing on the boundary, and the part with known $\mathbf{n} \times \underline{\mathbf{H}}$ using an electric current sheet flowing on the boundary.

10.4.2. The integral formulation of scattering problems involving thin wires

Let us analyse one perfectly conductive cylindrical tube of finite length placed in an external field, as shown in Figure 10.4.1. Since the tube is perfectly conductive, currents are flowing only on the conductor surface. We prescribe the electric field strength as being

$$\underline{\mathbf{E}}^I = \underline{E}_r^I(r, z) \mathbf{a}_r + \underline{E}_z^I(r, z) \mathbf{a}_z, \quad (10.4.1)$$

which means that it is circumferentially invariant and possesses no circumferential component in a cylindrical coordinate system. Since the geometry of the problem is circumferentially invariant as well, it follows that all quantities in a problem are also circumferentially invariant. Considering these assumptions, we may conclude that the surface current can possess only a z -component. Otherwise, if it possessed a circumferential component, a similar component of scattered electric field would result, thus violating the circumferential invariance of the field quantities.

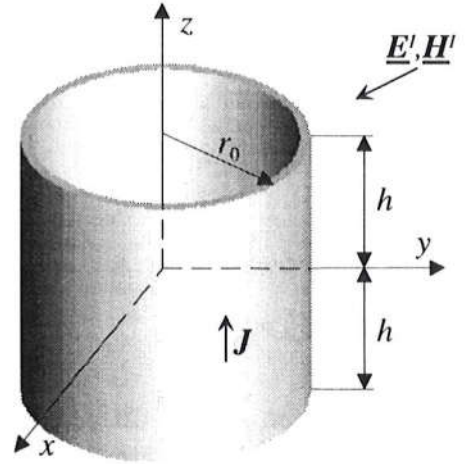


Figure 10.4.1. Perfectly conductive cylindrical tube placed in an external field

On the surface of a perfect conductor, the tangential component of electric field strength is equal to zero and on the tube we get that $\underline{E}_z^s(z) - \underline{E}_z^I(z) = 0$. The magnetic vector potential in this problem possesses only a z -component because the currents on the surface of the conductor possess only this component, and so we obtain

$$\underline{E}_z^s = -j\omega \underline{A}_z - \frac{d\varphi}{dz}. \quad (10.4.2)$$

The Lorentz gauge in a phasor domain is

$$\frac{dA_z}{dz} = -j\omega\mu\epsilon\varphi . \quad (10.4.3)$$

Differentiation of this equation and substitution into (10.4.2) and $\underline{E}_z^s(z) - \underline{E}_z^l(z) = 0$ results in

$$j\frac{\omega}{k^2} \left(\frac{d^2 A_z}{dz^2} + k^2 A_z \right) = \underline{E}_z^l . \quad (10.4.4)$$

We derived Green's function for free space in Unit 8.8. as the magnetic vector potential of a unit Hertzian dipole in a phasor domain, and so the general solution for the magnetic vector and scalar electric potential here is

$$\underline{A}_z = \frac{\mu}{4\pi} \int_{-h}^h \underline{J}_z(z') \int_0^{2\pi} \frac{e^{-jkR}}{R} r_0 d\alpha' dz' , \quad \varphi = \frac{1}{4\pi\epsilon} \int_{-h}^h \underline{\sigma}(z') \int_0^\pi \frac{e^{-jkR}}{R} r_0 d\alpha' dz' .$$

Here R is the distance between the point where we have a source (r_0, α', z') and the point where we calculate the field (r, α, z) . Since $\underline{J}_z$ is circumferentially invariant, we can introduce the total longitudinal current $\underline{I}$ as $\underline{I}(z) = 2r_0\pi\underline{J}(z)$. By using the continuity equation, we determine the relationship between the surface charge density $\underline{\sigma}$ and the total current

$$\frac{d\underline{I}(z)}{dz} = -j2r_0\pi\omega\underline{\sigma}(z) . \quad (10.4.5)$$

Substitution into equations for $\underline{A}_z$ and for φ results in

$$\underline{A}_z(z) = \frac{\mu}{4\pi} \int_{-h}^h \underline{I}(z') F(z-z') dz' , \quad \varphi(z) = j\frac{Z'}{4\pi k} \int_{-h}^h \frac{d}{dz'} \underline{I}(z') F(z-z') dz' .$$

In these expressions

$$F(z-z') = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} d\alpha' . \quad (10.4.6)$$

We obtain the integro-differential equation for unknown current $\underline{I}(z')$ by substitution of equations for $\underline{A}_z$ and for φ into (10.4.2) as

$$j\frac{Z'}{4\pi k} \left\{ k^2 \int_{-h}^h \underline{I}(z') F(z-z') dz' + \frac{d}{dz} \int_{-h}^h \frac{d}{dz'} \underline{I}(z') F(z-z') dz' \right\} = \underline{E}_z^l(z) , \quad (10.4.7)$$

or by substitution of the equation for $\underline{A}_z$ into (10.4.4)

$$j \frac{Z'}{4\pi k} \left(\frac{d^2}{dz^2} + k^2 \right) \int_{-h}^h I(z') F(z-z') dz' = \underline{E}_z^I(z) . \quad (10.4.8)$$

Equations (10.4.7,8) are alternative forms, which we refer to as *Pocklington's equations*. We solve them taking into account the condition that the total current at the end of the tube is equal to zero $I(\pm h) = 0$.

An application of Pocklington's equation enables the identical formulation of both scattering and radiation problems involving thin-wire structures. In radiation problems, the external field $\underline{E}_z^I$ is the field of the source, and in scattering problems $\underline{E}_z^I$ is the field of a distant source not included in the problem.

10.4.3. Models of excitation for radiation sources

We can apply the above derived Pocklington's equation directly in scattering problems. If we want to analyse a thin-wire structure as a source of radiation, we have to derive equations for the current induced by an excitation.

Excitation sources can be modelled in such a way that we may substitute a proper external electric field $\underline{E}_z^I$ into Pocklington's equation. To illustrate this procedure, we shall analyse the excitation of a cylindrical antenna fed by a coaxial line through the ground plane, as shown in Figure 10.4.2.

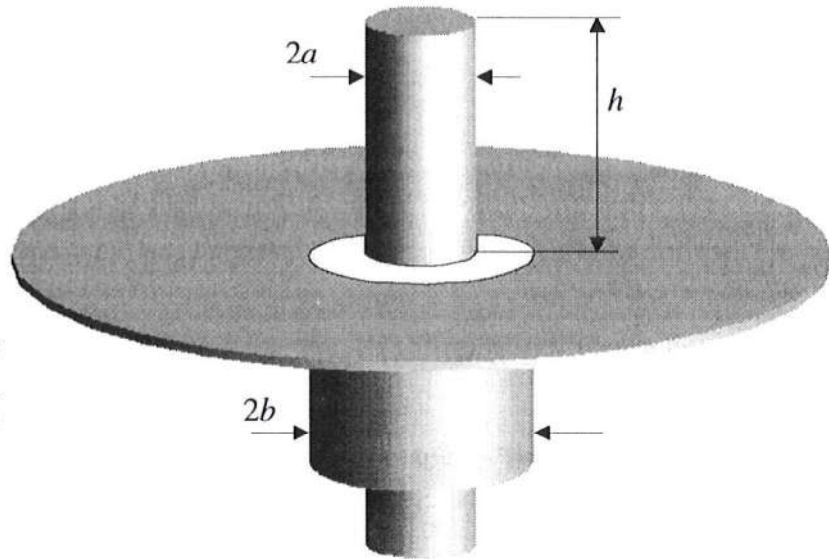


Figure 10.4.2. The excitation of a cylindrical antenna fed by a coaxial line through the ground plane

Here, the cylindrical antenna is excited by an electromagnetic field which penetrates the annular aperture placed at the join of the coaxial line and the ground plane. This means that we can regard the excitation of the cylindrical antenna as radiation from the field within this aperture. Since we are only interested in the half space above the ground plane, we shall replace the half space under the ground plane using equivalent sources within the aperture.

The tangential component of electric field strength has a discontinuity at the aperture. Therefore, we choose a magnetic current sheet $\underline{K}_m$ flowing on a perfect conductor placed

over the annular region formerly occupied by the aperture as our equivalent source. The tangential component of electric field strength in the ground plane possesses only a radial component due to the axial symmetry of the geometry, and from $\mathbf{n} \times (\mathbf{E}_1 - \mathbf{E}_2) = \mathbf{K}_m$ it follows that $\mathbf{K}_m \mathbf{a}_\alpha = -\mathbf{E}_r' \mathbf{a}_r \times \mathbf{a}_z$, and so the magnetic currents possess only circumferential components.

This choice of equivalent source results in a continuous ground plane, and we can apply the image theory in order to form an equivalent model of the upper half-space possessing $2\mathbf{K}_m$ and double the length of the tube. Our final model consists of a perfectly conductive tube excited by an equivalent magnetic current, as shown in Figure 10.4.3.

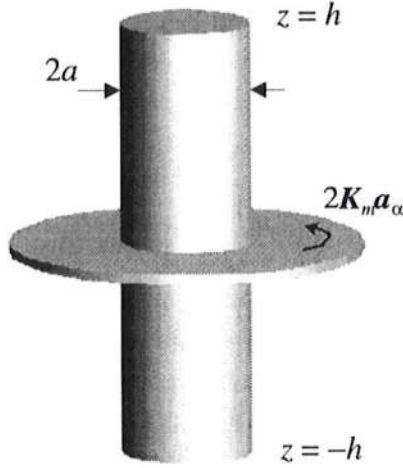


Figure 10.4.3. An equivalent model of a cylindrical antenna fed by a coaxial line

$\mathbf{E}_r'$ is generally unknown, but if $(b - a) \ll \lambda$, we can apply a quasi-static approximation and from the known voltage $\underline{U}$ on the coaxial line, we can determine

$$\mathbf{E}_r' = \frac{\underline{U}}{r' \ln \frac{b}{a}}. \quad (10.4.9)$$

By application of $\mathbf{n} \times (\mathbf{E}_1 - \mathbf{E}_2) = \mathbf{K}_m$, we evaluate the magnetic current sheet $\mathbf{K}_m$ as

$$\mathbf{K}_m = -\frac{\underline{U}}{r' \ln \frac{b}{a}} \mathbf{a}_\alpha. \quad (10.4.10)$$

From the main Maxwell's equations it follows that $-\nabla \times \nabla \times \mathbf{E} + k^2 \mathbf{E} = j\omega \mu \mathbf{J} - \nabla \times \mathbf{J}_m$, and since $\mathbf{J} = 0$, we determine the electric field strength caused by the magnetic currents as

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi} \int_{S'} \nabla \times \mathbf{K}_m(\mathbf{r}') \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} dS' = -\frac{\underline{U}}{2\pi \ln \frac{b}{a}} \int_0^{2\pi} \int_a^b \nabla \times \mathbf{a}_\alpha \frac{e^{-jkD}}{D} d\alpha' dr', \quad (10.4.11)$$

$$D = \sqrt{z^2 + r^2 + r'^2 - 2rr' \cos \alpha'}.$$

We may only calculate the electric field strength at a given point from this expression numerically. For points on the z -axis, the distance D is circumferentially invariant and the electric field strength in this case possesses only a z component that is equal to

$$\underline{E}_z = \frac{\underline{U}}{2 \ln \frac{b}{a}} \left(\frac{e^{-jk\sqrt{z^2+a^2}}}{\sqrt{z^2+a^2}} - \frac{e^{-jk\sqrt{z^2+b^2}}}{\sqrt{z^2+b^2}} \right). \quad (10.4.12)$$

We may construct similar equivalent models for other ways of excitation to enable a direct application of Pocklington's equation.

10.4.4. The solution of Pocklington's equation by the application of MOM

To solve this problem, we shall apply Galerkin's method. This means that we shall choose identical expansion and weighting functions. We shall describe here the application of triangular functions

$$w_m(z) = \begin{cases} 1 - \frac{|z - z_m|}{d} & ; \quad z_{m-1} \leq z \leq z_{m+1} , \\ 0 & ; \quad z < z_{m-1} \text{ ili } z > z_{m+1} . \end{cases} \quad (10.4.13)$$

Here $d = z_m - z_{m-1} = z_{m+1} - z_m$. These functions are shown in Figure 10.4.4.

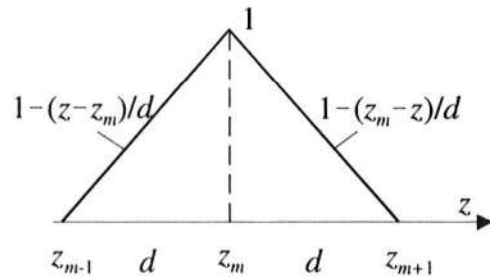


Figure 10.4.4. Triangular functions

Such a choice of expansion functions satisfies the condition $I(\pm h) = 0$. In the next step, we subdivide the tube into segments in the way presented in Figure 10.4.5.

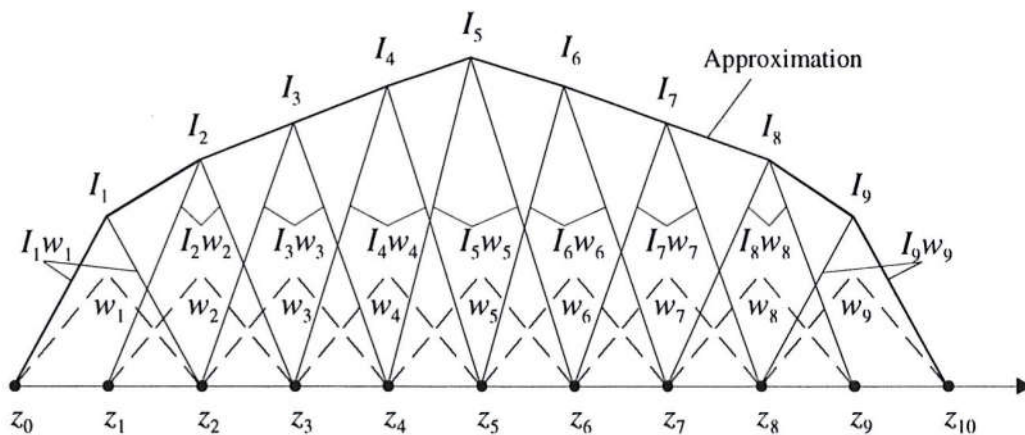


Figure 10.4.5. Expansion and weighting triangular functions for subdivision into 10 segments

Usage of Galerkin's method transforms Pocklington's equation (10.4.8) into

$$j \frac{Z'}{4\pi k} \int_{z_{m-1}}^{z_{m+1}} \left[\left(\frac{d^2}{dz^2} + k^2 \right) \int_{-h}^h I(z') K(z - z') dz' \right] w_m(z) dz =$$

$$= \int_{z_{m-1}}^{z_{m+1}} \underline{E}_z'(z) w_m(z) dz \quad ; \quad m = 1, \dots, N-1, \quad (10.4.14)$$

since the weighting function $w_m(z)$ are equal to zero outside the interval (z_{m-1}, z_{m+1}) . Double integration by parts and taking into account that $w_m(z_{m\pm 1}) = 0$ results in

$$\int_{z_{m-1}}^{z_{m+1}} \left[\frac{d^2 A(z)}{dz^2} \right] w_m(z) dz = \frac{2}{d} [A(z_{m+1}) - 2A(z_m) + A(z_{m-1})], \quad (10.4.15)$$

and by utilization of this identity, expression (10.4.14) transforms to

$$j \frac{Z'}{4\pi k} \frac{2}{d} \int_{-h}^h I(z') [F(z_{m+1} - z') - 2F(z_m - z') + F(z_{m-1} - z')] dz' +$$

$$+ j \frac{kZ'}{4\pi} \int_{z_{m-1}}^{z_{m+1}} \left[\int_{-h}^h I(z') F(z - z') dz' \right] w_m(z) dz = \int_{z_{m-1}}^{z_{m+1}} \underline{E}_z'(z) w_m(z) dz \quad ; \quad m = 1, \dots, N \quad (10.4.16)$$

We approximate the unknown current function using the triangular functions

$$I(z') = \sum_{i=1}^{N-1} I_i w_i(z'). \quad (10.4.17)$$

Substitution of this approximation into (10.4.16) results in a system of linear algebraic equations for unknown coefficients I_i , $[Z] \underline{I} = \underline{V}$. The terms of the matrix $[Z]$ are

$$Z_{ij} = j \frac{Z'}{4\pi k} \frac{2}{d} \int_{z_{j-1}}^{z_{j+1}} I(z') [F(z_{i+1} - z') - 2F(z_i - z') + F(z_{i-1} - z')] dz' +$$

$$+ j \frac{kZ'}{4\pi} \int_{z_{i-1}}^{z_{i+1}} \left[\int_{z_{j-1}}^{z_{j+1}} I(z') F(z - z') dz' \right] w_m(z) dz, \quad (10.4.18)$$

and terms of the vector on the right-hand side are

$$\underline{V}_i = \int_{z_{i-1}}^{z_{i+1}} \underline{E}_z'(z) w_m(z) dz. \quad (10.4.19)$$

For calculating the terms of the matrix $[Z]$ and the vector $\underline{V}$, we use a numerical integration. In the case where $z - z' = 0$, Green's function becomes singular. The geometrical

relationships used in calculating the kernel $F(z - z')$ are shown in Figure 10.4.6. and we can thus see that

$$D = \sqrt{(z - z')^2 + 2r_0^2 - 2r_0^2 \cos(\alpha - \alpha')} , \quad (10.4.20)$$

and so we can calculate the kernel $F(z - z')$ by

$$F(z - z') = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{-jk\sqrt{(z - z')^2 + 2r_0^2 - 2r_0^2 \cos(\alpha - \alpha')}}}{\sqrt{(z - z')^2 + 2r_0^2 - 2r_0^2 \cos(\alpha - \alpha')}} d(\alpha - \alpha') . \quad (10.4.21)$$

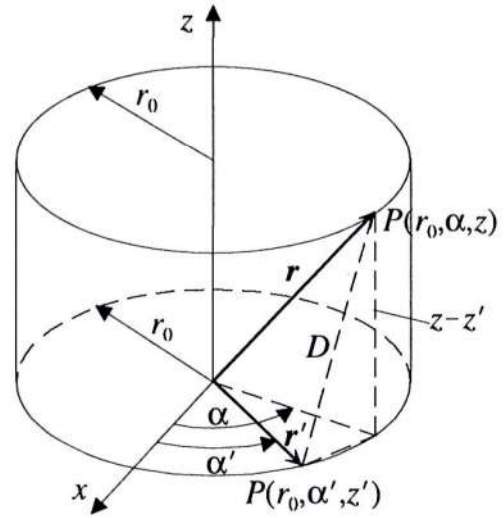


Figure 10.4.6. The geometrical relationships used in the calculation of the kernel of Pocklington's equation

In order to analyse the singular behaviour of the kernel $F(z - z')$, we shall write it as

$$F(z - z') = \frac{1}{2\pi} \int_0^{2\pi} \frac{d(\alpha - \alpha')}{\sqrt{(z - z')^2 + 2r_0^2 - 2r_0^2 \cos(\alpha - \alpha')}} + \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{-jk\sqrt{(z - z')^2 + 2r_0^2 - 2r_0^2 \cos(\alpha - \alpha')}} - 1}{\sqrt{(z - z')^2 + 2r_0^2 - 2r_0^2 \cos(\alpha - \alpha')}} d(\alpha - \alpha') . \quad (10.4.22)$$

The integrand of the second integral in this expression is bounded and continuous even at the point $(z - z') = 0$ because

$$\lim_{\alpha \rightarrow 0} \frac{e^{-jk\sqrt{2r_0^2 - 2r_0^2 \cos \alpha}} - 1}{\sqrt{2r_0^2 - 2r_0^2 \cos \alpha}} = \lim_{\alpha \rightarrow \pi} \frac{e^{-jk\sqrt{2r_0^2 - 2r_0^2 \cos \alpha}} - 1}{\sqrt{2r_0^2 - 2r_0^2 \cos \alpha}} = -jk . \quad (10.4.23)$$

Therefore we may calculate this integral analytically. We reduce the first integral in expression (10.4.31) to an elliptic integral of the first kind by the substitution $\alpha - \alpha' = \pi + 2\beta$

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{d(\alpha - \alpha')}{\sqrt{(z - z')^2 + 2r_0^2 - 2r_0^2 \cos(\alpha - \alpha')}} = \frac{kK(k)}{\pi r_0} \quad ; \quad k = \frac{2r_0}{\sqrt{(z - z')^2 + 4r_0^2}} . \quad (10.4.24)$$

This part of the kernel is singular for $k = 1$ and we shall expand it by adding and subtracting the small argument approximation of $K(k)$

$$\frac{1}{\pi r_0} \left\{ kK(k) + \ln \frac{|z - z'|}{8r_0} \right\} - \frac{1}{\pi r_0} \ln \frac{|z - z'|}{8r_0} . \quad (10.4.25)$$

We may integrate the first term of this expression numerically since, in the case where $k \rightarrow 1$, it tends to zero. A logarithmic singularity in the second term can be integrated analytically by usage of the expression

$$\lim_{x \rightarrow 0} x \ln x = 0 . \quad (10.4.26)$$

Finally, we use the form

$$\begin{aligned} F(z - z') = & \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{-jk\sqrt{(z - z')^2 + 2r_0^2 - 2r_0^2 \cos(\alpha - \alpha')}} - 1}{\sqrt{(z - z')^2 + 2r_0^2 - 2r_0^2 \cos(\alpha - \alpha')}} d(\alpha - \alpha') + \\ & + \frac{1}{\pi r_0} \left\{ kK(k) + \ln \frac{|z - z'|}{8r_0} \right\} - \frac{1}{\pi r_0} \ln \frac{|z - z'|}{8r_0} , \end{aligned} \quad (10.4.27)$$

to calculate integrals of the kernel F that possess a singularity.

10.4.4.1. Far radiation field

After solution of the system, we can determine the magnetic vector potential around the antenna by

$$\underline{\mathbf{A}} = \mu \int_S \underline{\mathbf{J}} \frac{e^{-jk|\mathbf{r} - \mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} dS . \quad (10.4.28)$$

By the usage of the usual approximations for a far-field region given by (10.3.37), we may determine the magnetic vector potential in a far-field region as

$$\underline{\mathbf{A}} = \underline{\mathbf{a}} \frac{\mu}{4\pi r} e^{-jkr} \sum_{i=1}^{N-1} \underline{\mathbf{I}}_i \int_{z_{i-1}}^{z_{i+1}} w(z') e^{-jkr' \cdot \frac{\mathbf{r}}{r}} dz' . \quad (10.4.29)$$

To calculate the electric field strength from electromagnetic potentials, we use $\underline{\mathbf{E}} = -j\omega \underline{\mathbf{A}} - \nabla \underline{\Phi}$. An application of the Lorentz gauge to this equation results in

$$\underline{E} = -j\frac{\omega}{k^2}[k^2\underline{A} + \nabla(\nabla \cdot \underline{A})] . \quad (10.4.30)$$

If we substitute equation (10.4.29) into this expression and take into account only those terms contributing to a far field (i.e. those varying by $1/r$), we obtain

$$\underline{E} = -j\omega \left[\underline{A} - \frac{\underline{r}}{r} \left(\frac{\underline{r}}{r} \cdot \underline{A} \right) \right] = j\omega \frac{\underline{r}}{r} \times \left(\frac{\underline{r}}{r} \times \underline{A} \right) , \quad (10.4.31)$$

and we use this expression to calculate the electric field strength in a far radiation zone.

Example 10.4.4.1. Radiation of a thin-wire antenna excited by a coaxial line

As an example, we have chosen the calculation of the radiation field of a thin-wire antenna excited by a coaxial line, as shown in Figure 10.4.2. We have shown the mathematical model of this problem in Figure 10.4.3.

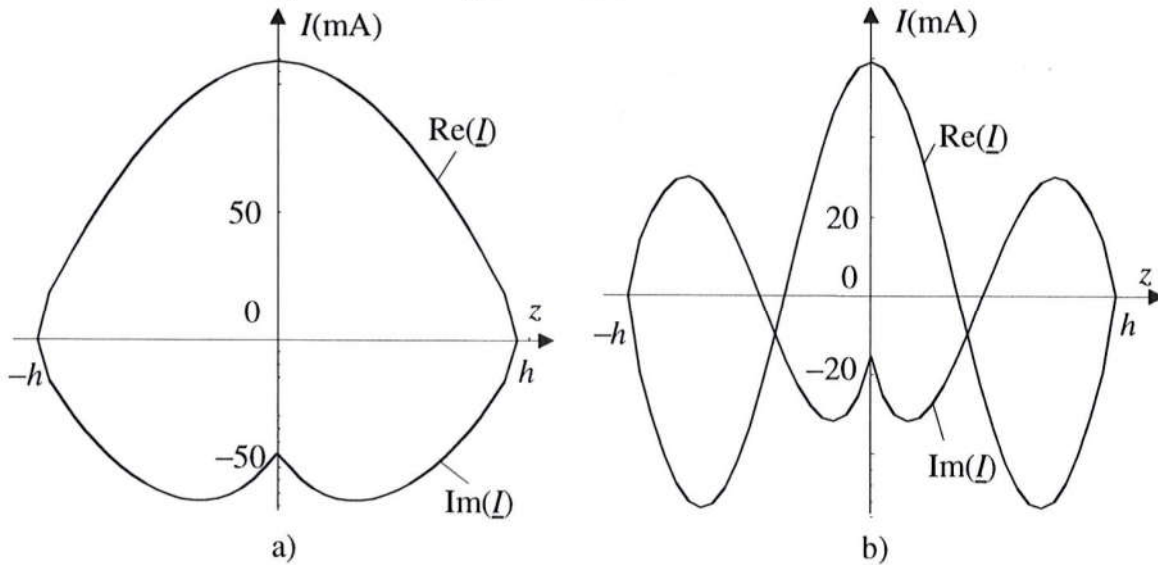


Figure 10.4.7. The distribution of current along our model of an antenna a) $2h = \lambda/2$, b) $2h = 3\lambda/2$

The distribution of current along the model of the antenna for heights $2h = \lambda/2$ and $2h = 3\lambda/2$, with $\underline{U} = 1$ V, $r_0 = 0.01\lambda$, $a = r_0$ and $b = 2r_0$ is shown in Figure 10.4.7. We have calculated the far radiation field by an application of expression (10.4.41). The distribution of the absolute value of electric field strength phasor in a far radiation zone is shown in Figure 10.4.8.

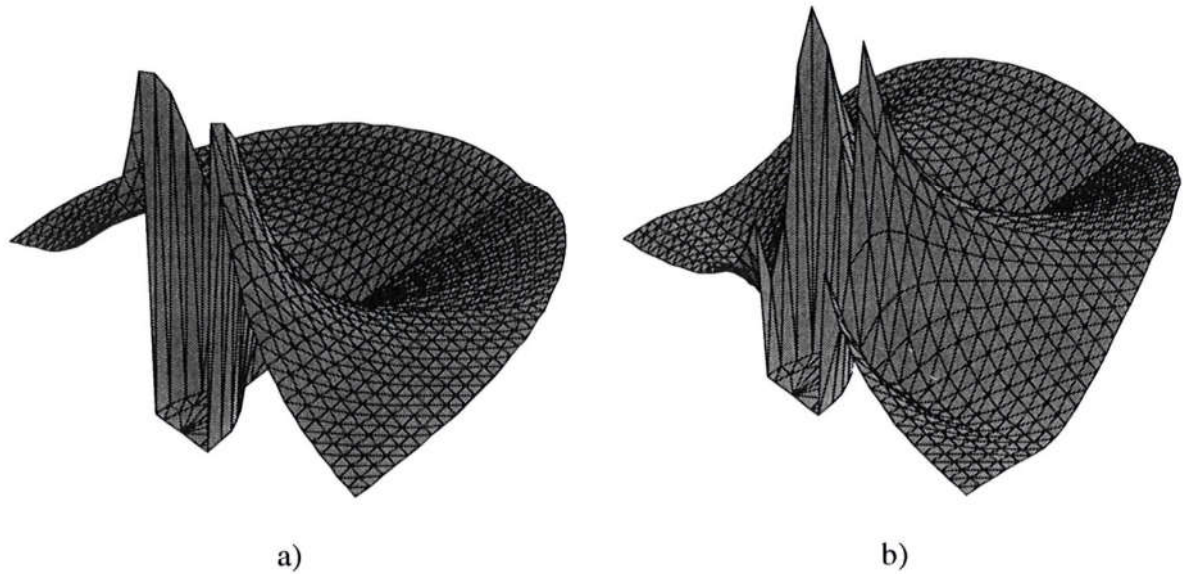


Figure 10.4.8. The distribution of the absolute value of electric field strength phasor in a far radiation zone of a thin-wire antenna excited by a coaxial line a) $2h = \lambda/2$, b) $2h = 3\lambda/2$

PROBLEMS

- 10.1. Derive the integral equation (10.2.6) for the dielectric-dielectric boundary in a static electric field!
Hint: use the derivation of the integral equation for scattering in a TE field!
- 10.2. Prove that the electric field strength of a unit line current in the space around the current satisfies the equation (10.3.3)
- 10.3. Analyse equation (10.3.35) for scattering in a TM field and prove that we may represent a conductive body in an external field by an N -port network!
- 10.4. Derive the boundary condition $\mathbf{n} \times (\mathbf{E}_1 - \mathbf{E}_2) = \mathbf{K}_m$!
- 10.5. Prove expression (10.4.30) for calculation of the electric field strength in a far radiation zone!

APPENDIX A

DIRAC δ -FUNCTION

In a macroscopic approach we describe field sources using continuous distributions in space. Sometimes it is useful to idealise the distribution of a source by assuming that a source of finite magnitude is concentrated at a point. This results in an infinite (or singular) source density. In order to describe such quantities, Dirac introduced δ -function in one dimension defined by

$$\delta(x - x') = 0 \quad ; \quad x \neq x' , \quad (\text{A.1})$$

under the condition

$$\int_l \delta(x - x') dx = 1 , \quad (\text{A.2})$$

if x' is within the domain of integration l . Therefore, this function is infinite at a point x' and equal to zero at all other points. The integral in (A.2) is a kind of extension of the integration toward such singularities. We may imagine that the δ -function emerges from a rectangular pulse if the width of the pulse tends to zero and the height of the pulse tends to infinity so that the area of the pulse is constant. The Dirac δ -function is not a function in the usual sense of the word, and we usually use the terms “generalized function” or “symbolic function” to describe such singular quantities. Important properties of the δ -function are

$$\int_l \psi(x) \delta(x - x') dl = \begin{cases} \psi(x') & ; \quad x' \in l , \\ 0 & ; \quad x' \notin l , \end{cases} \quad (\text{A.3})$$

and

$$\delta(x - x') = \delta(x' - x) . \quad (\text{A.4})$$

The three-dimensional δ -function is defined by

$$\delta(\mathbf{r} - \mathbf{r}') = 0 \quad ; \quad \mathbf{r} \neq \mathbf{r}' \quad \text{and} \quad \int_V \delta(\mathbf{r} - \mathbf{r}') d\mathbf{r} = 1 \quad ; \quad \mathbf{r} \in V . \quad (\text{A.5})$$

Accordingly to expressions (B.3,4) the three-dimensional δ -function satisfies the following

$$\int_V \psi(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}') dV = \begin{cases} \psi(\mathbf{r}') & ; \quad \mathbf{r}' \in V , \\ 0 & ; \quad \mathbf{r}' \notin V , \end{cases} \quad (\text{A.7})$$

and

$$\delta(\mathbf{r} - \mathbf{r}') = \delta(\mathbf{r}' - \mathbf{r}) . \quad (\text{A.8})$$

APPENDIX B

VECTOR CALCULUS

In this appendix, we shall present a short survey of the results of vector algebra and calculus that we most frequently use in electromagnetic theory. The magnitude A of a vector $\mathbf{A}$, which we denote as $|\mathbf{A}|$, possesses the following properties

$$\begin{aligned} |\mathbf{A}| \geq 0 \quad , \quad |\mathbf{A}| = 0 &\Leftrightarrow \mathbf{A} = 0 \quad , \\ |\lambda \mathbf{A}| = |\lambda| |\mathbf{A}| \quad , \quad |\mathbf{A} + \mathbf{B}| \leq |\mathbf{A}| + |\mathbf{B}| \quad . \end{aligned} \quad (\text{B.1})$$

Here, λ is a scalar. The vector

$$\mathbf{A}_0 = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\mathbf{A}}{A} \quad . \quad (\text{B.2})$$

is a unit vector in the direction of the vector $\mathbf{A}$.

B.1. Vector multiplication

We define the dot or inner product of two vectors $\mathbf{A}$ and $\mathbf{B}$ as being

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}| |\mathbf{B}| \cos \alpha \quad , \quad (\text{B.3})$$

where α is the angle between the vectors $\mathbf{A}$ and $\mathbf{B}$. Properties of the dot-product are

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= \mathbf{B} \cdot \mathbf{A} \quad , \quad \mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C} \quad , \\ (\lambda \mathbf{A}) \cdot \mathbf{B} &= \mathbf{A} \cdot (\lambda \mathbf{B}) = \lambda (\mathbf{A} \cdot \mathbf{B}) \quad , \quad |\mathbf{A}| = A = \sqrt{\mathbf{A} \cdot \mathbf{A}} \quad . \end{aligned} \quad (\text{B.4})$$

The cross or outer product of two vectors $\mathbf{A}$ and $\mathbf{B}$ is defined as being

$$\mathbf{A} \times \mathbf{B} = n |\mathbf{A}| |\mathbf{B}| \sin \alpha \quad , \quad (\text{B.5})$$

where α is the angle between the vectors $\mathbf{A}$ and $\mathbf{B}$, and $\mathbf{n}$ is a unit vector perpendicular to both $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{n}$ form a right-handed coordinate system. The properties of a cross product are

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= -\mathbf{B} \times \mathbf{A} \quad , \\ \mathbf{A} \times (\mathbf{B} + \mathbf{C}) &= \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \quad , \\ (\lambda \mathbf{A}) \times \mathbf{B} &= \mathbf{A} \times (\lambda \mathbf{B}) = \lambda (\mathbf{A} \times \mathbf{B}) \quad , \\ \mathbf{A} \times \mathbf{A} &= 0 \quad . \end{aligned} \quad (\text{B.6})$$

Vector relationships involving multiple products are

$$\begin{aligned}
\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) &= \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) , \\
\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) , \\
\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) - \mathbf{C} \times (\mathbf{B} \times \mathbf{A}) &= \mathbf{B} \times (\mathbf{A} \times \mathbf{C}) , \\
\mathbf{A} \times [\mathbf{B} \times (\mathbf{C} \times \mathbf{D})] &= (\mathbf{B} \cdot \mathbf{D})(\mathbf{A} \times \mathbf{C}) - (\mathbf{B} \cdot \mathbf{C})(\mathbf{A} \times \mathbf{D}) , \\
(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) &= (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}) , \\
(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) &= [(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{D}]\mathbf{C} - [(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}]\mathbf{D} , \\
(\mathbf{A} \times \mathbf{B}) \cdot [(\mathbf{B} \times \mathbf{C}) \times (\mathbf{C} \times \mathbf{A})] &= [\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})]^2 .
\end{aligned} \tag{B.7}$$

B.2. Differential relationships

Differentiation of a scalar field $\phi = \phi(\mathbf{r})$ with respect to a vector $\mathbf{c}$ at a point given by the position vector $\mathbf{r}$ is the limiting value of the quotient

$$\frac{\partial \phi}{\partial \mathbf{c}} = \lim_{\varepsilon \rightarrow 0} \frac{\phi(\mathbf{r} + \varepsilon \mathbf{c}) - \phi(\mathbf{r})}{\varepsilon} . \tag{B.8}$$

The derivative $\partial \phi / \partial c_0$ shows the rate of increase of the function ϕ in the direction of the unit vector $\mathbf{c}_0$ at each point. The biggest among all derivatives at any given point is the derivative $\partial \phi / \partial n$ in the direction of the unit vector $\mathbf{n}$ normal to the equipotential surface at this point.

The *gradient of a scalar field* ϕ is a *vector* in the direction of the normal to the equipotential surface with magnitude equal to $\partial \phi / \partial n$. We denote this vector as $\nabla \phi$, or as $\text{grad } \phi$, and define it as being

$$\nabla \phi(\mathbf{r}) = \lim_{\Delta V \rightarrow 0} \frac{\oint_S \mathbf{n} \phi dS}{\Delta V} . \tag{B.9}$$

The fundamental properties of the gradient are

$$\begin{aligned}
\nabla \lambda &= 0 , \quad \nabla(u + v) = \nabla u + \nabla v , \quad \nabla \lambda u = \lambda \nabla u , \\
\nabla(uv) &= v \nabla u + u \nabla v , \quad \nabla u(v) = u'(v) \nabla v .
\end{aligned} \tag{B.10}$$

Here, λ is a scalar constant and u and v are scalar fields.

The *divergence of a vector field* $\mathbf{F}(\mathbf{r})$ at a point $\mathbf{r}$, denoted by $\nabla \cdot \mathbf{F}(\mathbf{r})$ or $\text{div} \mathbf{F}(\mathbf{r})$, is defined as a *scalar* that is the limiting value of the quotient

$$\nabla \cdot \mathbf{F}(\mathbf{r}) = \lim_{\Delta V \rightarrow 0} \frac{\oint_S \mathbf{F} \cdot \mathbf{n} dS}{\Delta V} . \tag{B.11}$$

Here, ΔV is the volume enclosed by the *closed* surface S surrounding the point $\mathbf{r}$, and $\mathbf{n}$ is the unit vector normal to the elemental surface dS . Divergence represents the flux of

vector $\mathbf{F}$ from the volume ΔV per unit of this volume. If it is a value other than zero at any position, then $\mathbf{F}$ possesses its source or sink at this position. The fundamental properties of divergence are

$$\begin{aligned}
 \nabla \cdot \mathbf{C} &= 0 , \\
 \nabla \cdot (\mathbf{A} + \mathbf{B}) &= \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B} , \\
 \nabla \cdot \lambda \mathbf{A} &= \lambda \nabla \cdot \mathbf{A} , \\
 \nabla \cdot (u \mathbf{A}) &= u \nabla \cdot \mathbf{A} + (\nabla u) \cdot \mathbf{A} , \\
 \nabla \cdot (\mathbf{A} \times \mathbf{B}) &= \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}) .
 \end{aligned} \tag{B.12}$$

Here, $\mathbf{A}$ and $\mathbf{B}$ are vector fields, u is a scalar field, $\mathbf{C}$ is a constant vector, and λ is a scalar constant.

The *curl of a vector field* $\mathbf{F}(\mathbf{r})$ at a point $\mathbf{r}$, denoted by $\nabla \times \mathbf{F}(\mathbf{r})$ or $\text{curl } \mathbf{F}(\mathbf{r})$, is a *vector* that is the limiting value of the quotient

$$\nabla \times \mathbf{F}(\mathbf{r}) = \lim_{\Delta V \rightarrow 0} \frac{\oint_S \mathbf{n} \times \mathbf{F} dS}{\Delta V} . \tag{B.13}$$

Here, ΔV is the volume enclosed by the *closed* surface S surrounding the point $\mathbf{r}$, and $\mathbf{n}$ is a unit vector normal to the elemental surface dS . The fundamental properties of the curl are

$$\begin{aligned}
 \nabla \times (\mathbf{A} + \mathbf{B}) &= \nabla \times \mathbf{A} + \nabla \times \mathbf{B} , \\
 \nabla \times \lambda \mathbf{A} &= \lambda \nabla \times \mathbf{A} , \\
 \nabla \times (u \mathbf{A}) &= (\nabla u) \times \mathbf{A} + u (\nabla \times \mathbf{A}) , \\
 \nabla \times (\mathbf{A} \times \mathbf{B}) &= \mathbf{A} (\nabla \cdot \mathbf{B}) - \mathbf{B} (\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla) \mathbf{A} - (\mathbf{A} \cdot \nabla) \mathbf{B} .
 \end{aligned} \tag{B.14}$$

Here $\mathbf{A}$ and $\mathbf{B}$ are vector fields, u is a scalar field, and λ is a scalar constant. In all expressions, we have used the *del operator* ∇ . It is a partial differential vector operator and we have to obey the laws of vector algebra and of partial differentiation when we use it.

Some useful identities involving the gradient, divergence, and curl are

$$\begin{aligned}
 \nabla \cdot \mathbf{C} &= 0 , \\
 \nabla \cdot (\mathbf{A} + \mathbf{B}) &= \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B} , \\
 \nabla \cdot \lambda \mathbf{A} &= \lambda \nabla \cdot \mathbf{A} , \\
 \nabla \cdot (u \mathbf{A}) &= u \nabla \cdot \mathbf{A} + (\nabla u) \cdot \mathbf{A} , \\
 \nabla \cdot (\mathbf{A} \times \mathbf{B}) &= \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}) .
 \end{aligned} \tag{B.15}$$

Here $\mathbf{A}$ and $\mathbf{B}$ are vector fields, u and v are scalar fields, and the operator $\Delta = \nabla \cdot \nabla = \nabla^2$ is called the *Laplacian*.

B.3. Integral relationships

In the following expressions, V is a volume enclosed by a regular closed surface S . A unit vector $\mathbf{n}$ is normal to the elemental surface dS , and is of positive sign when directed outward of a closed surface. A fundamental expression for various integral relationships is

$$\int_V \frac{\partial f}{\partial u} dV = \oint_S f(\mathbf{n} \cdot \mathbf{a}_u) dS . \quad (\text{B.16})$$

Here, $\mathbf{a}_u$ is a unit vector in the direction of an arbitrary coordinate u , and f is a scalar field. By application of this expression in three-dimensional orthogonal coordinate systems, we obtain the following *Gauss' theorems*

$$\begin{aligned} \int_V \nabla \cdot \mathbf{A} dV &= \oint_S (\mathbf{A} \cdot \mathbf{n}) dS , & \int_V \nabla \times \mathbf{A} dV &= \oint_S (\mathbf{n} \times \mathbf{A}) dS \\ \int_V \nabla u dV &= \oint_S u \mathbf{n} dS . \end{aligned} \quad (\text{B.17})$$

Here, $\mathbf{A}$ is a vector field, and u is a scalar field. If we choose special vectors, eg. $u(\nabla v)$ where u and v are scalar fields, and if we introduce them into expressions (B.17) we may obtain a series of *Green's theorems*, of which we most frequently use the following two:

$$\begin{aligned} \int_V (u \Delta v - v \Delta u) dV &= \oint_S \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS , \\ \int_V [u \Delta v + (\nabla u) \cdot (\nabla v)] dV &= \oint_S u \frac{\partial v}{\partial n} dS . \end{aligned} \quad (\text{B.18})$$

If S is a regular two-sided surface outlined by a closed curve l , the following *Stokes's theorems*

$$\begin{aligned} \int_S [\mathbf{n} \cdot (\nabla \times \mathbf{A})] dS &= \oint_l \mathbf{A} \cdot d\mathbf{l} , & \int_S [\mathbf{n} \times (\nabla u)] dS &= \oint_l u d\mathbf{l} , \\ \int_S (\mathbf{n} \times \nabla) \times \mathbf{A} dS &= \oint_l d\mathbf{l} \times \mathbf{A} , \\ \int_S [(\nabla u) \times (\nabla v)] \cdot \mathbf{n} dS &= \oint_l u(\nabla v) \cdot d\mathbf{l} = - \oint_l v(\nabla u) \cdot d\mathbf{l} , \end{aligned} \quad (\text{B.19})$$

are valid.

APPENDIX C

COORDINATE SYSTEMS

C.1. A Cartesian coordinate system

A Cartesian or rectangular coordinate system is formed by the unit vectors $\mathbf{a}_x$, $\mathbf{a}_y$ and $\mathbf{a}_z$, as shown in Figure C.1.1.

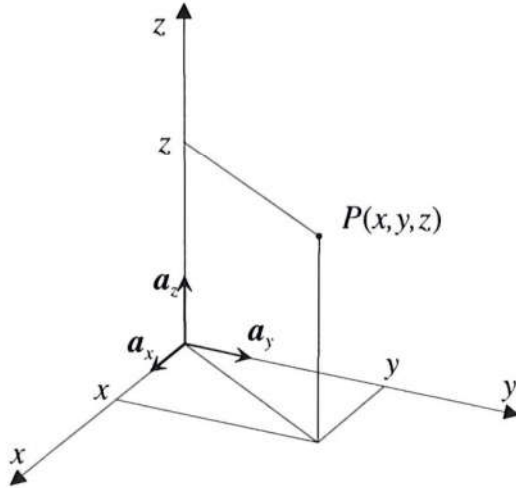


Figure C.1.1. A Cartesian coordinate system, the position of a point and unit vectors

The position vector of a point P with coordinates (x, y, z) is

$$\mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z, \quad (\text{C.1})$$

and the unit vectors $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$ satisfy the following

$$\begin{aligned} \mathbf{a}_x \cdot \mathbf{a}_y &= \mathbf{a}_y \cdot \mathbf{a}_z = \mathbf{a}_z \cdot \mathbf{a}_x = 0, \\ \mathbf{a}_x \times \mathbf{a}_y &= \mathbf{a}_z; \quad \mathbf{a}_y \times \mathbf{a}_z = \mathbf{a}_x; \quad \mathbf{a}_z \times \mathbf{a}_x = \mathbf{a}_y. \end{aligned} \quad (\text{C.2})$$

The del and Laplacian operators in a Cartesian coordinate system are of the form

$$\begin{aligned} \nabla &= \frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z, \\ \Delta &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \end{aligned} \quad (\text{C.3})$$

The scalar field $u(x, y, z)$ and the vector field $\mathbf{F}(x, y, z) = F_x(x, y, z)\mathbf{a}_x + F_y(x, y, z)\mathbf{a}_y + F_z(x, y, z)\mathbf{a}_z$ satisfy the following expressions:

$$\begin{aligned}
 \nabla u &= \frac{\partial u}{\partial x} \mathbf{a}_x + \frac{\partial u}{\partial y} \mathbf{a}_y + \frac{\partial u}{\partial z} \mathbf{a}_z, \quad \nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}, \\
 \nabla \times \mathbf{F} &= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \mathbf{a}_x + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \mathbf{a}_y + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \mathbf{a}_z, \\
 \Delta u &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}, \quad \Delta \mathbf{F} = \Delta F_x \mathbf{a}_x + \Delta F_y \mathbf{a}_y + \Delta F_z \mathbf{a}_z.
 \end{aligned} \tag{C.4}$$

Differential elements of length in the direction of the x -, y - and z -coordinate are dx , dy , and dz , and the elemental volume is thus $dV = dxdydz$.

C.2. A Cylindrical coordinate system

The unit vectors $\mathbf{a}_r$, $\mathbf{a}_\alpha$, and $\mathbf{a}_z$ forming a cylindrical coordinate system are shown in Figure C.2.1.

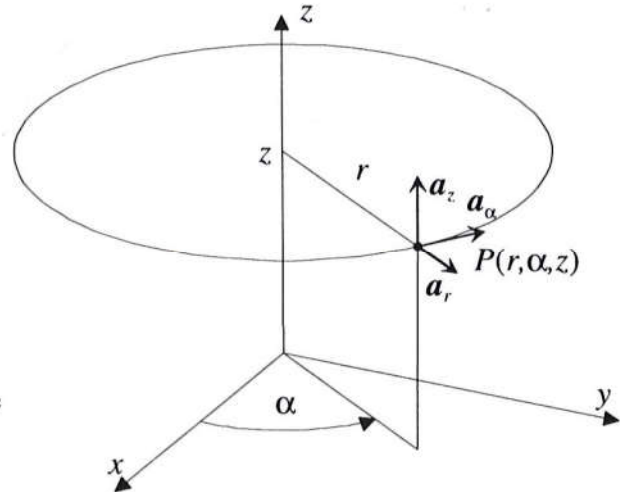


Figure C.2.1. A Cylindrical coordinate system, the position of a point and unit vectors

The relationships between Cartesian and cylindrical coordinates are:

$$\begin{aligned}
 \mathbf{a}_r &= \cos\alpha \mathbf{a}_x + \sin\alpha \mathbf{a}_y, \quad \mathbf{a}_\alpha = -\sin\alpha \mathbf{a}_x + \cos\alpha \mathbf{a}_y, \\
 \mathbf{a}_x &= \cos\alpha \mathbf{a}_r - \sin\alpha \mathbf{a}_\alpha, \quad \mathbf{a}_y = \sin\alpha \mathbf{a}_r + \cos\alpha \mathbf{a}_\alpha.
 \end{aligned} \tag{C.5}$$

The unit vectors $\mathbf{a}_r$, $\mathbf{a}_\alpha$, and $\mathbf{a}_z$ satisfy the following relationships:

$$\begin{aligned}
 \mathbf{a}_r \cdot \mathbf{a}_\alpha &= \mathbf{a}_\alpha \cdot \mathbf{a}_z = \mathbf{a}_z \cdot \mathbf{a}_r = 0, \\
 \mathbf{a}_r \times \mathbf{a}_\alpha &= \mathbf{a}_z; \quad \mathbf{a}_\alpha \times \mathbf{a}_z = \mathbf{a}_r; \quad \mathbf{a}_z \times \mathbf{a}_r = \mathbf{a}_\alpha, \\
 \frac{\partial \mathbf{a}_r}{\partial \alpha} &= \mathbf{a}_\alpha; \quad \frac{\partial \mathbf{a}_\alpha}{\partial \alpha} = -\mathbf{a}_r.
 \end{aligned} \tag{C.6}$$

The del operator in cylindrical coordinate system is of the form

$$\nabla = \frac{\partial}{\partial r} \mathbf{a}_r + \frac{1}{r} \frac{\partial}{\partial \alpha} \mathbf{a}_\alpha + \frac{\partial}{\partial z} \mathbf{a}_z . \quad (\text{C.7})$$

The vector field $\mathbf{F}(r, \alpha, z) = F_r(r, \alpha, z) \mathbf{a}_r + F_\alpha(r, \alpha, z) \mathbf{a}_\alpha + F_z(r, \alpha, z) \mathbf{a}_z$ and the scalar field $u(r, \alpha, z)$ satisfy the following relationships:

$$\begin{aligned} \nabla u &= \frac{\partial u}{\partial r} \mathbf{a}_r + \frac{1}{r} \frac{\partial u}{\partial \alpha} \mathbf{a}_\alpha + \frac{\partial u}{\partial z} \mathbf{a}_z , \\ \nabla \cdot \mathbf{F} &= \frac{1}{r} \frac{\partial (r F_r)}{\partial r} + \frac{1}{r} \frac{\partial F_\alpha}{\partial \alpha} + \frac{\partial F_z}{\partial z} , \\ \nabla \times \mathbf{F} &= \left(\frac{1}{r} \frac{\partial F_z}{\partial \alpha} - \frac{\partial F_\alpha}{\partial z} \right) \mathbf{a}_r + \left(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r} \right) \mathbf{a}_\alpha + \left(\frac{1}{r} \frac{\partial (r F_\alpha)}{\partial r} - \frac{1}{r} \frac{\partial F_r}{\partial \alpha} \right) \mathbf{a}_z , \\ \Delta u &= \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \alpha^2} + \frac{\partial^2 u}{\partial z^2} , \\ \Delta \mathbf{F} &= \left(\Delta F_r - \frac{F_r}{r^2} - \frac{2}{r^2} \frac{\partial F_\alpha}{\partial \alpha} \right) \mathbf{a}_r + \left(\Delta F_\alpha - \frac{F_\alpha}{r^2} + \frac{2}{r^2} \frac{\partial F_r}{\partial \alpha} \right) \mathbf{a}_\alpha + \Delta F_z \mathbf{a}_z . \end{aligned} \quad (\text{C.8})$$

The differential elements of length in the direction of the r -, α - and z -coordinate are dr , $r d\alpha$, and dz , and the elemental volume is thus $dV = r dr d\alpha dz$.

C.3. A Spherical coordinate system

A spherical coordinate system is formed by the unit vectors $\mathbf{a}_r$, $\mathbf{a}_\theta$, and $\mathbf{a}_\alpha$ in the direction of the coordinates r , ϑ , and α . The position of a point $P(r, \vartheta, \alpha)$ and the unit vectors are shown in Figure C.3.1.

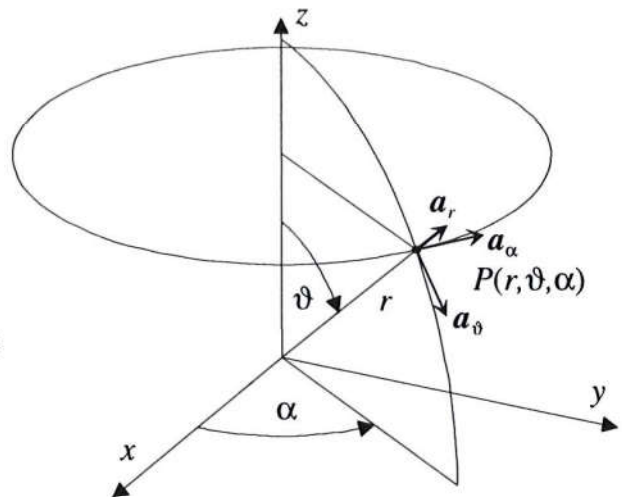


Figure C.3.1. A spherical coordinate system, the position of a point and unit vectors

Relationships between unit vectors of a Cartesian and spherical coordinate system are:

$$\begin{aligned}
\mathbf{a}_r &= \sin \vartheta \cos \alpha \mathbf{a}_x + \sin \vartheta \sin \alpha \mathbf{a}_y + \cos \vartheta \mathbf{a}_z, \\
\mathbf{a}_\vartheta &= \cos \vartheta \cos \alpha \mathbf{a}_x + \cos \vartheta \sin \alpha \mathbf{a}_y - \sin \vartheta \mathbf{a}_z, \quad \mathbf{a}_\alpha = -\sin \alpha \mathbf{a}_x + \cos \alpha \mathbf{a}_y, \\
\mathbf{a}_x &= \sin \vartheta \cos \alpha \mathbf{a}_r + \cos \vartheta \cos \alpha \mathbf{a}_\vartheta - \sin \alpha \mathbf{a}_\alpha, \\
\mathbf{a}_y &= \sin \vartheta \sin \alpha \mathbf{a}_r + \cos \vartheta \sin \alpha \mathbf{a}_\vartheta + \cos \alpha \mathbf{a}_\alpha, \quad \mathbf{a}_z = \cos \vartheta \mathbf{a}_r - \sin \vartheta \mathbf{a}_\vartheta.
\end{aligned} \tag{C.9}$$

The unit vectors satisfy the following relationships:

$$\begin{aligned}
\mathbf{a}_r \cdot \mathbf{a}_\vartheta &= \mathbf{a}_\vartheta \cdot \mathbf{a}_\alpha = \mathbf{a}_\alpha \cdot \mathbf{a}_r = 0, \\
\mathbf{a}_r \times \mathbf{a}_\vartheta &= \mathbf{a}_\alpha; \quad \mathbf{a}_\vartheta \times \mathbf{a}_\alpha = \mathbf{a}_r; \quad \mathbf{a}_\alpha \times \mathbf{a}_r = \mathbf{a}_\vartheta.
\end{aligned} \tag{C.10}$$

The del operator in a spherical coordinate system is of the form

$$\nabla = \frac{\partial}{\partial r} \mathbf{a}_r + \frac{1}{r} \frac{\partial}{\partial \vartheta} \mathbf{a}_\vartheta + \frac{1}{r \sin \vartheta} \frac{\partial}{\partial \alpha} \mathbf{a}_\alpha. \tag{C.11}$$

A vector field $\mathbf{F}(r, \vartheta, \alpha) = F_r(r, \vartheta, \alpha) \mathbf{a}_r + F_\vartheta(r, \vartheta, \alpha) \mathbf{a}_\vartheta + F_\alpha(r, \vartheta, \alpha) \mathbf{a}_\alpha$ and a scalar field $u(r, \vartheta, \alpha)$ in a spherical coordinate system satisfy the following relationships:

$$\begin{aligned}
\nabla u &= \frac{\partial u}{\partial r} \mathbf{a}_r + \frac{1}{r} \frac{\partial u}{\partial \vartheta} \mathbf{a}_\vartheta + \frac{1}{r \sin \vartheta} \frac{\partial u}{\partial \alpha} \mathbf{a}_\alpha, \\
\nabla \cdot \mathbf{F} &= \frac{1}{r^2} \frac{\partial(r^2 F_r)}{\partial r} + \frac{1}{r \sin \vartheta} \frac{\partial(\sin \vartheta F_\vartheta)}{\partial \vartheta} + \frac{1}{r \sin \vartheta} \frac{\partial F_\alpha}{\partial \alpha}, \\
\nabla \times \mathbf{F} &= \left\{ \frac{1}{r \sin \vartheta} \left[\frac{\partial(\sin \vartheta F_\alpha)}{\partial \vartheta} - \frac{\partial F_\vartheta}{\partial \alpha} \right] \right\} \mathbf{a}_r + \left[\frac{1}{r \sin \vartheta} \frac{\partial F_r}{\partial \alpha} - \frac{1}{r} \frac{\partial(r F_\alpha)}{\partial r} \right] \mathbf{a}_\alpha + \\
&\quad + \left[\frac{1}{r} \frac{\partial(r F_\vartheta)}{\partial r} - \frac{1}{r} \frac{\partial F_r}{\partial \alpha} \right] \mathbf{a}_\vartheta, \\
\Delta u &= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \sin \vartheta} \frac{\partial}{\partial \vartheta} \left(\sin \vartheta \frac{\partial u}{\partial \vartheta} \right) + \frac{1}{r^2 \sin^2 \vartheta} \frac{\partial^2 u}{\partial \alpha^2}, \\
\Delta \mathbf{F} &= \left(\Delta F_r - \frac{2 F_r}{r^2} - \frac{2 \cot \vartheta F_\vartheta}{r^2} - \frac{2}{r^2} \frac{\partial F_\vartheta}{\partial \vartheta} - \frac{2}{r^2 \sin \vartheta} \frac{\partial F_\alpha}{\partial \alpha} \right) \mathbf{a}_r + \\
&\quad + \left(\Delta F_\vartheta + \frac{2}{r^2} \frac{\partial F_r}{\partial \vartheta} - \frac{F_\vartheta}{r^2 \sin^2 \vartheta} - \frac{2 \cos \vartheta}{r^2 \sin^2 \vartheta} \frac{\partial F_\alpha}{\partial \alpha} \right) \mathbf{a}_\vartheta + \\
&\quad + \left(\Delta F_\alpha + \frac{2}{r^2 \sin \vartheta} \frac{\partial F_r}{\partial \alpha} - \frac{F_\alpha}{r^2 \sin^2 \vartheta} + \frac{2 \cos \vartheta}{r^2 \sin^2 \vartheta} \frac{\partial F_\vartheta}{\partial \alpha} \right) \mathbf{a}_\alpha.
\end{aligned} \tag{C.12}$$

The differential elements of length in the direction of the r -, ϑ - and α -coordinate are dr , $r d\vartheta$, and $r \sin \vartheta d\alpha$, and the elemental volume is thus $dV = r^2 \sin \vartheta dr d\vartheta d\alpha$.

APPENDIX D

BESSEL'S DIFFERENTIAL EQUATION

If we apply a separation of variables to Laplace's equation in a cylindrical coordinate system, we may frequently obtain a differential equation of the form

$$\frac{d^2 f}{dr^2} + \frac{1}{r} \frac{df}{dr} + \left(k^2 - \frac{p^2}{r^2} \right) f = 0 . \quad (\text{D.1})$$

The substitution $x = kr$ reduces this equation to the standard form of a *Bessel's differential equation* of order p

$$\frac{d^2 f}{dx^2} + \frac{1}{x} \frac{df}{dx} + \left(1 - \frac{p^2}{x^2} \right) f = x \frac{d}{dx} \left(x \frac{df}{dx} \right) + (x^2 - p^2) f = 0 . \quad (\text{D.2})$$

Solutions of this equation are Bessel functions $J_{\pm p}(x)$ that we can calculate by

$$J_p(x) = \left(\frac{x}{2} \right)^p \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+p+1)} \left(\frac{x}{2} \right)^{2m} . \quad (\text{D.3})$$

If p is not an integer, the functions J_p and J_{-p} are two independent solutions of the equation (D.2). When p is an integer, the function $\Gamma(m+p+1)$ is reduced to $(m+p)!$ and we get that $J_{-p} = (-1)^p J_p$, and these two solutions are no longer independent. In such a case, we introduce *Neumann functions*

$$N_p(x) = \lim_{n \rightarrow p} \frac{J_n(x) \cos n\pi - J_{-n}(x)}{\sin n\pi} , \quad (\text{D.4})$$

as the second independent solution instead of J_{-p} . When p is a positive integer, we get that

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{x}{2} \right)^{2m} , \quad N_0(x) = J_0(x) \frac{2}{\pi} \ln \frac{\gamma x}{2} + \frac{2}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{(m!)^2} \left(\frac{x}{2} \right)^{2m} \varphi(m) , \quad (\text{D.5})$$

for zero-order functions and

$$\begin{aligned} J_n(x) &= \sum_{m=0}^{\infty} \frac{(-1)^m}{m! (m+n)!} \left(\frac{x}{2} \right)^{2m+n} , \\ N_n(x) &= J_n(x) \frac{2}{\pi} \ln \frac{\gamma x}{2} - \frac{1}{\pi} \sum_{m=0}^{n-1} \frac{(n-m-1)!}{m!} \left(\frac{x}{2} \right)^{n-2m} - \\ &\quad - \frac{1}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{m! (m+n)!} \left(\frac{x}{2} \right)^{2m+n} [\varphi(m) + \varphi(m+n)] , \end{aligned} \quad (\text{D.6})$$

when n is a positive integer. In expressions (D.5,6) $\ln \gamma = 0.577216\dots$ which is Euler's constant, and the function $\varphi(m)$ is

$$\varphi(m) = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{m} . \quad (\text{D.7})$$

Bessel and Neumann functions are shown in Figures D.1 and D.2

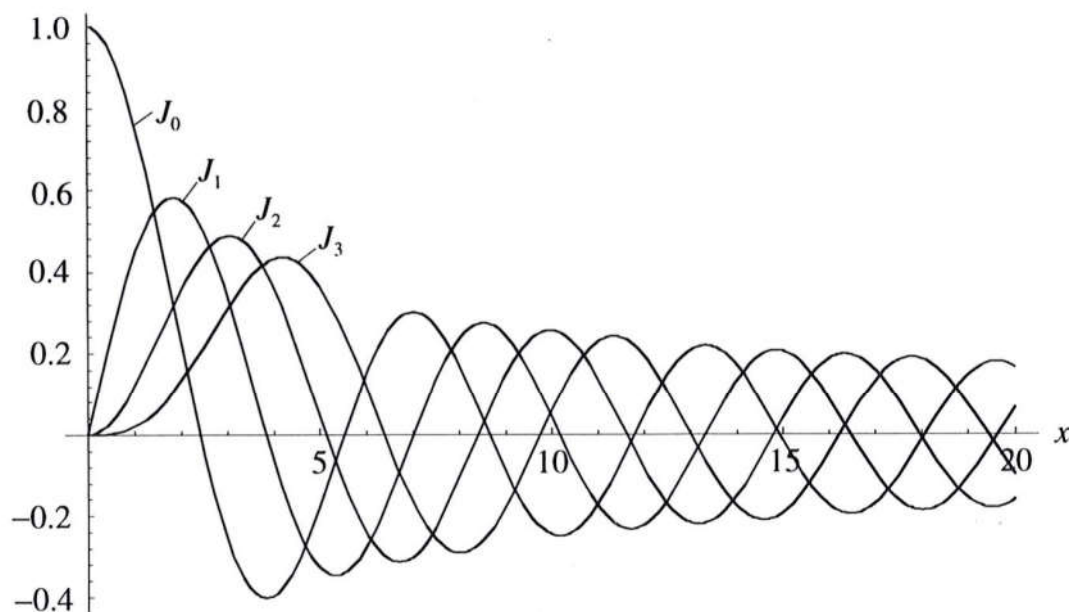


Figure D.1. Bessel function

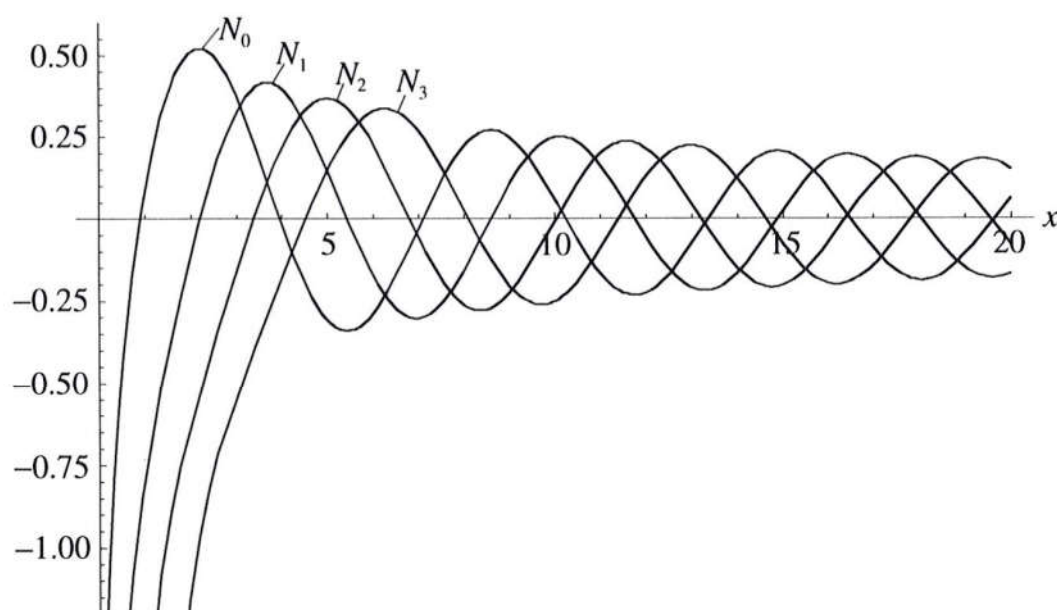


Figure D.2. Neumann function

We often use the small-argument approximations

$$\begin{aligned}
J_0(x) &\approx 1 \quad , \quad N_0(x) \approx \frac{2}{\pi} \ln \frac{\gamma x}{2} \quad , \\
J_n(x) &\approx \frac{1}{n!} \left(\frac{x}{2} \right)^n \quad , \quad N_n(x) \approx -\frac{(n-1)!}{\pi} \left(\frac{2}{x} \right)^n \quad ,
\end{aligned}
\tag{D.8}$$

and the large-argument approximations

$$J_n(x) \approx \sqrt{\frac{2}{\pi x}} \cos \left(x - \frac{\pi}{4} - \frac{n\pi}{2} \right) \quad , \quad N_n(x) \approx \sqrt{\frac{2}{\pi x}} \sin \left(x - \frac{\pi}{4} - \frac{n\pi}{2} \right) \quad .
\tag{D.9}$$

For the expression of wave phenomena, it is convenient to outline the linear combinations

$$H_n^{(1)}(x) = J_n(x) + jN_n(x) \quad , \quad H_n^{(2)}(x) = J_n(x) - jN_n(x) \quad ,
\tag{D.10}$$

which are called *Hankel functions* of the first and second kinds and which are solutions of Bessel's differential equations as well. We derive approximation equations of Hankel functions for small arguments from expression (D.8), and for large arguments from (D.9) as being

$$H_n^{(1)}(x) \approx \sqrt{\frac{2}{j\pi x}} j^{-n} e^{jx} \quad , \quad H_n^{(2)}(x) \approx \sqrt{\frac{2j}{\pi x}} j^n e^{jx} \quad .
\tag{D.11}$$

The wave-like character of Hankel functions is obvious from the approximations (D.9). If we denote any solution of Bessel's differential equation by $F_n(x)$, we can determine its derivative by

$$F_n'(x) = F_{n-1}(x) - \frac{n}{x} F_n(x) \quad , \quad F_n'(x) = -F_{n+1}(x) + \frac{n}{x} F_n(x) \quad ,
\tag{D.12}$$

which, in the special case where $n = 0$, results in

$$F_0'(x) = -F_1(x) \quad .
\tag{D.13}$$

A separation of variables in Helmholtz' equation in a spherical coordinate system often results in a differential equation of the form

$$\frac{d^2 f}{dr^2} + \frac{1}{r} \frac{df}{dr} - \left(k^2 + \frac{p^2}{r^2} \right) f = 0 \quad .
\tag{D.14}$$

By the substitution $x = kr$, we reduce this equation to a standard form of a *modified Bessel's differential equation* of order p

$$\frac{d^2 f}{dx^2} + \frac{1}{x} \frac{df}{dx} - \left(1 + \frac{p^2}{x^2} \right) f = x \frac{d}{dx} \left(x \frac{df}{dx} \right) - (x^2 + p^2) f = 0 \quad .
\tag{D.15}$$

When p is not an integer, solutions of this equation are p -th order modified Bessel functions of the first kind $I_p(x) = j^{-p} J_p(jx)$ and $I_{-p}(x) = j^p J_{-p}(jx)$. On the other hand, when p is an

integer, solutions are $I_p(x)$ together with p -th order modified Bessel function of the second kind

$$K_p(x) = \frac{\pi}{2} j^{p+1} H_p^{(1)}(jx) , \quad (\text{D.16})$$

Modified Bessel functions are shown in Figure D.3.

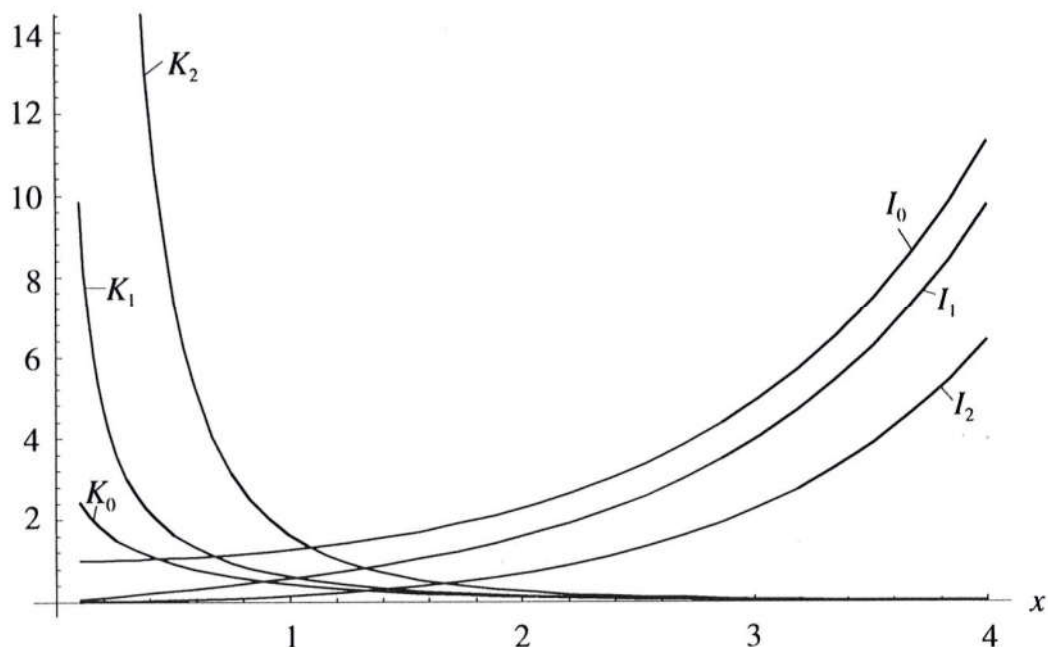


Figure D.3. Modified Bessel functions I_p and K_p

APPENDIX E

EXPANSION OF FUNCTIONS INTO EXPONENTIAL SERIES

A function $f(x)$ that is continuous and possesses all derivatives at $x = a$ can be expressed as the sum of an exponential series

$$f(x) = f(a) + \frac{x-a}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \dots + \frac{(x-a)^n}{n!}f^{(n)}(a) + \dots \quad (\text{E.1})$$

This series is called the *Taylor series* for functions of one variable. For a function of m variables, we write the Taylor series in a symbolical form as

$$f(x_1 + a_1, \dots, x_m + a_m) = f(x_1, \dots, x_m) + \sum_{i=1}^{\infty} \frac{1}{i!} \left(\sum_{j=1}^m \frac{\partial}{\partial x_j} a_j \right)^i f(x_1, \dots, x_m) \quad (\text{E.2})$$

If we choose $f(x) = (1 \pm x)^m$ and $m > 0$, from expression (E.1) the following *binomial series* results:

$$(1 \pm x)^m = 1 \pm mx + \frac{m(m-1)}{2!}x^2 \pm \frac{m(m-1)(m-2)}{3!}x^3 + \dots \quad (\text{E.3})$$

When m is an integer, this series is finite and consists of $m + 1$ terms. The binomial series (E.3) is convergent if $|x| \leq 1$. When the indices are negative, we obtain the binomial series

$$(1 \pm x)^{-m} = 1 \mp mx + \frac{m(m-1)}{2!}x^2 \mp \frac{m(m-1)(m-2)}{3!}x^3 + \dots \quad (\text{E.4})$$

which is convergent if $|x| < 1$. Some frequently used binomial series are

$$\begin{aligned} (1 \pm x)^{\frac{1}{2}} &= 1 \pm \frac{1}{2}x - \frac{1}{8}x^2 \pm \frac{3}{48}x^3 - \frac{15}{384}x^4 + \dots, \\ (1 \pm x)^{\frac{1}{3}} &= 1 \pm \frac{1}{3}x - \frac{2}{18}x^2 \pm \frac{10}{162}x^3 - \frac{80}{1944}x^4 + \dots, \\ (1 \pm x)^{-1} &= 1 \mp x + x^2 \mp x^3 + x^4 \mp \dots, \\ (1 \pm x)^{-\frac{1}{2}} &= 1 \mp \frac{1}{2}x + \frac{3}{8}x^2 \mp \frac{15}{48}x^3 + \frac{105}{384}x^4 \mp \dots, \\ (1 \pm x)^{-\frac{3}{2}} &= 1 \mp \frac{3}{2}x + \frac{15}{8}x^2 \mp \frac{105}{48}x^3 + \frac{945}{384}x^4 + \dots \end{aligned} \quad (\text{E.5})$$

APPENDIX F

ELLIPTIC INTEGRALS

If the symbol R denotes a rational function, then integrals of the form

$$\begin{aligned} \int R(x, \sqrt{ax^3 + bx^2 + cx + d}) dx, \\ \int R(x, \sqrt{ax^4 + bx^3 + cx^2 + dx + e}) dx, \end{aligned} \quad (\text{F.1})$$

are called *elliptic integrals* and in general cannot be expressed in terms of any elementary functions. By various transformations, we reduce these integrals to a *canonical* form for an elliptic integral of the first kind

$$\int \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}, \quad (\text{F.2})$$

for a elliptic integral of the second kind

$$\int \sqrt{1 - k^2 \sin^2 \varphi} d\varphi, \quad (\text{F.3})$$

and for an elliptic integral of the third kind

$$\int \frac{d\varphi}{(1 + h \sin^2 \varphi) \sqrt{1 - k^2 \sin^2 \varphi}}. \quad (\text{F.4})$$

The corresponding definite integrals are denoted as

$$\left. \begin{aligned} F(\varphi, k) &= \int_0^\varphi \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}, \\ E(\varphi, k) &= \int_0^\varphi \sqrt{1 - k^2 \sin^2 \varphi} d\varphi, \\ \Pi(\varphi, h, k) &= \int_0^\varphi \frac{d\varphi}{(1 + h \sin^2 \varphi) \sqrt{1 - k^2 \sin^2 \varphi}}, \end{aligned} \right\} \quad 0 < k < 1 \quad (\text{F.5})$$

and we call them incomplete elliptic integrals of the first, second, and third order. The parameter φ is called the amplitude, k is the modulus, and n is the characteristic of the elliptic integral. We call the integral $F(\pi/2, k)$ a complete elliptic integral of the first kind and denote it by $K(k)$, and the integral $E(\pi/2, k)$ is called the complete elliptic integral of the second kind and is denoted by $E(k)$. We calculate the complete elliptic integral using a series of the modulus k

$$\begin{aligned}
K(k) &= \frac{\pi}{2} \left[1 + \left(\frac{1}{2} \right)^2 k^2 + \left(\frac{3}{8} \right)^2 k^4 + \left(\frac{15}{48} \right)^2 k^6 + \dots \right], \\
E(k) &= \frac{\pi}{2} \left[1 - \left(\frac{1}{2} \right)^2 k^2 - \left(\frac{3}{8} \right)^2 \frac{k^4}{3} - \left(\frac{15}{48} \right)^2 \frac{k^6}{5} - \dots \right],
\end{aligned} \tag{F.6}$$

or of the complementary modulus $k' = (1 - k^2)^{1/2}$

$$\begin{aligned}
K(k') &= \ln \frac{4}{k'} + \left(\frac{1}{2} \right)^2 \left(\ln \frac{4}{k'} - 1 \right) + \left(\frac{3}{8} \right)^2 \left(\ln \frac{4}{k'} - 1 - \frac{2}{12} \right) + \dots, \\
E(k') &= 1 + \frac{1}{2} \left(\ln \frac{4}{k'} - 1 \right) k'^2 + \frac{3}{8} \left(\ln \frac{4}{k'} - 1 - \frac{2}{12} \right) k'^4 + \dots
\end{aligned} \tag{F.7}$$

The derivatives of complete elliptic integrals are

$$\begin{aligned}
\frac{dK(k)}{dk} &= \frac{E(k)}{kk'^2} - \frac{K(k)}{k}, \\
\frac{dE(k)}{dk} &= -\frac{E(k) - K(k)}{k}.
\end{aligned} \tag{F.8}$$

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